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CERTAIN DIFFERENTIAL SANDWICH-TYPE RESULTS

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Abstract. In this paper, we obtain certain subordination and superordination theorems. As special cases of main results, we find dominants and subordinants for $f(z)/z$, $f'(z)$ and $zf'(z)/f(z)$ and consequently we get certain sufficient conditions for close-to-convexity and starlikeness of normalized analytic functions. Mathematica 7.0 is used to plot the figures.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function f is said to be univalent in a domain $\mathbb{D}$ in the extended complex plane if and only if it is analytic in $\mathbb{D}$ except for at most one simple pole and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$ $(z_1, z_2 \in \mathbb{D})$. In this case, the equation $f(z) = w$ has at most one root in $\mathbb{D}$ for any complex number w. Such functions map $\mathbb{D}$ conformally onto a domain in the w-plane. Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{E}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \; z \in \mathbb{E}.$$

Let $\mathcal{S}^*$ denote the class of univalent starlike functions w.r.t. the origin.

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A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$ if there is a starlike function g (not necessarily normalized) such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [1] and Warchawski [2] obtained a simple criterion for univalence of analytic functions. They proved that if an analytic function f satisfies the condition $\Re\, f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

The main objective of this paper is find certain sandwich-type results for the members of the class $\mathcal{A}$ of normalized analytic functions and consequently we present certain sufficient conditions for close-to-convexity and starlikeness. Here, we use the dual concept of differential subordination and superordination:

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \Phi(p(0), 0; 0) = h(0), \tag{1.1}$$

then p is called a solution of the differential subordination (1.1). The univalent function q is called a dominant of the differential subordination (1.1) if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1).

Miller and Mocanu [4] introduced the concept of differential superordination.

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z),\ h(0) = \Psi(p(0), 0; 0). \tag{1.2}$$

An analytic function q is called a subordinant of the differential superordination (1.2), if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2), is said to be the best subordinant of (1.2).

2. Preliminaries

To prove the main results, we need the following definition and lemmas.

Definition 2.1. *([3], p.21). We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where*

$$\mathbb{B}(p) = \left\{\zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} p(z) = \infty\right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 2.2. *([3], p.132). Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

(i) h is convex, or

(ii) Q_1 is starlike.

In addition, assume that

(iii) $\Re\, \dfrac{zh'(z)}{Q_1(z)} > 0,\ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.3. *([5]). Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*

(i) Q_1 is starlike in $\mathbb{E}$ and

(ii) $\Re\left(\dfrac{\theta'(q(z))}{\phi(q(z))}\right) > 0,\ z \in \mathbb{E}$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],\ z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. Main Results

In what follows, all the powers taken, are the principle ones.

Theorem 3.1. *Let α, β, γ and λ be complex numbers such that $\alpha, \lambda \neq 0$. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\alpha} - 1\right)\frac{zq'(z)}{q(z)}\right] > \max\left\{0, -\Re\left(\frac{\beta\gamma}{\alpha\lambda}\right)\right\}. \tag{3.1}$$

If $p,\ p(z) \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(p(z))^{\beta}\left(\gamma + \lambda\frac{zp'(z)}{p(z)}\right)^{\alpha} \prec (q(z))^{\beta}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right)^{\alpha}, \tag{3.2}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Proof. The subordination in (3.2) can be rewritten as under:

$$(p(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zp'(z)}{p(z)}\right) \prec (q(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right). \tag{3.3}$$

Define the functions θ and ϕ as follows:

$$\theta(w) = \gamma w^{\frac{\beta}{\alpha}},$$

and

$$\phi(w) = \lambda w^{\frac{\beta}{\alpha}-1}.$$

It is clear that the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Now, define the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \lambda zq'(z)(q(z))^{\frac{\beta}{\alpha}-1},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = (q(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right).$$

Then in view of condition (3.1), we have

(1) Q_1 is starlike in $\mathbb{E}$ and

(2) $\Re \dfrac{zh'(z)}{Q_1(z)} > 0,\ z \in \mathbb{E}$.

Thus conditions (ii) and (iii) of Lemma 2.2, are satisfied. In view of (3.3), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 2.2. □

Theorem 3.2. *Let α, β, γ and λ be complex numbers such that $\Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$, where $\alpha, \lambda \neq 0$. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\beta}{\alpha} - 1\right)\frac{zq'(z)}{q(z)}\right] > 0. \tag{3.4}$$

If $p,\ p \in \mathcal{H}[q(0), 1] \cap Q$ with $p(z) \neq 0$ in $\mathbb{E}$, satisfies the differential superordination

$$(q(z))^{\beta}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right)^{\alpha} \prec (p(z))^{\beta}\left(\gamma + \lambda\frac{zp'(z)}{p(z)}\right)^{\alpha} = h(z), \tag{3.5}$$

where h is univalent in $\mathbb{E}$, then $q(z) \prec p(z)$ and q is the best subordinant.

Proof. The superordination in (3.5) can be rewritten as under:

$$(q(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right) \prec (p(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zp'(z)}{p(z)}\right). \tag{3.6}$$

Similar to the case of Theorem 3.1, define the functions θ and ϕ as follows:

$$\theta(w) = \gamma w^{\frac{\beta}{\alpha}},$$

and

$$\phi(w) = \lambda w^{\frac{\beta}{\alpha}-1}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Let Q_1 and h be defined as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \lambda zq'(z)(q(z))^{\frac{\beta}{\alpha}-1},$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = (q(z))^{\frac{\beta}{\alpha}}\left(\gamma + \lambda\frac{zq'(z)}{q(z)}\right).$$

In view of condition (3.4), Q_1 is starlike in $\mathbb{E}$ and $\Re\, \dfrac{\theta'(q(z))}{\phi(q(z))} = \Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$. Thus conditions (i) and (ii) of Lemma 2.3, are satisfied. In view of (3.6), we have

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)].$$

Therefore, the proof, now, follows from Lemma 2.3. □

Combining Theorem 3.1 and Theorem 3.2, we obtain the following sandwich-type theorem.

Theorem 3.3. *Let α, β, γ and λ be complex numbers such that $\Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$, where $\alpha, \lambda \neq 0$. Let $q_1,\ q_2$ be univalent functions in $\mathbb{E}$ such that $q_1(0) = q_2(0)$ with $q_1,\ q_2 \neq 0$ in $\mathbb{E}$. Suppose that*

$$\Re\left[1 + \frac{zq_k''(z)}{q_k'(z)} + \left(\frac{\beta}{\alpha} - 1\right)\frac{zq_k'(z)}{q_k(z)}\right] > 0,\ z \in \mathbb{E},\ k = 1, 2.$$

If $p,\ p\in\mathcal{H}[q(0),1]\cap Q$ *with* $p(z)\neq 0$ *in* $\mathbb{E}$, *satisfies*

$$(q_1(z))^\beta\left(\gamma+\lambda\frac{zq_1'(z)}{q_1(z)}\right)^\alpha\prec(p(z))^\beta\left(\gamma+\lambda\frac{zp'(z)}{p(z)}\right)^\alpha\prec(q_2(z))^\beta\left(\gamma+\lambda\frac{zq_2'(z)}{q_2(z)}\right)^\alpha,\tag{3.7}$$

where $(p(z))^\beta\left(\gamma+\lambda\frac{zp'(z)}{p(z)}\right)^\alpha$ *is univalent in* $\mathbb{E}$, *then* $q_1(z)\prec p(z)\prec q_2(z)$. *Moreover, the functions* q_1 *and* q_2 *are respectively the best subordinant and the best dominanat.*

4. Certain Special Cases of Theorem 3.1

In case $\alpha=\beta=\gamma=1$, the condition (3.1) reduces to

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{1}{\lambda}\right)\right\}.$$

If $\Re\left(\frac{1}{\lambda}\right)=\Re\left(\frac{\overline{\lambda}}{|\lambda|^2}\right)\geq 0$, then the above condition can be re-written as

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>0.$$

When we select the dominant $q(z)=\frac{1+z}{1-z}$, we have

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)=\Re\left(\frac{1+z}{1-z}\right)>0.$$

With these substitutions and by selecting respectively $p(z)=\frac{f(z)}{z}$ and $p(z)=f'(z)$ in Theorem 3.1, we immediately get the next two results.

Corollary 4.1. *Let* λ *be a non-zero complex number such that* $\Re\left(\frac{\overline{\lambda}}{|\lambda|^2}\right)\geq 0$. *If* $f\in\mathcal{A},\ \frac{f(z)}{z}\neq 0,\ z\in\mathbb{E}$, *satisfies*

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\prec\frac{1+z}{1-z}+\frac{2\lambda z}{(1-z)^2},$$

then $\frac{f(z)}{z}\prec\frac{1+z}{1-z}$ *in* $\mathbb{E}$.

Corollary 4.2. *Let* λ *be a non-zero complex number such that* $\Re\left(\frac{\overline{\lambda}}{|\lambda|^2}\right)\geq 0$. *If* $f\in\mathcal{A},\ f'(z)\neq 0,\ z\in\mathbb{E}$, *satisfies*

$$f'(z)+\lambda zf''(z)\prec\frac{1+z}{1-z}+\frac{2\lambda z}{(1-z)^2},\ z\in\mathbb{E},$$

then $f'(z)\prec\frac{1+z}{1-z}$ *i.e.* f *is close-to-convex.*

In case $\alpha = \lambda = 1$ and $\beta = \gamma = 0$ and the dominant $q(z) = \dfrac{1+z}{1-z}$, the condition (3.1) becomes

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+z^2}{1-z^2}\right) > 0,\ z \in \mathbb{E}.$$

Now on writing writing $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem 3.1, we get:

Corollary 4.3. *Suppose* $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2},\ z \in \mathbb{E},$$

then $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+z}{1-z}$ *i.e.* $f \in \mathcal{S}^*$.

5. Deduction of Certain Sandwich-type Results

In this section, we shall use Theorem 3.3 to obtain certain sandwich-type results and consequently get some criteria for close-to-convexity and starlikeness of functions $f \in \mathcal{A}$.

Writing $p(z) = \dfrac{f(z)}{z}$ in Theorem 3.3, we obtain the following result.

Theorem 5.1. *Let* α, β, γ *and* λ *be complex numbers such that* $\Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$ *in* $\mathbb{E}$ *where* $\alpha, \lambda \neq 0$. *Let* $q_1,\ q_2$ *be same as in Theorem 3.3. If* $f \in \mathcal{A}$, $\dfrac{f(z)}{z} \in \mathcal{H}[q(0), 1] \cap Q$ *with* $\dfrac{f(z)}{z} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$(q_1(z))^\beta \left(\gamma + \lambda\frac{zq_1'(z)}{q_1(z)}\right)^\alpha \prec \left(\frac{f(z)}{z}\right)^\beta \left[\gamma + \lambda\left(\frac{zf'(z)}{f(z)} - 1\right)\right]^\alpha \prec (q_2(z))^\beta \left(\gamma + \lambda\frac{zq_2'(z)}{q_2(z)}\right)^\alpha,\ z \in \mathbb{E},$$

then

$$q_1(z) \prec \frac{f(z)}{z} \prec q_2(z).$$

Setting $p(z) = f'(z)$ in Theorem 3.3, we get:

Theorem 5.2. *Let* α, β, γ *and* λ *be complex numbers such that* $\Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$, *where* $\alpha, \lambda \neq 0$. *Let* $q_1,\ q_2$ *be same as in Theorem 3.3. If* $f \in \mathcal{A}$, $f'(z) \in \mathcal{H}[q(0), 1] \cap Q$ *with* $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$(q_1(z))^\beta \left(\gamma + \lambda\frac{zq_1'(z)}{q_1(z)}\right)^\alpha \prec (f'(z))^\beta \left(\gamma + \lambda\frac{zf''(z)}{f'(z)}\right)^\alpha \prec (q_2(z))^\beta \left(\gamma + \lambda\frac{zq_2'(z)}{q_2(z)}\right)^\alpha,$$

then

$$q_1(z) \prec f'(z) \prec q_2(z).$$

Taking $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem 3.3, we obtain:

Theorem 5.3. *Let* $\alpha,\ \beta, \gamma$ *and* λ *be complex numbers such that* $\Re\left(\dfrac{\beta\gamma}{\alpha\lambda}\right) > 0$, *where* $\alpha, \lambda \neq 0$. *Let* $q_1,\ q_2$ *be same as in Theorem 3.3. If* $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *with* $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$(q_1(z))^\beta \left(\gamma + \lambda \frac{zq_1'(z)}{q_1(z)}\right)^\alpha \prec \left(\frac{zf'(z)}{f(z)}\right)^\beta \left[\gamma + \lambda\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right]^\alpha$$
$$\prec (q_2(z))^\beta \left(\gamma + \lambda \frac{zq_2'(z)}{q_2(z)}\right)^\alpha,\ z \in \mathbb{E},$$

then

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Remark 5.4. *In the next part of the paper, we shall present certain special cases of above sandwich-type results by selecting particular values of* $\alpha,\ \beta, \gamma$ *and* λ *and by taking the subordinant* $q_1(z) = 1 + az$ *and the dominant* $q_2(z) = 1 + bz$ *where* $0 < a < b$.

Setting $\alpha = \beta = \gamma = 1$ in Theorem 5.1, we obtain the following result.

Corollary 5.5. *Let* λ *be a non-zero complex number such that* $\Re\left(\dfrac{\overline{\lambda}}{|\lambda|^2}\right) > 0$ *and let* a *and* b *be real numbers with* $0 < a < b$. *If* $f \in \mathcal{A},\ \dfrac{f(z)}{z} \in \mathcal{H}[q(0), 1] \cap Q$ *with* $\dfrac{f(z)}{z} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$1 + a(1 + \lambda)z \prec (1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) \prec 1 + b(1 + \lambda)z,$$

then

$$1 + az \prec \frac{f(z)}{z} \prec 1 + bz.$$

Taking $\alpha = \beta = \gamma = 1$ in Theorem 5.2, we get:

Corollary 5.6. *Let* λ *be a non-zero complex number such that* $\Re\left(\dfrac{\overline{\lambda}}{|\lambda|^2}\right) > 0$ *and let* a *and* b *be real numbers with* $0 < a < b$. *If* $f \in \mathcal{A},\ f'(z) \in \mathcal{H}[q(0), 1] \cap Q$ *with* $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$1 + a(1 + \lambda)z \prec f'(z) + \lambda z f''(z) \prec 1 + b(1 + \lambda)z,$$

then

$$1 + az \prec f'(z) \prec 1 + bz.$$

Keeping $\alpha = \beta = \gamma = \lambda = 1$ in Theorem 5.3, we obtain:

Corollary 5.7. *Let a and b be real numbers with $0 < a < b$ and let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ with $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$1 + 2az \prec \frac{zf'(z)}{f(z)}\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec 1 + 2bz,$$

then

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz.$$

Remark 5.8. *We select, in particular, $a = 1/2$, $b = 1$ and $\lambda = 2$ in Corollary 5.5 and Corollary 5.6, we deduce the following results respectively.*

Let f be same as in Corollary 5.5. Then for all z in $\mathbb{E}$, we have

$$1 + \frac{3}{2}z \prec 2f'(z) - \frac{f(z)}{z} \prec 1 + 3z \ \Rightarrow 1 + \frac{1}{2}z \prec \frac{f(z)}{z} \prec 1 + z. \tag{5.1}$$

For all z in $\mathbb{E}$, if f is same as in Corollary 5.6, then

$$1 + \frac{3}{2}z \prec f'(z) + 2zf''(z) \prec 1 + 3z \ \Rightarrow 1 + \frac{1}{2}z \prec f'(z) \prec 1 + z. \tag{5.2}$$

From (5.1), we observe that if for all z in $\mathbb{E}$, the operator $2f'(z) - \dfrac{f(z)}{z}$ takes values in the light shaded portion of Figure 5.1, then $\dfrac{f(z)}{z}$ takes values in the light shaded portion of Figure 5.2. Similarly using (5.2), we see whenever the operator $f'(z) + 2zf''(z)$ varies in the light shaded portion of Figure 5.1, then $f'(z)$ can take values in the light shaded portion of Figure 5.2 and hence f is univalent close-to-convex in $\mathbb{E}$.

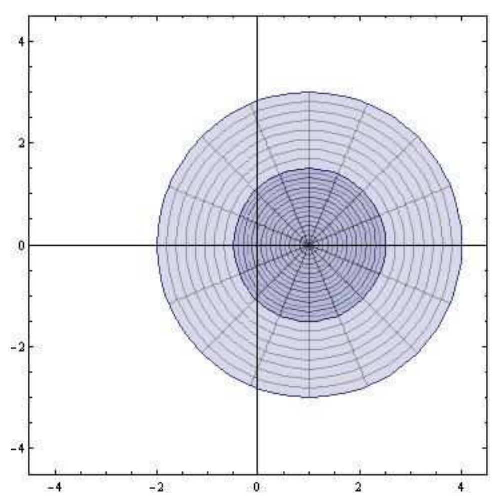

Figure 5.1

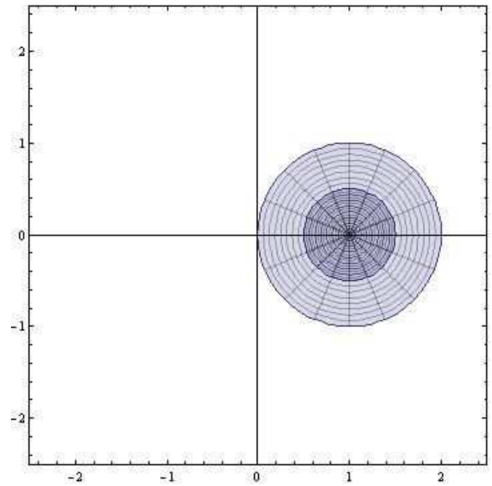

Figure 5.2

Remark 5.9. *By setting $a = 1/4$ and $b = 1$ in Corollary 5.7, we deduce:*

Let f be same as in Corollary 5.7. Then for all z in $\mathbb{E}$, we have

$$1+\frac{1}{2}z \prec \frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec 1+2z \;\Rightarrow 1+\frac{1}{4}z \prec \frac{zf'(z)}{f(z)} \prec 1+z. \tag{5.3}$$

In view of (5.3), we notice that if the values taken by the operator

$$\frac{zf'(z)}{f(z)}\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)$$

for all z in $\mathbb{E}$, lies in the light shaded portion of Figure 5.3, then $\frac{zf'(z)}{f(z)}$ takes values in the light shaded portion of Figure 5.4 and consequently f is starlike in $\mathbb{E}$.

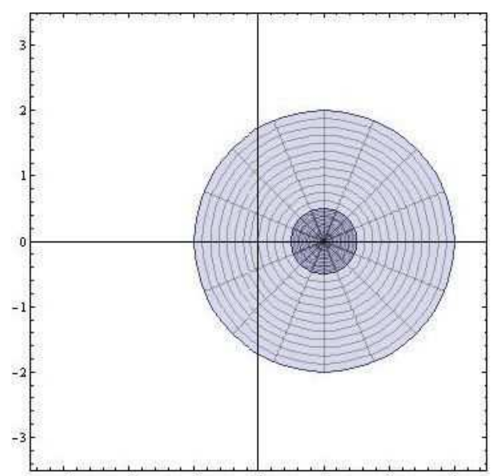

Figure 5.3

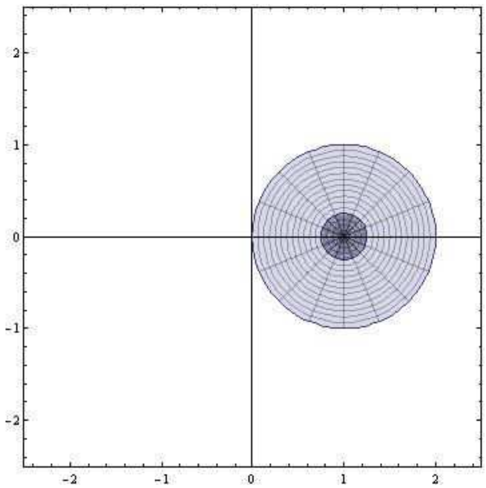

Figure 5.4

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