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EXTENSIONS OF STEFFENSEN'S INEQUALITY

W. T. SULAIMAN

Abstract. In this paper we generalized the result of Pecaric concerning Steffensen's inequality. For another result we extend the scope of this result. As well as we presented the reverse inequality. Other results are also deduced as corollaries.

1. Introduction

Steffensen's inequality reads as follows :

Theorem 1.1. *Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval (a,b), that $f(t)$ non-increasing and that $0 \le g(t) \le 1$ in (a,b). Then*

$$\int_{a-\lambda}^{b} f(t)dt \le \int_{a}^{b} f(t)g(t)dt \le \int_{a}^{a+\lambda} f(t)dt, \tag{1.1}$$

where $\lambda = \int_a^b g(t)dt$.

2. Known Results

Bellman [1] gives the following generalization via new proof

Theorem 2.1. *Let $f(t)$ be non-negative and monotone decreasing in $[a,b]$ and $\in L^p[a,b]$ and let $g(t) \ge 0$ in $[a,b]$ and $\int_a^b g(t)dt \le 1$, where $p > 1$ and $(1/p)+(1/q) = 1$. Then*

$$\left(\int_a^b f(t)g(t)dt\right)^p \le \int_a^{a+\lambda} f^p(t)dt \qquad \left(\lambda = \left(\int_a^b g(t)dt\right)^p\right). \tag{2.1}$$

It may be mentioned that Bellman's result is not correct as has been mentioned by Levin [4]. A generalization in a different sense is made for $p \le 1$. Inequality for $p \ge 1$, which similar to the inequality (2.1) given in [3].

Pecaric [6], however, through some modification, gives the following

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Theorem 2.2. *Let $f : [0,1] \to \Re$ be nonnegative and non-increasing function and let $g : [0,1] \to \Re$ be an integrable function such that $0 \leq g(t) \leq 1$ for all $t \in [0,1]$. If $p \geq 1$, then*

$$\left(\int_0^1 f(t)g(t)dt\right)^p \leq \int_0^\lambda f^p(t)dt \tag{2.2}$$

where $\lambda = \left(\int_0^1 g(t)dt\right)^p$.

A mapping $\phi : \Re \to \Re$ is said to be convex on $[a,b]$ if

$$\phi(tx+(1-t)y) \leq t\phi(x)+(1-t)\phi(y), \qquad x,y \in [a,b], \quad 0 \leq t \leq 1. \tag{2.3}$$

If (2.3) reverses, then ϕ is called concave.

We have proved the following (see [9])

Theorem 2.3. *Let $f,g,h : [a,b] \to \Re$ be non-negative, f is non-increasing, g and h are integrable functions satisfying*

(1) $\lambda^{1-1/p}g(t) \leq h(t)$, *where* $\lambda = \left(\int_a^b g(t)dt\right)^p$, $p > 0$,

(2) $\int_{k_2(\lambda)}^b h(t)dt \leq \lambda \leq \int_a^{k_1(\lambda)} h(t)dt$, *for some k_1 and k_2 functions of λ with $a \leq k_1(\lambda)$; $k_2(\lambda) \leq b$. Then we have*

$$\int_{k_2(\lambda)}^b f^p(t)h(t)dt \leq \lambda^{1-1/p}\int_a^b f^p(t)g(t)dt \leq \int_a^{k_1(\lambda)} f^p(t)h(t)dt.$$

Very recently Srivastava [8] proved the following

Theorem 2.4. *Let f,g and h be integrable functions defined on $[a,b]$ with f non-increasing . Also let $0 \leq g(t) \leq h(t), t \in [a,b]$. Then the following integral inequalities hold*

$$\begin{aligned}
\int_{b-\lambda}^b f(t)h(t)dt &\leq \int_{b-\lambda}^b [f(t)h(t)-[f(t)-f(b-\lambda)][h(t)-g(t)]]dt \\
&\leq \int_a^b f(t)g(t)dt \\
&\leq \int_a^{a+\lambda} [f(t)h(t)-[f(t)-f(a+\lambda)][h(t)-g(t)]]dt \\
&\leq \int_a^{a+\lambda} f(t)h(t)dt,
\end{aligned}$$

where λ is given by

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt = \int_{b-\lambda}^b h(t)dt.$$

The aim of this paper is to give a generalization of Theorem 1.2, as well as other results including a reverse of Steffensen's inequality.

The following Lemma is needed

Lemma 2.5. *The mapping $\phi(x) = x^p$ is convex for $p \geq 1$, and concave for $0 \leq p \leq 1$.*

Proof. As

$$\phi''(x) = p(p-1)x^{p-2} \geq 0,$$

then ϕ is convex . Also for $0 < p \leq 1$, the concavity of ϕ follows from the inequality $\phi''(x) \leq 0$. □

3. New Results

The following gives a generalization of Theorem 1.2 [6].

Theorem 3.1. *Let $f, g, \phi \geq 0, 0 \leq g \leq 1$, $p \geq 1, \phi(\lambda^{1/p}) \leq \lambda$, where $\lambda = \left(\int_0^1 g(t)dt\right)^p$, f is non-increasing , ϕ is non-decreasing. Then*

$$\int_0^\lambda \phi o f(x)dx \geq \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 \phi o f(x)g(x)dx. \tag{3.1}$$

Proof. As $\phi \geq 0$ and f is non-increasing, then $\phi o f$ is non-increasing. Also

$$\lambda^{-1/p}\phi(\lambda^{1/p})g(x) \leq \lambda^{-1/p}\phi(\lambda^{1/p}) \leq \lambda^{1-1/p} \leq 1,$$

then

$$\begin{aligned}
& \int_0^\lambda \phi o f(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 \phi o f(x)g(x)dx \\
= & \int_0^\lambda \phi o f(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \left(\int_0^\lambda + \int_\lambda^1\right) \phi o f(x)g(x)dx \\
= & \int_0^\lambda \phi o f(x)\left(1 - \lambda^{-1/p}\phi(\lambda^{1/p})g(x)\right)dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_\lambda^1 \phi o f(x)g(x)dx \\
\geq & \phi o f(\lambda)\left(\int_0^\lambda \left(1 - \lambda^{-1/p}\phi(\lambda^{1/p})g(x)\right)dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_\lambda^1 g(x)dx\right) \\
= & \phi o f(\lambda)\left(\lambda - \lambda^{-1/p}\phi(\lambda^{1/p}) \left(\int_0^\lambda + \int_\lambda^1\right) g(x)dx\right) \\
= & \phi o f(\lambda)\left(\lambda - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 g(x)dx\right) \\
= & \phi o f(\lambda)\left(\lambda - \phi(\lambda^{1/p})\right) \geq 0.
\end{aligned}$$

□

The following result is deals with Steffensen's inequality for $p > 0$.

Theorem 3.2. *Let* $f, g, \phi \geq 0$, $0 \leq g \leq 1$, $p > 0, f$ *is non-increasing.* $\phi(p) > 0$, $\lambda^{\phi(p)} g \leq 1$, $\int\limits_0^1 g(x)dx \leq \lambda^{1-\phi(p)}$, ϕ *is non-decreasing .Then*

$$\int\limits_0^\lambda \phi o f(x)dx \geq \lambda^{\phi(p)} \int\limits_0^1 \phi o f(x)g(x)dx. \tag{3.2}$$

Proof. As $\phi \geq 0$ and f is non-increasing, then $\phi o f$ is non-increasing.
Also, $\lambda^{\phi(p)} g \leq 1 \Longrightarrow \lambda^{\phi(p)} \int\limits_0^1 g(x)dx \Longrightarrow \lambda \leq 1$. Then, we have

$$\begin{aligned}
& \int\limits_0^\lambda \phi o f(x)dx - \lambda^{\phi(p)} \int\limits_0^1 \phi o f(x)g(x)dx \\
= & \int\limits_0^\lambda \phi o f(x)dx - \lambda^{\phi(p)} \left(\int\limits_0^\lambda + \int\limits_\lambda^1 \right) \phi o f(x)g(x)dx \\
= & \int\limits_0^\lambda \phi o f(x) \left(1 - \lambda^{\phi(p)} g(x)\right) dx - \lambda^{\phi(p)} \int\limits_\lambda^1 \phi o f(x)g(x)dx \\
\geq & \phi o f(x) \left(\int\limits_0^\lambda \left(1 - \lambda^{\phi(p)} g(x)\right) dx - \lambda^{\phi(p)} \int\limits_\lambda^1 g(x)dx \right) \\
= & \phi o f(\lambda) \left(\lambda - \lambda^{\phi(p)} \left(\int\limits_0^\lambda + \int\limits_\lambda^1 \right) g(x)dx \right) \\
= & \phi o f(\lambda) \left(\lambda - \lambda^{\phi(p)} \int\limits_0^1 g(x)dx \right) \\
\geq & \phi o f(\lambda) (\lambda - \lambda) = 0.
\end{aligned}$$

□

Theorem 3.3. *Let* $f, g, h, \phi \geq 0, 0 \leq g \leq 1$, f *is non-increasing,* ϕ *is non-decreasing* $p \geq 1, \phi(\lambda^{1/p}) \leq \int\limits_0^\lambda h(x)dx$, $\lambda^{-1/p}\phi(\lambda^{1/p}) \leq h$, *where* $\lambda = \left(\int\limits_0^1 g(t)dt \right)^p$.
Then

$$\int\limits_0^\lambda \phi o f(x)h(x)dx \geq \lambda^{-1/p}\phi(\lambda^{1/p}) \int\limits_0^1 \phi o f(x)g(x)dx. \tag{3.3}$$

Proof. As before, $\phi o f$ is non-increasing. Therefore

$$
\begin{aligned}
& \int_0^\lambda \phi o f(x) h(x) dx - \lambda^{-1/p} \phi(\lambda^{1/p}) \int_0^1 \phi o f(x) g(x) dx \\
= & \int_0^\lambda \phi o f(x) h(x) dx - \lambda^{-1/p} \phi(\lambda^{1/p}) \left(\int_0^\lambda + \int_\lambda^1 \right) \phi o f(x) g(x) dx \\
= & \int_0^\lambda \phi o f(x) \left(h(x) - \lambda^{-1/p} \phi(\lambda^{1/p}) g(x) \right) dx - \lambda^{-1/p} \phi(\lambda^{1/p}) \int_\lambda^1 \phi o f(x) g(x) dx \\
\geq & \ \phi o f(\lambda) \left(\int_0^\lambda \left(h(x) - \lambda^{-1/p} \phi(\lambda^{1/p}) g(x) \right) dx - \lambda^{-1/p} \phi(\lambda^{1/p}) \int_\lambda^1 g(x) dx \right) \\
= & \ \phi o f(\lambda) \left(\int_0^\lambda h(x) dx - \lambda^{-1/p} \phi(\lambda^{1/p}) \left(\int_0^\lambda + \int_\lambda^1 \right) g(x) dx \right) \\
= & \ \phi o f(\lambda) \left(\int_0^\lambda h(x) dx - \phi(\lambda^{1/p}) \right) \geq 0.
\end{aligned}
$$

□

The following gives a reverse inequality

Theorem 3.4. *Let $f, g, \phi \geq 0$, ϕ is concave with $\phi(0) = 0$, f is non-decreasing, ϕ is non-decreasing , $0 \leq g \leq 1$, $p \geq 1$ and $\lambda = \left(\int_0^1 g(t) dt \right)^p$. Then*

$$
\int_0^\lambda \phi o f(x) dx \leq \phi \left(\lambda^{1-1/p} \int_0^1 f(x) g(x) dx \right). \tag{3.4}
$$

Proof.

$$\begin{aligned}
&\int_0^\lambda \phi o f(x)dx - \phi\left(\lambda^{1-1/p}\int_0^1 f(x)g(x)dx\right)\\
= &\int_0^\lambda \phi o f(x)dx - \phi\left(\lambda\frac{1}{\lambda^{1/p}}\int_0^1 f(x)g(x)dx\right)\\
\leq &\int_0^\lambda \phi o f(x)dx - \lambda\phi\left(\frac{1}{\lambda^{1/p}}\int_0^1 f(x)g(x)dx\right) \quad \text{(as } \phi \text{ is concave with } \phi(0)=0)\\
\leq &\int_0^\lambda \phi o f(x)dx - \lambda^{1-1/p}\left(\int_0^1 \phi o f(x)g(x)dx\right) \quad \text{(by Jensen's inequality)}\\
= &\int_0^\lambda \phi o f(x)dx - \lambda^{1-1/p}\left(\int_0^\lambda + \int_\lambda^1\right)\phi o f(x)g(x)dx\\
= &\int_0^\lambda \left(1-\lambda^{1-1/p}g(x)\right)dx - \lambda^{1-1/p}\int_\lambda^1 \phi o f(x)g(x)dx\\
\leq &\ \phi o f(\lambda)\left(\int_0^\lambda \left(1-\lambda^{1-1/p}g(x)\right)dx - \lambda^{1-1/p}\int_\lambda^1 g(x)dx\right)\\
= &\ \phi o f(\lambda)\left(\lambda - \lambda^{1-1/p}\left(\int_0^\lambda + \int_\lambda^1\right)g(x)dx\right)\\
= &\ \phi o f(\lambda)\left(\lambda - \lambda^{1-1/p}\int_0^1 g(x)dx\right)\\
= &\ \phi o f(\lambda)(\lambda-\lambda) = 0.
\end{aligned}$$

□

4. Applications

For an application of Theorem 3.1, we have the following

(1) If we are put $\phi(x) = x^p$, $p \geq 1$, we obtain the inequality (2.2) as follows

$$\int_0^\lambda f^p(x)dx \geq \left(\int_0^1 g(t)dt\right)^{p-1}\int_0^1 f^p(x)g(x)dx \geq \left(\int_0^1 f(x)g(x)dx\right)^p.$$

(2) If we are take $\phi(x) = sinx, 0 \leq x \leq \pi/2$, we obtain

$$\int_0^\lambda sin\left(f(x)\right)dx \geq \lambda^{-1/p}\sin\left(\lambda^{1/p}\right)\int_0^1 \sin\left(f(x)\right)g(x)dx.$$

(3) If we are take $\phi(x) = lnx$, $x > 0$, we obtain

$$\int_0^\lambda \ln f(x)dx \geq \lambda^{-1/p} \ln\left(\lambda^{1/p}\right) \int_0^1 (\ln f(x))\, g(x)dx.$$

Corollary 4.1. *Let* $f, g, \phi \geq 0$, $\lambda^{1/p-1} g \leq 1$, f *is non-increasing,* $p > 0$, *where* $\lambda = \left(\int_0^1 g(t)dt\right)^{\frac{p}{2p-1}}$ *.*ϕ *is non-decreasing. Then*

$$\int_0^\lambda \phi o f(x)dx \geq \lambda^{1/p-1} \int_0^1 \phi o f(x) g(x)dx. \tag{4.1}$$

Proof. The proof follows from Theorem 2.2, by putting $\phi(p) = 1/p - 1$, $0 < p < 1$. □

The following gives another extension of Theorem 2.1

Corollary 4.2. *Let* $f, g \geq 0$, f *is non-decreasing,* $0 \leq g \leq 1$, $p \geq 1$, $0 < q < 1$, *and* $\lambda = \left(\int_0^1 g(t)dt\right)^p$ *.Then*

$$\int_0^\lambda f^q(x)dx \leq \lambda^{q-q/p} \left(\int_0^1 f(x)g(x)dx\right)^q. \tag{4.2}$$

Proof. The proof follows from Theorem 2.4 via Lemma 1.5, by putting $\phi(x) = x^q$, $0 < q < 1$. □

5. Conclusion

Several results are proved in the present paper. The first gives a generalization of the main result of [6] in which $p \geq 1$. The second result is dealing with Steffensen's inequality with $p > 0$. The third result present the reverse of Steffensen's inequality. Other results give some examples.

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W. T. Sulaiman, Department of Computer Engineering, College of Engineering, University of Mosul, Iraq.

E-mail address: waadsulaiman@hotmail.com