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RICCI SOLITONS IN KENMOTSU MANIFOLDS

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Abstract. In this paper we study Ricci solitons in Kenmotsu manifolds. We consider quasi conformal, conharmonic and projective curvature tensors in a Kenmotsu manifold admitting Ricci solitons and prove the conditions for the Ricci solitons to be shrinking, steady and expanding.

1. Introduction

In [5], Ramesh Sharma started the study of the Ricci solitons in contact geometry. Later Mukut Mani Tripathi [6], Cornelia Livia Bejan and Mircea Crasmareanu [2] and others extensively studied Ricci solitons in contact metric manifolds. A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is a Ricci tensor of M. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. In this paper, we prove conditions for Ricci solitons in Kenmotsu manifolds to be shrinking, steady and expanding. Section 2 is a very brief review of Kenmotsu manifolds and Ricci solitons. In section 3, we prove the conditions for Ricci solitons in a Kenmotsu manifold to be shrinking or expanding. The last section is devoted to the study of steady Ricci solitons in a Kenmotsu manifold.

2. Preliminaries

An n dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \ \eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

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An almost contact metric manifold is said to be a Kenmotsu manifold [4] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi(X), \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold the following relations hold [4].

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(\phi X, \phi Y), \tag{2.6}$$

$$\eta(R(X,Y)Z) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.9}$$

$$R(\xi,X)\xi = X - \eta(X)\xi. \tag{2.10}$$

Let (g, V, λ) be a Ricci soliton in an n dimensional Kenmotsu manifold M.
From (2.4) we have

$$(L_\xi g)(X,Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.11}$$

From (1.1) and (2.11) we see that

$$S(X,Y) = \eta(X)\eta(Y) - (\lambda+1)g(X,Y). \tag{2.12}$$

The above equation yields

$$QX = \eta(X)\xi - (\lambda+1)X, \tag{2.13}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{2.14}$$

$$r = -\lambda n - (n-1) \tag{2.15}$$

where Q and r are respectively the Ricci operator and scalar curvature on M.

3. Shrinking and Expanding Ricci solitons in a Kenmotsu manifold

Let M be an n dimensional Kenmotsu manifold admitting a Ricci soliton (g, V, λ).

3.1. **Shrinking Ricci solitons.**

The quasi-conformal curvature tensor $\tilde{C}$ in M is defined by [1]

$$\begin{aligned}\tilde{C}(X,Y)Z =& aR(X,Y)Z + b\left(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right) \\ &- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{3.1}$$

where a, b are constants.
Taking $Z = \xi$ in (3.1) and using (2.8), (2.13), (2.14), we obtain

$$\tilde{C}(X,Y)\xi = [a + b(2\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)](\eta(X)Y - (\eta(Y)X). \tag{3.2}$$

Similarly using (2.7), (2.12), (2.13), in (3.1), we get

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) =& -[a + b(2\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)][g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)].\end{aligned} \tag{3.3}$$

We assume that the condition

$$R(\xi, X).\tilde{C} = 0 \tag{3.4}$$

holds in M.
We can write (3.4) as

$$0 = R(\xi, X)(\tilde{C}(Y,Z)W) - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W - \tilde{C}(Y,Z)R(\xi,X)W, \tag{3.5}$$

for all vector fields X, Y, Z, W on M.
Using (2.9) in (3.5), we find

$$\begin{aligned} 0 =&\eta(\tilde{C}(Y,Z)W)X - {}'\tilde{C}(Y,Z,W,X)\xi \\ &- \eta(Y)\tilde{C}(X,Z)W + g(Y,X)\tilde{C}(\xi,Z)W \\ &- \eta(Z)\tilde{C}(Y,X)W + g(X,Z)\tilde{C}(Y,\xi)W \\ &- \eta(W)\tilde{C}(Y,Z)X + g(W,X)\tilde{C}(Y,Z)\xi, \end{aligned} \tag{3.6}$$

where

$${}'\tilde{C}(Y,Z,W,X) = g(\tilde{C}(Y,Z)W,X).$$

(3.6) may be rewritten as

$$\begin{aligned} 0 =&\eta(\tilde{C}(Y,Z)W)\eta(X) - {}'\tilde{C}(Y,Z,W,X) - \eta(Y)\eta(\tilde{C}(X,Z)W) \\ &+ g(Y,X)\eta(\tilde{C}(\xi,Z)W) - \eta(Z)\eta(\tilde{C}(Y,X)W) + g(X,Z)\eta(\tilde{C}(Y,\xi)W) \\ &- \eta(W)\eta(\tilde{C}(Y,Z)X) + g(W,X)\eta(\tilde{C}(Y,Z)\xi). \end{aligned} \tag{3.7}$$

In view of (3.1) - (3.3), the equation (3.7) reduces to

$$0 = g(\tilde{C}(Y,Z)W,X) + [a + b(2\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)][g(X,Y)g(W,Z) - g(X,Z)g(Y,W)]. \tag{3.8}$$

Using (3.1) in (3.8), we obtain

$$\begin{aligned} 0 =&a\,{}'R(Y,Z,W,X) + b[S(Z,W)g(Y,X) - S(Y,W)g(Z,X) + g(Z,W)g(QY,X) \\ &- g(Y,W)g(QZ,X)] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Z,W)g(Y,X) - g(Y,W)g(Z,X)] \\ &+ [a + b(2\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)][g(X,Y)g(W,Z) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.9}$$

Taking $X = Y = e_i$ in (3.9) and summing over $i = 1, 2, ..., n$, we get

$$[a + (n-2)b]S(Z,W) = -[a + (2\lambda+1)b](n-1)g(Z,W). \tag{3.10}$$

Taking $Z = W = e_i$ in (3.10) and summing over $i = 1, 2, ..., n$, we obtain from (2.15),

$$\lambda(2bn(n-1) + n[a + b(n-2)]) = (a+b)n(n-1) - [a + b(n-2)](n-1).$$

If $a + (n-2)b = 0$ then $\lambda = -\frac{(n-1)}{2}$.
i.e. λ is negative for $n \geq 2$ or Ricci soliton is shrinking for $n \geq 2$.
Hence we can state that

Theorem 3.1. *A Ricci soliton in a quasi conformally semi symmetric Kenmotsu manifold is shrinking for $n \geq 2$, provided $a + (n-2)b = 0$.*

3.2. **Expanding Ricci solitons.** We consider the projective curvature tensor[7] which is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{3.11}$$

From (3.1), (2.7), (2.13) and (2.14) and (3.15), we find

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) = {}& [a + b\lambda + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ & + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)],\end{aligned} \tag{3.12}$$

$$\begin{aligned}\eta(\tilde{C}(\xi,X)Y) = {}& [a + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)][\eta(Y)\eta(X) - g(X,Y)] \\ & + b[S(X,Y) + 2\lambda\eta(Y)\eta(X)],\end{aligned} \tag{3.13}$$

$$\eta(\tilde{C}(X,Y)\xi) = 0, \tag{3.14}$$

and

$$\begin{aligned}P(\xi,X)Y = {}& \frac{1}{n-1}[[(\lambda+2-n)g(X,Y) - \eta(X)\eta(Y)]\xi \\ & - (\lambda-n+1)\eta(Y)X].\end{aligned} \tag{3.15}$$

Assume that in M
$P(\xi,X).\tilde{C} = 0$ holds.
We can write

$$0 = P(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(P(\xi,X)Y,Z)W - \tilde{C}(Y,P(\xi,X)Z)W - \tilde{C}(Y,Z(P(\xi,X)W, \tag{3.16}$$

for all vector fields X, Y, Z, W on M.
Using (3.15) in (3.16) we obtain

$$\begin{aligned}0 = {}& [(\lambda+2-n)'\tilde{C}(Y,Z,W,X) - \eta(X)\eta(\tilde{C}(Y,Z)W)]\xi - (\lambda-n+1)\eta(\tilde{C}(Y,Z)W)X \\ & - [(\lambda+2-n)g(X,Y) - \eta(X)\eta(Y)]\tilde{C}(\xi,Z)W + (\lambda-n+1)\eta(Y)\tilde{C}(X,Z)W \\ & - [(\lambda+2-n)g(X,Z) - \eta(X)\eta(Z)]\tilde{C}(Y,\xi)W + (\lambda-n+1)\eta(Z)\tilde{C}(Y,X)W \\ & - [(\lambda+2-n)g(X,W) - \eta(X)\eta(W)]\tilde{C}(Y,Z)\xi + (\lambda-n+1)\eta(W)\tilde{C}(Y,Z)X.\end{aligned}$$

Contracting the above equation with ξ, we obtain

$$\begin{aligned}& [(\lambda+2-n)'\tilde{C}(Y,Z,W,X) - \eta(X)\eta(\tilde{C}(Y,Z)W)] \\ & - (\lambda-n+1)\eta(\tilde{C}(Y,Z)W)\eta(X) \\ & - [(\lambda+2-n)g(X,Y) - \eta(X)\eta(Y)]\eta(\tilde{C}(\xi,Z)W) \\ & + (\lambda-n+1)\eta(Y)\eta(\tilde{C}(X,Z)W) \\ & - [(\lambda+2-n)g(X,Z) - \eta(X)\eta(Z)]\eta(\tilde{C}(Y,\xi)W) \\ & + (\lambda-n+1)\eta(Z)\eta(\tilde{C}(Y,X)W) \\ & - [(\lambda+2-n)g(X,W) - \eta(X)\eta(W)]\eta(\tilde{C}(Y,Z)\xi) \\ & + (\lambda-n+1)\eta(W)\eta(\tilde{C}(Y,Z)X) = 0.\end{aligned} \tag{3.17}$$

From (3.12)-(3.14), the equation (3.17) is written as

$$\begin{aligned}&(\lambda+2-n)\,'\tilde{C}(Y,Z,W,X)+b\lambda[g(Z,W)\eta(Y)\eta(X)-g(Y,W)\eta(X)\eta(Z)]\\&+[(\lambda-n+2)(l+2b\lambda)-(\lambda-n+1)k][g(Z,X)\eta(Y)\eta(W)-g(X,Y)\eta(Z)\eta(W)]\\&+(\lambda-n+2)l[g(X,Y)g(Z,W)-g(X,Z)g(Y,W)]\\&+(\lambda-n+2)b[g(X,Z)S(Y,W)-g(X,Y)S(Z,W)]\\&+(\lambda-n+1)b[S(Z,X)\eta(Y)\eta(W)-S(X,Y)\eta(Z)\eta(W)]=0,\end{aligned} \tag{3.18}$$

$$where \qquad k=a+b\lambda+\frac{r}{n}\left(\frac{a}{n-1}+2b\right),$$
$$l=a+\frac{r}{n}\left(\frac{a}{n-1}+2b\right).$$

In view of (3.1) and (3.18), we have

$$\begin{aligned}&(\lambda+2-n)a\,'R(Y,Z,W,X)+(\lambda+2-n)b[g(Z,W)S(Y,X)-g(Y,W)S(Z,X)]\\&+b\lambda[g(Z,W)\eta(Y)\eta(X)-g(Y,W)\eta(X)\eta(Z)]\\&+[(\lambda-n+2)(l+2b\lambda)-(\lambda-n+1)k][g(Z,X)\eta(Y)\eta(W)-g(X,Y)\eta(Z)\eta(W)]\\&+(\lambda-n+2)a[g(X,Y)g(Z,W)-g(X,Z)g(Y,W)]\\&+(\lambda-n+1)b[S(Z,X)\eta(Y)\eta(W)-S(X,Y)\eta(Z)\eta(W)]=0.\end{aligned} \tag{3.19}$$

Taking $X=Y=e_i$ and summing over $i=1,.....,n$, we obtain

$$\begin{aligned}&(\lambda+2-n)(a-b)S(Z,W)+[(\lambda+2-n)br+b\lambda+(\lambda+2-n)a(n-1)]g(Z,W)\\&-[b\lambda+(n-1)[(\lambda-n+2)(l+2b\lambda)-(\lambda-n+1)k]+(\lambda-n+1)br\\&+\lambda(\lambda-n+1)b]\eta(Z)\eta(W)=0.\end{aligned} \tag{3.20}$$

Taking $Z=W=e_i$ and summing over $i=1,.....,n$, we obtain

$$\begin{aligned}&(\lambda+2-n)(a-b)r+[(\lambda+2-n)br+b\lambda+(\lambda+2-n)a(n-1)]n\\&-[b\lambda+(n-1)[(\lambda-n+2)(l+2b\lambda)-(\lambda-n+1)k]+(\lambda-n+1)br\\&+\lambda(\lambda-n+1)b]=0.\end{aligned}$$

If $a+(n-1)b=0$ then we have

$$\lambda=\frac{n^4-3n^3+2n^2-1}{n^3-n^2-n}.$$

Therefore λ is positive for all $n\geq 3$. i.e. the Ricci soliton is expanding.
Thus we can state that

Theorem 3.2. *A Ricci soliton in an n-dimensional Kenmotsu manifold satisfying $P(\xi,X).C=0$ is expanding for all $n\geq 3$ provided $a+(n-1)b=0$.*

4. Steady Ricci solitons in a Kenmotsu manifold

Let M be an n dimensional Kenmotsu manifold admitting a Ricci soliton (g, V, λ). The conharmonic curvature tensor [3] on M is defined by

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{4.1}$$

From (2.9), (2.12), (2.13), (2.14) and (4.1), we obtain

$$\begin{aligned} H(\xi,X)Y = \eta(Y)X - g(X,Y)\xi - \frac{1}{n-2}([\eta(X)\eta(Y) - (\lambda+1)g(X,Y)]\xi \\ + \lambda\eta(Y)X - \eta(Y)QX), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \eta(H(\xi,X)Z) = \eta(Z)\eta(X) - g(X,Z) - \frac{1}{n-2}((2\lambda+1)\eta(X)\eta(Z) \\ - (\lambda+1)g(X,Z)). \end{aligned} \tag{4.3}$$

We assume that $H(\xi, X).S = 0$, holds. Then we have

$$S(H(\xi,X)Y,Z) + S(Y,H(\xi,X)Z) = 0. \tag{4.4}$$

In view of (2.12), (4.4) becomes

$$\eta(H(\xi,X)Y)\eta(Z) - (\lambda+1)g(H(\xi,X)Y,Z) + \eta(Y)\eta(H(\xi,X)Z) - (\lambda+1)g(Y,H(\xi,X)Z) = 0, \tag{4.5}$$

$$i.e.\ \eta(H(\xi,X)Y)\eta(Z) + \eta(Y)\eta(H(\xi,X)Z) = (\lambda+1)[g(H(\xi,X)Y,Z) + g(Y,H(\xi,X)Z)]. \tag{4.6}$$

Using (4.2) and (4.3) in (4.6), we have

$$\begin{aligned} & 2\eta(X)\eta(Y)\eta(Z)\left(1 - \frac{1}{n-2} - \frac{2\lambda}{n-2}\right) \\ & + \left(\frac{\lambda+1}{n-2} - 1 + \frac{\lambda(\lambda+1)}{n-2}\right) \\ & [g(X,Y)\eta(Z) + g(X,Z)\eta(Y)] = 0. \end{aligned} \tag{4.7}$$

Taking $X = Y = e_i$ and summing over $i = 1, ..., n$, we obtain

$$(n+1)\lambda^2 + 2(n-1)\lambda - (n-1)(n-3) = 0,$$
$$when\ \ n = 3,\ \ 4\lambda^2 + 4\lambda = 0$$
$$\therefore\ \lambda = 0\ \ or\ \ \lambda = -1$$

Conversely for $n = 3$ *and* $\lambda = 0$, we have

$$H(\xi,X).S = S(H(\xi,X)Y,Z) + S(Y,H(\xi,X)Z).$$

In view of (2.1), (4.2) and (4.3), we obtain
$H(\xi, X).S = 0$.
Thus the converse is true when $\lambda = 0$.
Hence we can state that

Theorem 4.1. *A Ricci soliton in a three dimensional Kenmotsu manifold is steady iff* $H(\xi, X).S = 0$.

The condition $\tilde{C}(\xi, X).S = 0$, implies that

$$S(\tilde{C}(\xi, X)Y, Z) + S(Y, \tilde{C}(\xi, X)Z) = 0. \tag{4.8}$$

In view of (3.1), (4.8) becomes

$$[a+b(2\lambda+1)+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)][2\eta(X)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)-g(X,Y)\eta(Z)] = 0. \tag{4.9}$$

From (4.9), by a contraction we get

$$[a + b(2\lambda + 1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)](1 - n)\eta(Y) = 0. \tag{4.10}$$

From (4.10), we have

$$a + b(2\lambda + 1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) = 0. \tag{4.11}$$

If $\lambda = 0$, then from (4.11) we get $a = \frac{b(n-2)}{(n-1)}$.
The converse is also true.
Thus we have

Theorem 4.2. *If a Ricci soliton in a Kenmotsu manifold is steady, then M satisfies the condition $\tilde{C}(\xi, X).S = 0$ iff $a = \frac{n-2}{n-1}b$.*

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