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GENERALIZED CAUCHY-RIEMANN LIGHTLIKE SUBMANIFOLDS

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Abstract. We extend the study of Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kaehler manifolds and characterize GCR-lightlike product of indefinite Kaehler manifolds. We investigate totally umbilical and mixed geodesic GCR-lightlike submanifolds of indefinite Kaehler manifolds and prove that if a GCR-lightlike submanifold is mixed foliate then indefinite Kaehler manifold is a complex space form.

1. Introduction

In the process of generalization of holomorphic and totally real submanifolds of Kaehler manifolds, Bejancu [1] introduced Cauchy-Riemann (CR)-submanifolds of Kaehler manifolds in 1978 and which are further studied by Bejancu and et al. [3], Chen [6], Blair and Chen [4], Yano and Kon [16, 17] and others. At that time, the study of CR-submanifolds was confined to a positive definite metric. Sharma and Duggal [14] generalized the theory of CR-submanifolds of Kaehler manifolds with Riemannian metric to semi-Riemannian (indefinite) metric so that the theory of CR-submanifolds can be applied to other branches of mathematics and physics where the metric is not necessarily definite. Duggal [7, 8] also studied CR-submanifolds with Lorentzian metric. Later on, Duggal and Bejancu [9] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds, which exclude the holomorphic and totally real cases. To include holomorphic and totally real submanifolds as subcases, Duggal and Sahin [10] introduced Screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds. But there was no inclusion relation between SCR and CR classes. For such inclusion relations, recently Duggal and Sahin [11] introduced Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kaehler manifolds. In present paper, we extend the existing theory of GCR-lightlike submanifolds and study GCR-lightlike product of indefinite Kaehler manifolds and mixed foliate GCR-lightlike submanifolds of complex space form.

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2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [9] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the metric induced by $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{2.1}$$

is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2.2}$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = RadTM \perp S(TM), \tag{2.3}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.4}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$, where $\{\xi_1, ..., \xi_r\}$, $\{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(RadTM|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$ respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1. *([9]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{2.6}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.5), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla_X^{\perp} U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form and A_U is a linear operator on M and known as the shape operator.

According to (2.4), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (2.7) and (2.8) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.9}$$

$$\bar{\nabla}_X U = -A_U X + D_X^l U + D_X^s U, \tag{2.10}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D_X^l U = L(\nabla_X^{\perp} U)$, $D_X^s U = S(\nabla_X^{\perp} U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M, respectively. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X,N), \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X,W), \tag{2.12}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.4)-(2.5) and (2.9)-(2.12), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.13}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.14}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X), \tag{2.15}$$

for any $\xi \in \Gamma(RadTM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.
Using the properties of linear connection, we have the following covariant derivatives

$$(\nabla_X h^l)(Y,Z) = \nabla_X^l(h^l(Y,Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{2.16}$$

$$(\nabla_X h^s)(Y,Z) = \nabla_X^s(h^l(Y,Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{2.17}$$

Denoting $\bar{R}$ and R as the curvature tensor of $\bar{\nabla}$ and ∇, respectively then using (2.9)-(2.12) and (2.16)-(2.17) we obtain

$$\begin{aligned} \bar{R}(X,Y)Z \quad = \quad & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & -A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) \\ & -(\nabla_Y h^s)(X,Z) + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) \\ & +D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)). \end{aligned} \tag{2.18}$$

Barros-Romero [5] defined indefinite Kaehler manifolds as

Definition 2.2. *Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is*

$$(\bar{\nabla}_X \bar{J})Y = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}). \tag{2.19}$$

3. Generalized Cauchy-Riemann (GCR)-lightlike Submanifolds

Definition 3.1. *([11]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:*

(A) *There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that*

$$Rad(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM). \tag{3.1}$$

(B) *There exist two subbundles D_0 and D' of $S(TM)$ such that*

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2, \tag{3.2}$$

where D_0 is a non-degenerate distribution on M, L_1 and L_2 are vector bundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then the tangent bundle TM of M is decomposed as

$$TM = D \oplus D', \quad D = Rad(TM) \oplus D_0 \oplus \bar{J}D_2. \tag{3.3}$$

M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$.

Let Q, P_1 and P_2 be the projection on D, $\bar{J}L_1 = M_1 \subset D'$ and $\bar{J}L_2 = M_2 \subset D'$, respectively then for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X, \tag{3.4}$$

applying $\bar{J}$ to above equation, we obtain

$$\bar{J}X = TX + wP_1X + wP_2X, \tag{3.5}$$

where $TX \in \Gamma(D), wP_1X \in \Gamma(L_1)$ and $wP_2X \in \Gamma(L_2)$ and we can write the equation (3.5) as

$$\bar{J}X = TX + wX, \tag{3.6}$$

where TX and wX are the tangential and transversal components of $\bar{J}X$, respectively. Similarly

$$\bar{J}V = BV + CV, \quad V \in \Gamma(tr(TM)), \tag{3.7}$$

where BV and CV are the sections of TM and $tr(TM)$, respectively.

Differentiating (3.5) and using (2.9)-(2.12) and (3.7) we have

$$D^s(X, wP_1Y) = -\nabla^s_X wP_2Y + wP_2\nabla_X Y - h^s(X, TY) + Ch^s(X, Y). \tag{3.8}$$

$$D^l(X, wP_2Y) = -\nabla^l_X wP_1Y + wP_1\nabla_X Y - h^l(X, TY) + Ch^l(X, Y). \tag{3.9}$$

Using Kaehlerian property of $\bar{\nabla}$ with (2.19) and (2.10), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y = A_{wY}X + Bh(X, Y), \tag{3.10}$$

and

$$(\nabla^t_X w) = Ch(X, Y) - h(X, TY), \tag{3.11}$$

where $X, Y \in \Gamma(TM)$and

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{3.12}$$

$$(\nabla^t_X)Y = \nabla^t_X wY - w\nabla_X Y. \tag{3.13}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Kaehlerian manifold $\bar{M}$. Then we have*

$$(\nabla_X B)V = A_{CV}X - TA_V X, \tag{3.14}$$

and

$$(\nabla^t_X C)V = -wA_V X - h(X, BV), \tag{3.15}$$

where $X \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$ and

$$(\nabla_X B)V = \nabla_X BV - B\nabla^t_X V, \tag{3.16}$$

$$(\nabla^t_X C)V = \nabla^t_X CV - C\nabla^t_X V. \tag{3.17}$$

Definition 3.4. *The distribution D (respectively D') is said to define a totally geodesic foliation in M if and only if $\nabla_X Y \in \Gamma(D)$ (respectively $\in \Gamma(D')$), for any $X, Y \in \Gamma(D)$ (respectively $\in \Gamma(D')$).*

Duggal and Sahin [11] investigated the conditions for the definition of totally geodesic foliation by D and D' in M as

Theorem 3.5. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. The distribution D defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0$, for $X, Y \in \Gamma(D)$.*

Theorem 3.6. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. The distribution D' defines a totally geodesic foliation in M if and only if $A_{wY}X \in \Gamma(D')$, for $X, Y \in \Gamma(D')$.*

Definition 3.7. *A GCR-lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called a GCR-lightlike product if both the distributions D and D' define totally geodesic foliations in M.*

Chen [6] characterized the CR-product of Kaehler manifolds as

Theorem 3.8. *A CR-submanifold of a Kaehler manifold is a CR-product if and only if $\nabla T = 0$.*

In present paper, we also characterize the GCR-lightlike product of indefinite Kaehler manifolds as

Theorem 3.9. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is a GCR-lightlike product if and only if $\nabla T = 0$.*

Proof. Let T be parallel then (3.10) implies

$$Bh(X, Y) = -A_{wY}X, \tag{3.18}$$

for any $X, Y \in \Gamma(TM)$. Let $X, Y \in \Gamma(D)$ then $wY = 0$ and (3.18) implies that $Bh(X, Y) = 0$. Hence using the Theorem (3.5), the distribution D defines a totally geodesic foliation in M.

Next, let $X, Y \in \Gamma(D')$, since $BV \in \Gamma(D')$ for any $V \in \Gamma(tr(TM))$ then (3.18) implies that $A_{wY}X \in \Gamma(D')$. Hence using the Theorem (3.6), the distribution D' defines a totally geodesic foliation in M. Since both the distributions D and D' define the totally geodesic foliation in M therefore M is a GCR-lightlike product of an indefinite Kaehler manifold.

Conversely, let M be a GCR-lightlike product of indefinite Kaehler manifold therefore the distribution D defines a totally geodesic foliation in M. Using the

Kaehlerian property of $\bar{\nabla}$, for any $X, Y \in \Gamma(D)$ we obtain $\bar{\nabla}_X \bar{J}Y = \bar{J}\bar{\nabla}_X Y$ then comparing the transversal components both sides, we get $h(X, \bar{J}Y) = \bar{J}h(X, Y)$. Thus $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y = \bar{\nabla}_X \bar{J}Y - h(X, \bar{J}Y) - \bar{J}\bar{\nabla}_X Y + h(X, \bar{J}Y) = 0$, for any $X, Y \in \Gamma(D)$.

Let the distribution D' defines a totally geodesic foliation in M. Using the Kaehlerian property of $\bar{\nabla}$, we have $\bar{\nabla}_X \bar{J}Y = \bar{J}\bar{\nabla}_X Y$, for $X, Y \in \Gamma(D')$. Compare the tangential components both sides, we obtain $-A_{wY}X = Bh(X, Y)$ then (3.10) implies that $(\nabla_X T)Y = 0$, which completes the proof. □

Corollary 3.10. *Let the distribution D' defines a totally geodesic foliation in M then $(\nabla_X T)Y = 0$. Hence (3.10) implies that $A_{wY}X = -Bh(X, Y)$. Since the second fundamental form h is symmetric therefore $A_{wY}X = A_{wX}Y$, for any $X, Y \in \Gamma(D')$.*

Theorem 3.11. *([11]) Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then*

(i) *The distribution D is integrable if and only if $h(X, \bar{J}Y) = h(\bar{J}X, Y)$, for $X, Y \in \Gamma(D)$.*
(ii) *The distribution D' is integrable if and only if $A_{\bar{J}Z}V = A_{\bar{J}V}Z$, for $Z, V \in \Gamma(D')$.*

Theorem 3.12. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation in M if and only if D is integrable.*

Proof. Let $X, Y \in \Gamma(D)$ then using (3.8) and (3.9), we have

$$wP\nabla_X Y = h(X, \bar{J}Y) - Ch(X, Y). \tag{3.19}$$

Taking into account that h is symmetric, we obtain

$$h(X, \bar{J}Y) - h(\bar{J}X, Y) = wP\nabla_X Y - wP\nabla_Y X. \tag{3.20}$$

Hence from (3.20) the assertion follows. □

Definition 3.13. *([12]) A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there exists a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that*

$$h(X, Y) = Hg(X, Y), \tag{3.21}$$

for $X, Y \in \Gamma(TM)$. Using (2.9) it is clear that M is totally umbilical if and only if on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0, \tag{3.22}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$.

Definition 3.14. *([15]) A CR-lightlike submanifold of an indefinite almost Hermitian manifold is called a mixed foliate CR-lightlike submanifold if the second fundamental form h satisfied*

$$h(X, Y) = 0, \tag{3.23}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Lemma 3.15. *([11]) Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $H^s \in \Gamma(L_2)$.*

Lemma 3.16. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $A^*_\xi X = X\bar{g}(H^l, \xi)$, $\nabla^{*t}_X \xi = 0$ and $A_W X = 0$, for $X \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(L_2^\perp)$.*

Proof. Since M is a totally umbilical GCR-lightlike submanifold of $\bar{M}$ therefore for any $W \in \Gamma(L_2^\perp), X \in \Gamma(TM)$ and $Y \in \Gamma(D_0)$ from (2.13), we have $g(X, Y)\bar{g}(H^s, W) = g(A_W X, Y)$. Using the non degeneracy of the distribution D_0, we get $A_W X = X\bar{g}(H^s, W)$, then using the Lemma (3.15) we obtain $A_W X = 0$. Similarly from (2.14) we have $g(X, Y)\bar{g}(H^l, \xi) = -g(Y, \nabla_X \xi)$, then the non degeneracy of the distribution D_0 implies that $\nabla_X \xi = -X\bar{g}(H^l, \xi)$ or $A^*_\xi X - \nabla^{*t}_X \xi = X\bar{g}(H^l, \xi)$ then by comparing the tangential components, the assertion follows. □

Theorem 3.17. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is mixed geodesic if and only if $A_{\bar{J}Z} X \in \Gamma(D)$ and $\nabla^t_X \bar{J}Z \in \Gamma(L_1 \perp L_2)$, for $X \in \Gamma(D)$ and $Z \in \Gamma(D')$.*

Proof. Let $X \in \Gamma(D)$ and $Z \in \Gamma(D')$ then $\bar{\nabla}_X Z = -\bar{J}\bar{\nabla}_X \bar{J}Z$. Using (2.7), (3.5) and (3.7) we obtain $\nabla_X Z + h(X, Z) = TA_{\bar{J}Z}X + wP_1 A_{\bar{J}Z}X + wP_2 A_{\bar{J}Z}X - B\nabla^t_X \bar{J}Z - C\nabla^t_X \bar{J}Z$. Comparing the tangential and transversal components, we have

$$\nabla_X Z = TA_{\bar{J}Z}X - B\nabla^t_X \bar{J}Z, \tag{3.24}$$

and

$$h(X, Z) = wP_1 A_{\bar{J}Z}X + wP_2 A_{\bar{J}Z}X - C\nabla^t_X \bar{J}Z. \tag{3.25}$$

Hence the proof follows form (3.25). □

Definition 3.18. *A GCR-lightlike submanifold M of an indefinite Kaehler manifold is said to be a mixed foliate GCR-lightlike submanifold if the distribution D is integrable and M is a mixed geodesic GCR-lightlike submanifold.*

Lemma 3.19. *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If the distribution D defines a totally geodesic foliation in M then $h^s(X, \bar{J}Y) \in \Gamma(L_2^\perp)$, for $X, Y \in \Gamma(D)$.*

Proof. Let $W \in \Gamma(L_2)$ and use the hypothesis that the distribution D defines a totally geodesic foliation in M then $\bar{g}(h^s(X, \bar{J}Y), W) = -\bar{g}(\bar{\nabla}_X Y, \bar{J}W) = -g(\nabla_X Y, \bar{J}W) - \bar{g}(h(X, Y), \bar{J}W) = 0$, then the non degeneracy of $S(TM^\perp)$ implies that $h^s(X, \bar{J}Y) \in \Gamma(L_2^\perp)$, for $X, Y \in \Gamma(D)$. □

Let $\bar{M}$ be an indefinite Kaehler manifold of constant holomorphic sectional curvature c then the curvature tensor is given by

$$\bar{R}(X, Y)Z = \frac{c}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y + \bar{g}(\bar{J}Y, Z)\bar{J}X - \bar{g}(\bar{J}X, Z)\bar{J}Y + 2\bar{g}(X, \bar{J}Y)\bar{J}Z\}.$$

and $\bar{M}$ is called an indefinite complex space form and denoted by $\bar{M}(c)$.

Theorem 3.20. . *Let M be a mixed foliate GCR-lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then $\bar{M}$ is a complex semi-Euclidean space.*

Proof. From (3.26) we have

$$\bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) \quad = \quad -\frac{c}{2}g(X, X)g(Z, Z), \tag{3.26}$$

for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$. For a mixed foliate GCR-lightlike submanifold M, we also have

$$\bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) \quad = \quad \bar{g}((\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z), \bar{J}Z), \quad (3.27)$$

where $(\nabla_X h^s)(\bar{J}X, Z) = \nabla^s_X(h^s(\bar{J}X, Z)) - h^s(\nabla_X \bar{J}X, Z) - h^s(\bar{J}X, \nabla_X Z)$ and $(\nabla_{\bar{J}X} h^s)(X, Z) = \nabla^s_{\bar{J}X}(h^s(X, Z)) - h^s(\nabla_{\bar{J}X} X, Z) - h^s(X, \nabla_{\bar{J}X} Z)$. Since M is mixed geodesic, therefore $(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z) = h^s(\nabla_{\bar{J}X} X, Z) + h^s(X, \nabla_{\bar{J}X} Z) - h^s(\nabla_X \bar{J}X, Z) - h^s(\bar{J}X, \nabla_X Z)$. Using the hypothesis with the Theorem (3.12), the distribution D defines a totally geodesic foliation in M, therefore $\nabla_X \bar{J}X, \nabla_{\bar{J}X} X \in \Gamma(D)$, for any $X \in \Gamma(D)$ then we get

$$\begin{aligned}(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z) &= h^s(X, \nabla_{\bar{J}X} Z) - h^s(\bar{J}X, \nabla_X Z)\\ &= h^s(X, TA_{\bar{J}Z}\bar{J}X - B\nabla^t_{\bar{J}X}\bar{J}Z) - h^s(\bar{J}X, TA_{\bar{J}Z}X - B\nabla^t_X \bar{J}Z)\\ = h^s(X, TA_{\bar{J}Z}\bar{J}X) - h^s(X, B\nabla^t_{\bar{J}X}\bar{J}Z) &- h^s(\bar{J}X, TA_{\bar{J}Z}X) + h^s(\bar{J}X, B\nabla^t_X \bar{J}Z).\end{aligned} \quad (3.28)$$

Using the Theorem (3.17) and the Lemma (3.19) with the hypothesis of the theorem, we obtain

$$(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z) \quad = \quad 0. \quad (3.29)$$

Thus from (3.26), (3.27) and (3.29) we get $\frac{c}{2} g(X, X) g(Z, Z) = 0$. Since D_0 and $\bar{J}L_2$ are non degenerate therefore $c = 0$, this completes the proof. □

Next, let $(\nabla^s_X wP_2)Y = \nabla^s_X wP_2 Y - wP_2(\nabla_X Y)$ and $(\nabla^l_X wP_1)Y = \nabla^l_X wP_1 Y - wP_1(\nabla_X Y)$, then from (3.8) and (3.9) we get

$$(\nabla^s_X wP_2)Y = Ch^s(X, Y) - h^s(X, TY) - D^s(X, wP_1 Y). \quad (3.30)$$

$$(\nabla^l_X wP_1)Y = Ch^l(X, Y) - h^l(X, TY) - D^l(X, wP_2 Y). \quad (3.31)$$

Hence we have

Theorem 3.21. . *Let M be a GCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If M is a totally geodesic GCR-lightlike submanifold then*

(i) $(\nabla_X T)Y = 0$, *for* $Y \in \Gamma(D)$.
(ii) $(\nabla^l_X wP_1)Z = 0$, *for* $Z \in M_1 = \bar{J}L_1 \subset D'$.
(iii) $(\nabla^s_X wP_2)Z = 0$, *for* $Z \in M_2 = \bar{J}L_2 \subset D'$.

Proof. Using the hypothesis with (3.10), we have $(\nabla_X T)Y = A_{wY} X$. Let $Y \in \Gamma(D)$ then $wY = 0$, therefore $(\nabla_X T)Y = 0$.

Let $Z \in M_1 = \bar{J}L_1 \subset D'$ then $wP_2 Z = 0$. Since M is a totally geodesic therefore (3.31) implies that $(\nabla^l_X wP_1)Z = Ch^l(X, Z) - h^l(X, TZ) = 0$. Similarly for $Z \in M_2 = \bar{J}L_2 \subset D'$ we have $wP_1 Z = 0$ then (3.30) implies that $(\nabla^s_X wP_2)Z = Ch^s(X, Z) - h^s(X, TZ) = 0$. Hence the result. □

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