Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 3 Issue 2(2012), Pages 1-9

SOME NEW SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION AND AN INFINITE MATRIX IN A SEMİNORMED SPACE

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Abstract. Let $\hat{c}$ denotes the space of almost convergent sequences introduced by G.G. Lorentz [A contribution to the theory of divergent sequences, *Acta Math.* **80**(1948), 167–190]. The main purpose of the present paper is to introduce the sequence spaces $w_0\left(\hat{A}, p, f, q, s\right)$, $w\left(\hat{A}, p, f, q, s\right)$ and $w_\infty\left(\hat{A}, p, f, q, s\right)$ defined by a modulus function f. Some topological properties of that spaces are examined. Also we exposed some inclusion relations among the variations of the space.

1. Introduction

Some definitions and conventions are made this section and some lemmas will be given as they become necessary. By a sequence space, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \cdots\}$. We write l_∞, c, c_0 and l_p for the classical sequence spaces of all bounded, convergent, null and absolutely $p-$summable sequences, respectively, where $1 \le p < \infty$. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. bv is the space consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in l_1 and bv_0 is the intersection of the spaces bv and c_0. w_0^p, w^p and w_∞^p are the spaces of sequences that are strongly summable to zero, summable and bounded of index $p \ge 1$ by the Cesàro method of order 1.

Let λ denotes any of the sets l_∞, c, c_0, l_p, bs, cs, bv, bv_0, w_0^p, w^p and w_∞^p. It is a routine verification that λ is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequences.

A sequence space λ with a linear topology is called a $K-$space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $K-$space λ is called an FK-space provided λ is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space.

2000 *Mathematics Subject Classification.* 46A45,40A05.
Key words and phrases. Sequence spaces; modulus function.

Submitted April 20, 2012. Published May 14, 2012.

* The main results of this paper were partially presented at the *International Conference on Applied Analysis and Algebra (ICAAA 1)* to be held on 29-30 June and 1-2 July, 2011 in istanbul, Turkey at the Yildiz Technical University.

Now, we focus on the $\alpha-, \beta-$ and $\gamma-$duals of the classical sequence spaces. For the sequence spaces λ and μ, the set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_kz_k)\in\mu \text{ for all } x=(x_k)\in\lambda\} \tag{1.1}$$

is called the multiplier space of λ and μ. One can easily observe for a sequence space ν that the inclusions $S(\lambda,\mu)\subset S(\nu,\mu)$ if $\nu\subset\lambda$ and $S(\lambda,\mu)\subset S(\lambda,\nu)$ if $\mu\subset\nu$ hold. With the notation of (1.1) the $\alpha-, \beta-$ and $\gamma-$duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by $\lambda^\alpha = S(\lambda,l_1)$, $\lambda^\beta=S(\lambda,cs)$, $\lambda^\gamma=S(\lambda,bs)$. The $\alpha-$dual, $\beta-$dual and $\gamma-$dual are also refered to as *Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual* and *Garling dual,* respectively [2].

We give a short survey on the concept of almost convergence. A linear functional φ on l_∞ is said to be a Banach limit if it has the properties, $\varphi(x)\geqslant 0$ when the sequence $x=(x_n)$ has $x_n\geqslant 0$ for all n, $\varphi(e)=1$, where $e=(1,1,1,\cdots)$ and $\varphi(x_{n+1})=\varphi(x_n)$ for all $x\in l_\infty$ [1]. For more detail on the Banach limit, the reader may refer to Çolak and Çakar [5], and Das [6]. The concept of almost convergence was defined by Lorentz in [9], using the idea of Banach limits. A sequence $x=(x_k)\in l_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are coincide and are equal to α [9], this is denoted by $f-\lim x_k=\alpha$. Lorentz [9] proved that $f-\lim x_k=\alpha$ if and only if $\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=1}^{m}x_{n+k}=\alpha$, uniformly in n. In the case $\alpha=0$, the sequence x is called almost null. The spaces of almost convergent and almost null sequences are denoted by $\hat{c}$ and $\hat{c}_0$, respectively. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal.

Maddox [12, 13] defined the strong almost convergence of a complex sequence x to number l by $\frac{1}{m}\sum_{k=0}^{m}|x_{n+k}-l|\to 0$, as $m\to\infty$, uniformly in n which leads to the concept of strong almost convergence. By $[\hat{c}]$, we denote the space of all strongly almost convergence sequences. It is immediate that the inclusion $[\hat{c}]\subset\hat{c}$ strictly holds. Also $[\hat{c}]$ is a closed subspace of l_∞ and the inclusions $c\subset[\hat{c}]\subset\hat{c}\subset l_\infty$ strictly hold.

Notation of modulus function introduced by Nakano [14] in 1953 and used to solve some structural problems of the scalar FK-spaces theory. For example, the question; "is there an $FK-$space in which the sequence of coordinate vectors is bounded", exposed by A. Wilansky, was solved by W. H. Ruckle with negative answer [17]. The problem was treated by constructing a class of scalar $FK-$spaces $L(f)$ where f is a modulus function. $L(f)$, in fact, is a generalization of the spaces $l_p\ (0<p\leq 1)$. Another extension of $l_p\ (p>0)$ spaces with respect to a positive real sequence $r=(r_k)$ was given by Simons [20]. For the definition of modulus function and some related results, the reader may refer to [17].

Ruckle [17] proved that the inclusion $L(f)\subset l_1$ holds for any modulus f and $L(f)^\alpha=l_\infty$.

A sequence $x=(x_k)$ is said to be summable $(C,1)$ iff $\lim_n\frac{1}{n}\sum_{i=1}^{n}x_i$ exists. Spaces of strongly Cesàro summable sequences were discussed by Kuttner [8] and this concept was generalized by Maddox [10] and some others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [11] as an extension of the concept of the strong Cesàro summability. Connor [4] extended this definition by replacing the Cesàro matrix with an arbitrary nonnegative regular summability method. In [18], following Connor [4], Savaş defined the concept of strongly almost $A-$summability with respect to a modulus, but the definition introduced there is not very satisfactory and seems to be unnatural. By specialising the infinite matrix in the definition introduced in [18], we don't get strongly almost convergent sequences with respect to a modulus. In [19] Savaş introduced an alternative definition of strongly almost $A-$summability with respect to a modulus. This definition seems to be more natural and contains the definition of strongly almost convergence with respect to a modulus as a special case. The sets $w_0\left(\hat{A}, f, p\right)$, $w\left(\hat{A}, f, p\right)$ and $w_\infty\left(\hat{A}, f, p\right)$ will called the spaces of strongly almost summable to zero, strongly almost summable and strongly almost bounded with respect to the modulus f respectively [19].

The argument s, that is, the factor k^{-s} was used by Bulut and Çakar [3], to generalize the Maddox sequence space $l(p)$, where $p = (p_k)$ is a bounded sequence of positive real numbers and $s \geq 0$. It performs an extension mission. For example, the space $l(p, s) = \{x \in w : \sum_{k=1}^{\infty} k^{-s} |x_k|^{p_k} < \infty\}$ contains $l(p)$ as a subspace for $s > 0$, and it coincides with $l(p)$ only for $s = 0$.

In the present note, we introduce some new sequence spaces defined by using a modulus function.

2. The sequence space $w(\hat{A}, p, f, q, s)$

Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $S(X)$ we denote the set of all $X-$valued sequences $x = (x_k)$ which is the linear space under the usual coordinatewise operations. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$. Taking $X = \mathbb{C}$ we get w, the space of all complex-valued sequences. This case is called scalar-valued case.

Let $A = (a_{mk})$ be a nonnegative matrix and suppose that $p = (p_k)$ be a sequence of positive real numbers and f be a modulus function. We define the sequence spaces over the complex field $\mathbb{C}$ as follows

$$w_0(\hat{A}, p, f, q, s) = \left\{ x \in S(X) : \lim_{m\to\infty} \sum_k \frac{a_{mk}}{k^s} [f(q(x_{k+n}))]^{p_k} = 0, \text{ uniformly in } n,\ s \geq 0 \right\}$$

$$w(\hat{A}, p, f, q, s) = \left\{ x \in S(X) : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_k \frac{a_{mk}}{k^s} [f(q(x_{k+n} - le))]^{p_k} = 0, \text{ uniformly in } n, l \in \mathbb{C},\ s \geq 0 \right\}$$

$$w_\infty(\hat{A}, p, f, q, s) = \left\{ x \in S(X) : \sup_{m,n} \sum_k \frac{a_{mk}}{k^s} [f(q(x_{k+n}))]^{p_k} < \infty,\ s \geq 0 \right\}$$

If $\phi(X)$ is the space of finite sequences in X, then we have $\phi(X) \subseteq w(\hat{A}, p, f, q, s)$. The following inequality and the sequence $p = (p_k)$ will be frequently used throughout this paper.

These spaces are reduced to some sequence spaces in the literature in the special case. For example, taking $(X, q) = (\mathbb{C}, |,|)$, $A = (C, 1)$, the Cesàro matrix, $p_k = 1$, for all k and $s = 0$, we get the spaces $[\hat{c}_0(f)]$, $[\hat{c}(f)]$ and $[\hat{c}(f)]_\infty$ introduced by

Pehlivan [16]. Moreover, we derive the spaces investigated in [7, 12, 13, 15, 19] as a special case.

If $a_k, b_k \in \mathbb{C}$ and $0 < p_k \leq \sup p_k = H$ for each k, we have (see Maddox [10, p.346])

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}), \tag{2.1}$$

where $C = \max\,(1, 2^{H-1})$.

We now establish a number of useful theorems about the sequence spaces which were defined above. We now have

Theorem 2.1. *Let f be a modulus, $p = (p_k) \in \ell_\infty$ and $A = (a_{mk})$ be a nonnegative regular matrix. Then $w_0(\hat{A}, p, f, q, s) \subset w(\hat{A}, p, f, q, s) \subset w_\infty(\hat{A}, p, f, q, s)$.*

Proof. It is obvious that $w_0(\hat{A}, p, f, q, s) \subset w(\hat{A}, p, f, q, s)$. Suppose that $x \in w(\hat{A}, p, f, q, s)$. Since q is a seminorm, there exists an integer R such that $q\,(l) \leq R$. Then, because of f is a modulus function and $A = (a_{mk})$ is a nonnegative regular matrix we can write from (2.1) that

$$\sum_k a_{mk} k^{-s} \left[f\left(q\left(x_{k+n}\right)\right)\right]^{p_k} \leq C \left\{ \sum_k a_{mk} k^{-s} \left[f\left(q\left(x_{k+n} - l\right)\right)\right]^{p_k} + \left[Rf\left(1\right)\right]^H \sum_k a_{mk} k^{-s} \right\}.$$

Therefore $x \in w_\infty(\hat{A}, p, f, q, s)$ and this completes the proof. □

Theorem 2.2. *Let $p = (p_k)$ be a bounded, then $w_0(\hat{A}, p, f, q, s)$, $w(\hat{A}, p, f, q, s)$ and $w_\infty(\hat{A}, p, f, q, s)$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We consider only $w(\hat{A}, p, f, q, s)$. Others can be treated similarly. Let $x, y \in w(\hat{A}, p, f, q, s)$ and $\lambda, \mu \in \mathbb{C}$, suppose that $x \to l_1 \left[w(\hat{A}, p, f, q, s)\right]$ and $y \to l_2 \left[w(\hat{A}, p, f, q, s)\right]$. For λ, μ there exist the integers M_λ and N_μ such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Combining (2.1) with the definitions of f and q, we have

$$a_{mk} k^{-s} \{f(q(\lambda x_{k+n} + \mu y_{k+n} - (\lambda l_1 + \mu l_2)))\}^{p_k} \leq C a_{mk} k^{-s} M_\lambda^H [f(q(x_{k+n} - l_1))]^{p_k} + C a_{mk} k^{-s} N_\mu^H [f(q(y_{k+n} - l_2))]^{p_k}$$

which leads us by summing over $1 \leq k \leq \infty$ that we get $\lambda x + \mu y \in w(\hat{A}, p, f, q, s)$. □

Theorem 2.3. *The spaces $w_0(\hat{A}, p, f, q, s)$ and $w(\hat{A}, p, f, q, s)$ are paranormed spaces by g defined by*

$$g\,(x) = \sup_m \left\{ \sum_k a_{mk} k^{-s} [f(q(x_{k+n}))]^{p_k} \right\}^{\frac{1}{M}}$$

where $M = \max\left(1, H = \sup_k p_k\right)$.

Proof. From Teorem 2.1, $g\,(x)$ exists for each $x \in w_0(\hat{A}, p, f, q, s)$. Clearly $g\,(\theta) = 0$, $g\,(x) = g\,(-x)$ and by Minkowski's inequality $g\,(x + y) \leq g\,(x) + g\,(y)$. For the continuity of scalar multiplication suppose that (μ^t) is a sequence of scalars such that $|\mu^t - \mu| \to 0$ and $g(x^t - x) \to 0$ for arbitrary sequence $(x^t) \in w_0(\hat{A}, p, f, q, s)$. We shall show that $g(\mu^t x^t - \mu x) \to 0$ as $t \to \infty$. Say $\tau_t = |\mu^t - \mu|$ then

$$\left\{\sum_k a_{mk} k^{-s} [f(q(\mu^t x^t_{k+n} - \mu x_{k+n}))]^{p_k}\right\}^{\frac{1}{M}} \leq \left\{\sum_k \left\{a_{mk}^{\frac{1}{M}} k^{-\frac{s}{M}} \left[A(t,k,n)\right]^{\frac{p_k}{M}} + a_{mk}^{\frac{1}{M}} k^{-\frac{s}{M}} \left[B(t,k,n)\right]^{\frac{p_k}{M}}\right\}^M\right\}^{\frac{1}{M}}$$

where $A(t,k,n) = Rf\left(q\left(x^t_{k+n} - x_{k+n}\right)\right),\ B(t,k,n) = f(\tau_t q(x_{k+n}))$ and $R = 1 + \max\{1, \sup|\mu^t|\}$.

$$\begin{aligned} g\left(\mu^t x^t - \mu x\right) &\leq R^{\frac{H}{M}} \sup_{m,n}\left\{\sum_k a_{mk} k^{-s}\left[\frac{A(t,k,n)}{R}\right]^{p_k}\right\}^{\frac{1}{M}} \\ &\quad + \sup_{m,n}\left\{\sum_k a_{mk} k^{-s}\left[B(t,k,n)\right]^{p_k}\right\}^{\frac{1}{M}} \\ &= R^{\frac{H}{M}} g\left(x^t - x\right) + \sup_{m,n}\left\{\sum_k a_{mk} k^{-s}\left[B(t,k,n)\right]^{p_k}\right\}^{\frac{1}{M}}. \end{aligned}$$

Because of $g(x^t - x) \to 0$ we must only show that $\sup_{m,n}\left\{\sum_k a_{mk} k^{-s}\left[B(t,k,n)\right]^{p_k}\right\}^{\frac{1}{M}} \to 0$ as $t \to \infty$. There exists a positive integer t_0 such that $0 \leq \tau_t \leq 1$ for $t > t_0$. Write

$$\sup_{m,n}\left\{\sum_{k=m+1}^{\infty} a_{mk} k^{-s}\left[f(q(x_{k+n}))\right]^{p_k}\right\}^{\frac{1}{M}} \to 0 \ \ (m \to \infty).$$

Hence, for every $\epsilon > 0$, there exists a positive integer m_0 such that

$$\sup_{m,n}\left\{\sum_{k=m_0+1}^{\infty} a_{mk} k^{-s}\left[f(q(x_{k+n}))\right]^{p_k}\right\}^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

For $t > t_0$, since $\tau_t q(x_{k+n}) \leq q(x_k)$, we get

$$a_{mk} k^{-s}\left[f(\tau_t(q(x_{k+n})))\right]^{p_k} \leq a_{mk} k^{-s}\left[f(q(x_{k+n}))\right]^{p_k}$$

for each n and k. This implies

$$\begin{aligned} \sup_{m,n}\left\{\sum_{k=m_0+1}^{\infty} a_{mk} k^{-s}\left[f(\tau_t(q(x_{k+n})))\right]^{p_k}\right\}^{\frac{1}{M}} &\leq \sup_{m,n}\left\{\sum_{k=m_0+1}^{\infty} a_{mk} k^{-s}\left[f(q(x_{k+n}))\right]^{p_k}\right\}^{\frac{1}{M}} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Now, the function $\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk} k^{-s}\left[f(\tau_t(q(x_{k+n})))\right]^{p_k}\right\}$ is continuous. Hence, there exists a δ $(0 < \delta < 1)$ such that $\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk} k^{-s}\left[f(\tau_t(q(x_{k+n})))\right]^{p_k}\right\} \leq \left(\frac{\varepsilon}{2}\right)^M$ for $0 < \tau_t < \delta$. Also we can find a number Δ such that $\tau_t < \delta$ for $t > \Delta$. So, for

$t > \Delta$, we have $\sup_{m,n} \{\sum_{k=1}^{m_0} a_{mk} k^{-s} [f(\tau_t(q(x_{k+n})))]^{p_k}\}^{\frac{1}{M}} < \frac{\varepsilon}{2}$, so eventually,

$$\begin{aligned}\sup_{m,n}\left\{\sum_k a_{mk}k^{-s}\left[f\left(\tau_t\left(q\left(x_{k+n}\right)\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}} &\leq \sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f\left(\tau_t\left(q\left(x_{k+n}\right)\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}} \\ &+ \sup_{m,n}\left\{\sum_{k=m_0+1}^{\infty} a_{mk}k^{-s}\left[f\left(\tau_t\left(q\left(x_{k+n}\right)\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

This shows that $\sup_{m,n} \{\sum_k a_{mk} k^{-s} [f(\tau_t(q(x_{k+n})))]^{p_k}\}^{\frac{1}{M}} \to 0 \ (t \to \infty)$. Thus $w_0(\hat{A}, p, f, q, s)$ is paranormed space by g. □

Theorem 2.4. *$w_0\left(\hat{A}, p, f, q, s\right)$ is complete with respect to its paranorm whenever (X, q) is complete.*

Proof. Suppose (x^i) is a Cauchy sequence in $w_0(\hat{A}, p, f, q, s)$. Therefore

$$g(x^i - x^j) = \sup_{m,n}\left\{\sum_k a_{mk}k^{-s}\left[f\left(q\left(x^i_{k+n}-x^j_{k+n}\right)\right)\right]^{p_k}\right\}^{\frac{1}{M}} \to 0 \text{ as } i,j\to\infty \tag{2.2}$$

also, for each n and k

$$k^{-s}\left[f\left(q\left(x^i_{k+n}-x^j_{k+n}\right)\right)\right]^{p_k} \to 0 \text{ as } i,j\to\infty$$

and so $q\left(x^i_{k+n}-x^j_{k+n}\right) \to 0 \ (i,j\to\infty)$ from the continuity of f. It follows that the sequence $\left(x^i_{k+n}\right)$ is a Cauchy in (X,q) for each fixed n and k. Then by the completeness of (X,q) we get the sequence $(x_{k+n}) \in X$ such that

$$q\left(x^i_{k+n}-x_{k+n}\right) \to 0 \ \ (j\to\infty). \tag{2.3}$$

It is easy to see the validity of the inequality

$$\left|q\left(x^i_{k+n}-x^j_{k+n}\right)-q\left(x^i_{k+n}-x_{k+n}\right)\right| \leq q\left(x^i_{k+n}-x_{k+n}\right).$$

We have

$$q\left(x^i_{k+n}-x^j_{k+n}\right) \to q\left(x^i_{k+n}-x_{k+n}\right) \ \ (j\to\infty).$$

from (2.3). Now, for each $\varepsilon > 0$ there exist $i_0(\varepsilon)$ such that $\left[g(x^i-x^j)\right]^M < \epsilon^M$ for $i, j > i_0$. Also

$$\begin{aligned}\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f\left(q\left(x^i_{k+n}-x^j_{k+n}\right)\right)\right]^{p_k}\right\} &\leq \sup_{m,n}\left\{\sum_k a_{mk}k^{-s}\left[f\left(q\left(x^i_{k+n}-x^j_{k+n}\right)\right)\right]^{p_k}\right\} \\ &= \left[g(x^i-x^j)\right]^M.\end{aligned}$$

Letting $j\to\infty$ we have

$$\begin{aligned}\sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f\left(q\left(x^i_{k+n}-x^j_{k+n}\right)\right)\right]^{p_k}\right\} &\to \sup_{m,n}\left\{\sum_{k=1}^{m_0} a_{mk}k^{-s}\left[f\left(q\left(x^i_{k+n}-x_{k+n}\right)\right)\right]^{p_k}\right\} \\ &< \epsilon^M\end{aligned}$$

for $i > i_0$. Since m_0 is arbitrary, by taking $m_0 \to \infty$ we obtain

$$\sup_{m,n} \left\{ \sum_k a_{mk} k^{-s} \left[f\left(q\left(x^i_{k+n} - x_{k+n} \right) \right) \right]^{p_k} \right\}^{\frac{1}{M}} < \epsilon$$

for all m and n that is

$$g\left(x^i - x \right) \to 0 \text{ as } i \to \infty \, .$$

We first need to show $x \in w_0(\hat{A}, p, f, q, s)$. We know that $g(x^i)$ is bounded, say, $g(x^i) \le K$. Futhermore we have

$$a_{mk} k^{-s} \left[f\left(q\left(x^i_{k+n} - x_{k+n} \right) \right) \right]^{p_k} \to 0 \ \ (i \to \infty) \, .$$

Now we can determine a sequence $\eta_k \in c_0 \ \left(0 < \eta^i_k \le 1 \right)$ for each k, such that

$$a_{mk} k^{-s} \left[f\left(q\left(x^i_{k+n} - x_{k+n} \right) \right) \right]^{p_k} \le \eta^i_k a_{mk} k^{-s} \left[f\left(q\left(x^i_{k+n} \right) \right) \right]^{p_k} .$$

On the other hand,

$$\left[f\left(q\left(x_{k+n} \right) \right) \right]^{p_k} \le C \left\{ \left[f\left(q\left(x^i_{k+n} - x_{k+n} \right) \right) \right]^{p_k} + \left[f\left(q\left(x^i_{k+n} \right) \right) \right]^{p_k} \right\}$$

where $C = \max\left(1, 2^{H-1} \right)$; $H = \sup p_k$. Also we have

$$\begin{aligned} a_{mk} k^{-s} \left[f\left(q\left(x_{k+n} \right) \right) \right]^{p_k} &\le C a_{mk} k^{-s} \left\{ \left[f\left(q\left(x^i_{k+n} - x_{k+n} \right) \right) \right]^{p_k} + \left[f\left(q\left(x^i_{k+n} \right) \right) \right]^{p_k} \right\} \\ &\le C \left(\eta^i_k + 1 \right) a_{mk} k^{-s} \left[f\left(q\left(x^i_{k+n} \right) \right) \right]^{p_k} \end{aligned}$$

from the last inequality above, we obtain $x \in w_0(\hat{A}, p, f, q, s)$ and this completes the proof of the theorem. □

Lemma 2.5. *Let f_1, f_2 are modulus function and $0 < \delta < 1$. If $f_1(t) > \delta$ for $t \in [0, \infty)$ then*

$$\left(f_2 \circ f_1 \right)(t) \le \frac{2 f_2(1)}{\delta} f_1(t)$$

[11].

Theorem 2.6. *Let f_1, f_2 are the modulus function and $s, s_1, s_2 > 0$. Then*

- **i)** $\limsup \frac{f_1(t)}{f_2(t)} < \infty$ implies $w_0(\hat{A}, p, f_2, q, s) \subset w_0(\hat{A}, p, f_1, q, s)$,
- **ii)** $w_0(\hat{A}, p, f_1, q, s) \cap w_0(\hat{A}, p, f_2, q, s) \subseteq w_0(\hat{A}, p, f_1 + f_2, q, s)$,
- **iii)** If the matrix $A = (a_{mk})$ is a regular matrix and $s > 1$, then $w_0(\hat{A}, p, f_1, q, s) \subseteq w_0(\hat{A}, p, f_1 \circ f_2, q, s)$,
- **iv)** $s_1 \le s_2$ implies $w_0(\hat{A}, p, f, q, s_1) \subseteq w_0(\hat{A}, p, f, q, s_2)$.

Proof. i) Since there exists a $K > 0$ such that $f_1(t) \le f_2(t)$ by the hypothesis, therefore we can write that

$$a_{mk} k^{-s} \left[f_1\left(q\left(x_{k+n} \right) \right) \right]^{p_k} \le K^H a_{mk} k^{-s} \left[f_2\left(q\left(x_{k+n} \right) \right) \right]^{p_k} .$$

Let $x \in w_0(\hat{A}, p, f_2, q, s)$. When adding the above inequality from $k = 1$ to ∞, we have $x \in w_0(\hat{A}, p, f_1, q, s)$.

ii) The relation follows from the inequality

$$\begin{aligned} a_{mk} k^{-s} \left[\left(f_1 + f_2 \right)\left(q\left(x_{k+n} \right) \right) \right]^{p_k} &= a_{mk} k^{-s} \left[f_1\left(q\left(x_{k+n} \right) \right) + f_2\left(q\left(x_{k+n} \right) \right) \right]^{p_k} \\ &\le C a_{mk} k^{-s} \left\{ \left[f_1\left(q\left(x_{k+n} \right) \right) \right]^{p_k} + \left[f_2\left(q\left(x_{k+n} \right) \right) \right]^{p_k} \right\} \end{aligned}$$

where $C = \max\left(1, 2^{H-1} \right)$.

iii) Let $0 < \delta < 1$, and define the sets $N_1 = \{k \in \mathbb{N} : f_1(q(x_{k+n})) \leq \delta\}$ and $N_2 = \{k \in \mathbb{N} : f_1(q(x_{k+n})) > \delta\}$. It follows from Lemma 2.5 that

$$(f_2 \circ f_1)(q(x_{k+n})) \leq \frac{2f_2(1)}{\delta} f_1(q(x_{k+n}))$$

when $k \in N_2$. If $k \in N_1$ then

$$(f_2 \circ f_1)(q(x_{k+n})) \leq f_2(\delta),$$

and so

$$k^{-s}[(f_2 \circ f_1)(q(x_{k+n}))]^{p_k} \leq \epsilon_1 k^{-s}$$

for $x \in w_0(\hat{A}, p, f_1, q, s)$, where $\epsilon_1 = \max\left\{[f_2(\delta)]^{\inf p_k}, [f_2(\delta)]^{\sup p_k}\right\}$. On the other hand

$$\begin{aligned} a_{mk}k^{-s}[(f_2 \circ f_1)(q(x_{k+n}))]^{p_k} &\leq a_{mk}k^{-s}\left[\frac{2f_2(1)}{\delta} f_1(q(x_{k+n}))\right]^{p_k} \\ &\leq \epsilon_2 a_{mk}k^{-s}[f_1(q(x_{k+n}))]^{p_k} \end{aligned}$$

for $k \in N_2$. Where $\epsilon_2 = \max\left\{\left[\frac{2f_2(1)}{\delta}\right]^{\inf p_k}, \left[\frac{2f_2(1)}{\delta}\right]^{\sup p_k}\right\}$. Now, say $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and we get

$$\sum_k a_{mk}k^{-s}[(f_2 \circ f_1)(q(x_{k+n}))]^{p_k} \leq \varepsilon\left\{\sum_k a_{mk}k^{-s} + \sum_k a_{mk}k^{-s}[f(q(x_{k+n}))]^{p_k}\right\}$$

for $k \in N_1 \cup N_2 = \mathbb{N}$. This implies $x \in w_0(\hat{A}, p, f_1 \circ f_2, q, s)$. □

Theorem 2.7. *Let $s > 1$ and f be bounded and A be a nonnegative regular matrix. When $x \in w_\infty\left(\hat{A}, p, f, q, s\right)$*

$$\sum_k a_k x_k \text{ is convergent iff } (a_k) \in \phi.$$

Proof. The sufficiency is trivial.

For the necessity, suppose that $a \notin \phi$. Then there is an increasing sequence (m_k) of positive integers such that $|a_{m_k}| > 0$.

Let us define

$$y_k = \begin{cases} \frac{u}{q(u)a_{m_k}}, & k = m_k \\ \theta, & k \neq m_k \end{cases}$$

where $u \in X$ such that $q(u) > 0$. Since f is bounded and $s > 1$,

$$\sum_k a_{mk}k^{-s}[f(q(y_{k+n}))]^{p_k} < \infty$$

hence $y \in w_\infty(\hat{A}, p, f, q, s)$ but $\sum_k a_k y_k = 1 + 1 + 1 + \cdots = \infty$, a contradiction the fact $(a_k y_k) \in cs$. This completes the proof. □

Corollary 2.8. *Let $s > 1$ and f be bounded. Then, $w\left(\hat{A}, p, f, q, s\right)^\beta = \phi$.*

Proof. One can easily show this fact by the similar way used in proving Theorem 2.7. So, we omit the detail. □

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