INEQUALITIES BETWEEN LOGARITHMIC, HARMONIC, ARITHMETIC AND CENTROIDAL MEANS

YU-MING CHU, SHOU-WEI HOU, WEI-MING GONG

Abstract. For all \(a, b > 0\) with \(a \neq b\) we prove that
\[L(a, b) < H(a, b) + \frac{\alpha a + \beta b}{a + b} + (1 - \alpha - \beta)C(a, b)\]
if and only if \(4\alpha + \beta \leq 2\) and \(L(a, b) > H(a, b) + \frac{\alpha a + \beta b}{a + b} + (1 - \alpha - \beta)C(a, b)\)
if and only if \(4\alpha + \beta \geq 4\), where \(L(a, b) = \frac{a - b}{\log a - \log b}\),
\(H(a, b) = 2ab/(a + b)\), \(A(a, b) = (a + b)/2\) and \(C(a, b) = 2(a^2 + ab + b^2)/[3(a + b)]\)
are the logarithmic, harmonic, arithmetic, and centroidal means of \(a\) and \(b\), respectively.

1. Introduction

The classical logarithmic mean \(L(a, b)\) of two positive real numbers \(a\) and \(b\) with \(a \neq b\) is defined by
\[L(a, b) = \frac{a - b}{\log a - \log b}.

In the recent past, the bivariate means have been the subject of intensively research. In particular, many remarkable inequalities for \(L(a, b)\) can be found in the literature \([1, 2, 5, 6, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 30]\). It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology \([9, 13, 14]\). In \([9]\) the authors study a variant of Jensen’s functional equation involving the logarithmic mean, which appears in a heat conduction problem. A representation of \(L(a, b)\) as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in \([6]\). In \([7, 8]\) it is shown that \(L(a, b)\) can be expressed in terms of Gauss’s hypergeometric function \(_2F_1\). And, in \([8]\) the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every \(n \times n\) determinant with elements \(1/L(a_i, b_i)\), where \(0 < a_1 < a_2 < \cdots < a_n\) and \(0 < b_1 < b_2 < \cdots < b_n\), is positive for all \(n \geq 1\).

Let \(G(a, b) = \sqrt{ab}\), \(H(a, b) = \frac{2ab}{a + b}\), \(I(a, b) = \frac{1}{e} \left(\frac{b}{a}\right)^{\frac{b}{a}}\), \(A(a, b) = \frac{1}{2}(a + b)\),
\(C(a, b) = 2(a^2 + ab + b^2)/[3(a + b)]\), \(M_p(a, b) = ((a^p + b^p)/2)^{1/p}(p \neq 0)\) and \(M_0(a, b) =

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where $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ be the geometric, harmonic, identric, arithmetic, $p$-th power, and $p$-th Lehmer means of two positive numbers $a$ and $b$, respectively. Then it is well known that both $L_p(a,b)$ and $M_p(a,b)$ are strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and inequalities

$$
\min\{a, b\} < H(a, b) = M_{-1}(a, b) = L_{-1}(a, b) < G(a, b) = M_0(a, b) = L_{-1/2}(a, b)
$$

$$
L(a, b) < I(a, b) < A(a, b) = M_1(a, b) = L_0(a, b) < C(a, b) = \max\{a, b\}
$$

holds for $a, b > 0$ with $a \neq b$.

Carlson [3] proved that the double inequality

$$
\sqrt{G(a, b)(A(a, b) + G(a, b))/2} < L(a, b) < \frac{1}{2}(A(a, b) + G(a, b))
$$

holds for $a, b > 0$ with $a \neq b$.

In [10], Lin found the best possible upper and lower power bounds for the logarithmic mean as follows:

$$
M_0(a, b) < L(a, b) < M_{1/3}(a, b)
$$

holds for $a, b > 0$ with $a \neq b$.

Sándor [16] established that

$$
\sqrt{G(a, b)I(a, b)} < L(a, b) < A(a, b) + G(a, b) - I(a, b)
$$

for all $a, b > 0$ with $a \neq b$.

In [3], Alzer gave the optimal Lehmer mean bounds for $L$, $(LI)^{1/2}$, and $(L + I)/2$ as follows:

$$
L_{-1/3}(a, b) < L(a, b) < L_0(a, b),
$$

$$
L_{-1/4}(a, b) < \sqrt{L(a, b)I(a, b)} < L_0(a, b)
$$

and

$$
L_{-1/4}(a, b) < \frac{1}{2}(L(a, b) + I(a, b)) < L_0(a, b)
$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for $(LI)^{1/2}$ and $(L + I)/2$ in terms of power mean were presented in [43]:

$$
M_0(a, b) < \sqrt{L(a, b)I(a, b)} < M_{1/2}(a, b)
$$

and

$$
M_{\log 2/(1+\log 2)}(a, b) < \frac{1}{2}(L(a, b) + I(a, b)) < M_{1/2}(a, b)
$$

for all $a, b > 0$ with $a \neq b$.

For any $\alpha \in (0, 1)$, in [26, 29] the authors obtained the sharp bounds for the products $A^\alpha(a, b)L^{1-\alpha}(a, b)$ and $G^\alpha(a, b)L^{1-\alpha}(a, b)$, and the sum $\alpha A(a, b) + (1 - \alpha) L(a, b)$ in terms of power mean as follows:

$$
M_0(a, b) < A^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1+2\alpha)/3}(a, b),
$$

$$
M_0(a, b) < G^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1-\alpha)/3}(a, b)
$$

and

$$
M_{\log 2/(\log 2-\log \alpha)}(a, b) < \alpha A(a, b) + (1 - \alpha) L(a, b) < M_{(1+2\alpha)/3}(a, b)
$$

for all $a, b > 0$ with $a \neq b$. 
It is the aim of this paper to answer the question: what are the values of \( \alpha \) and \( \beta \) such that the inequality \( L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \) or \( L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \) holds for all \( a, b > 0 \) with \( a \neq b \)?

2. Main Results

**Theorem 2.1.** For all \( a, b > 0 \) with \( a \neq b \) we have

1. \( L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \) if and only if \( 4\alpha + \beta \leq 2 \);

2. \( L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \) if and only if \( 4\alpha + \beta \geq 4 \).

**Proof.** Without loss of generality, we assume \( a > b \). Let \( t = a/b > 1 \), then it is not difficult to verify that

\[
\alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) - L(a,b) = \frac{b}{6(t + 1) \log t} \left\{ [(4 - 4\alpha - \beta)t^2 + 2(2 + 4\alpha + \beta)t + (4 - 4\alpha - \beta)] \log t - 6t^2 + 6 \right\}. \tag{2.1}
\]

Let

\[
f(t) = [(4 - 4\alpha - \beta)t^2 + 2(2 + 4\alpha + \beta)t + (4 - 4\alpha - \beta)] \log t - 6t^2 + 6. \tag{2.2}
\]

Then simple computations lead to

\[
f(1) = 0, \tag{2.3}
\]

\[
f'(t) = [2(4 - 4\alpha - \beta)t + 2(2 + 4\alpha + \beta)] \log t - (8 + 4\alpha + \beta)t + \frac{4 - 4\alpha - \beta}{t} + 2(2 + 4\alpha + \beta), \tag{2.4}
\]

\[
f'(1) = 0, \tag{2.5}
\]

\[
f''(t) = 2(4 - 4\alpha - \beta) \log t + \frac{2(2 + 4\alpha + \beta)}{t} - \frac{4 - 4\alpha - \beta}{t^2} - 3(4\alpha + \beta), \tag{2.6}
\]

\[
f''(1) = 0, \tag{2.7}
\]

\[
f'''(t) = \frac{2[(4 - 4\alpha - \beta)t^2 - (2 + 4\alpha + \beta)t + (4 - 4\alpha - \beta)]}{t^2}. \tag{2.8}
\]

We divide the proof into three cases.

**Case 1:** \( 4\alpha + \beta \leq 2 \). Then equation (2.8) leads to

\[
f'''(t) = \frac{2(4 - 4\alpha - \beta)}{t^2} \left[ \left( t - \frac{2 + 4\alpha + \beta}{2(4 - 4\alpha - \beta)} \right)^2 + \frac{3(2 - 4\alpha - \beta)(10 - 4\alpha - \beta)}{4(4 - 4\alpha - \beta)^2} \right] > 0 \tag{2.9}
\]

for \( t > 1 \).

From equations (2.1)-(2.3), (2.5) and (2.7) together with inequality (2.9) we clearly see that \( L(a,b) < \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \).

**Case 2:** \( 4\alpha + \beta \geq 4 \). Then equation (2.8) leads to

\[
f'''(t) = -\frac{2[(4\alpha + \beta - 4)t^2 + (2 + 4\alpha + \beta)t + (4\alpha + \beta - 4)]}{t^2} < 0 \tag{2.10}
\]

for \( t > 1 \).

Therefore, inequality \( L(a,b) > \alpha H(a,b) + \beta A(a,b) + (1 - \alpha - \beta)C(a,b) \) follows from equations (2.1)-(2.3), (2.5) and (2.7) together with inequality (2.10).
Case 3: $2 < 4\alpha + \beta < 4$. Then from equations (2.2), (2.4) and (2.6) we get
\[
\lim_{t \to +\infty} f(t) = +\infty, \tag{2.11}
\]
\[
\lim_{t \to +\infty} f'(t) = +\infty \tag{2.12}
\]
and
\[
\lim_{t \to +\infty} f''(t) = +\infty. \tag{2.13}
\]
Let
\[
g(t) = (4 - 4\alpha - \beta)t^2 - (2 + 4\alpha + \beta)t + (4 - 4\alpha - \beta). \tag{2.14}
\]
Then
\[
g(1) = -3(4\alpha + \beta - 2) < 0, \tag{2.15}
\]
\[
\lim_{t \to +\infty} g(t) = +\infty, \tag{2.16}
\]
\[
g'(t) = 2(4 - 4\alpha - \beta)t - (2 + 4\alpha + \beta), \tag{2.17}
\]
\[
g'(1) = -3(4\alpha + \beta - 2) < 0 \tag{2.18}
\]
and
\[
\lim_{t \to +\infty} g'(t) = +\infty. \tag{2.19}
\]
From equation (2.17) we know that $g'(t)$ is strictly increasing in $[1, +\infty)$. Then inequality (2.18) and equation (2.19) lead to the conclusion that there exists $\lambda_1 > 1$ such that $g'(t) < 0$ for $t \in [1, \lambda_1]$ and $g'(t) > 0$ for $t \in (\lambda_1, +\infty)$. Therefore, $g(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

It follows from inequality (2.15) and equation (2.16) together with the piecewise monotonicity of $g(t)$ that there exists $\lambda_2 > \lambda_1 > 1$ such that $g(t) < 0$ for $t \in [1, \lambda_2)$ and $g(t) > 0$ for $t \in (\lambda_2, +\infty)$. Then equations (2.8) and (2.14) lead to the conclusion that $f''(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$.

From equations (2.7) and (2.13) together with the piecewise monotonicity of $f''(t)$ we conclude that there exists $\lambda_3 > \lambda_2 > 1$ such that $f''(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$. Then equations (2.5) and (2.12) lead to the conclusion that there exists $\lambda_4 > \lambda_3 > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$.

It follows from equations (2.3) and (2.11) together with the piecewise monotonicity of $f(t)$ that there exists $\lambda > \lambda_4 > 1$ such that $f(t) < 0$ for $t \in (1, \lambda)$ and $f(t) > 0$ for $t \in (\lambda, +\infty)$. Then equations (2.1) and (2.2) lead to the conclusion that $L(a, b) > \alpha H(a, b) + \beta A(a, b) + (1 - \alpha - \beta)C(a, b)$ for $a/b \in (1, \lambda)$ and $L(a, b) < \alpha H(a, b) + \beta A(a, b) + (1 - \alpha - \beta)C(a, b)$ for $a/b \in (\lambda, +\infty)$. \qed

References


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