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A NEW FUZZY BANACH-TYPE DEFICIT CONTRACTION AND ITS APPLICATIONS TO ORDERED INTERPOLATIVE FIXED POINTS AND FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We introduce a new fuzzy Banach-type contraction that acts directly on the fuzzy distance deficit $1 - M(x, y, t)$, and show by an explicit example that it is genuinely fuzzy and not induced by any classical metric. Extending this to an ordered setting via weak fuzzy partial orders, we define ordered fuzzy Hardy–Rogers interpolative contractions and prove a fixed point theorem for such mappings on complete fuzzy metric spaces. As an application, we prove existence and uniqueness for a fuzzy Caputo fractional initial value problem ${}^{C}D^{\alpha}u(t) = -\lambda u(t)$ on $[0, 1]$, for every $\alpha \in (0, 1)$ and $\lambda > 0$, using a Mittag-Leffler-weighted fuzzy metric. A worked numerical example confirms geometric convergence of the Picard iterates to the exact Mittag-Leffler solution.

1. Introduction

Fixed point theory in ordered metric spaces has evolved significantly since the pioneering work of Ran and Reurings [33], who established an analogue of Banach's fixed point theorem in partially ordered sets with applications to linear and nonlinear equations. Their work opened new avenues for solving matrix equations using order-theoretic methods, particularly for equations arising in control theory, interpolation, and systems analysis [1, 11, 21, 31, 32].

In recent years, interpolative contraction principles have become one of the most active research directions in nonlinear analysis and fixed point theory. The pioneering work of Karapınar [20] introduced the concept of interpolative Kannan-type contractions, opening a new direction for obtaining fixed point results under weaker contractive assumptions than the classical Banach contraction principle. Since then, numerous extensions and generalizations have been developed in different generalized metric settings. These include interpolative Reich-Rus-Ćirić type contractions on partial metric spaces [19], interpolative Ćirić-Reich-Rus type contractions via the Branciari distance [22], ω-interpolative Ćirić-Reich-Rus type

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contractions [3], interpolative Hardy-Rogers type contractions [20], interpolative Rus-Reich-Ćirić type contractions via simulation functions [19], Boyd Wong and Matkowski type interpolative contractions [25], and a unified framework for interpolative contractions [13]. More recently, interpolative techniques have been extended to $(\alpha, \beta, \psi, \phi)$-interpolative contractions [23], Perov-interpolative contractions of Suzuki type [22], interpolative (φ, ψ)-type Z-contractions [24], and interpolative Kannan-Meir-Keeler type contractions [26]. These studies demonstrate that interpolation techniques provide a flexible framework for establishing fixed point theorems under substantially weaker hypotheses while maintaining broad applicability to nonlinear problems.

The extension of metric fixed point theory to fuzzy settings began with the introduction of fuzzy metric spaces by Kramosil and Michalek [28] and later refined by George and Veeramani [10]. These spaces provide a natural framework for handling uncertainty and imprecision in mathematical models. The incorporation of partial orders into fuzzy metric spaces was systematically studied by various authors, including Beg and Butt [5], who introduced ordered fuzzy metric spaces and proved fixed point results for monotone mappings.

Recently, interpolative contractions have gained considerable attention as generalizations of classical Banach contractions. Karapinar [20] introduced interpolative contractions in metric spaces, while Hussain et al. [16] extended these to partial metric spaces. The Hardy-Rogers type contractions, originally introduced in [14], have been similarly generalized to interpolative forms by several authors [4, 18].

In the fuzzy context, Mihet [30] introduced fuzzy contractive mappings and proved fixed point theorems, while Gregori and Sapena [12] established fuzzy Banach contraction principles. The combination of fuzzy metrics with partial orders was further developed by Diric et al. [9] and later by Hierro et al. [35], who considered various types of fuzzy ordered contractions.

Graded mathematical structures, which allow for degrees of membership in relations, have been increasingly important in fuzzy mathematics. Zadeh [36] originally introduced the concept of fuzzy sets with graded membership, while later developments by Rodabaugh [34] and Hohle [15] established theoretic foundations for graded mathematics. The concept of graded partial orders was formalized by Demirci [7] and has since been applied in various contexts including fixed point theory.

Motivated by these developments, the present work extends the theory of interpolative contractions to the framework of complete fuzzy metric spaces with fuzzy partial order. Unlike the aforementioned results, which are mainly formulated in metric, partial metric, Branciari metric, or other generalized metric spaces, our approach establishes a new interpolative fuzzy contraction principle and applies it to prove the existence and uniqueness of solutions for a fuzzy fractional differential equation involving the Caputo fractional derivative. Consequently, the present work further enlarges the scope of interpolative fixed point theory and demonstrates its effectiveness in the analysis of fuzzy fractional dynamical systems.

2. Main Results

In this section we introduced a new way to define fuzzy contractions, and later with examples we show that these fuzzy contractions are valid for fuzzy spaces not necessarily for crisp cases.

2.1. **New Fuzzy Banach-Type Contractions.**

Definition 2.1 (New Fuzzy Banach-Type Contraction)**.** *Let $(S, \digamma_M, \odot)$ be a fuzzy metric space. A mapping $T : S \to S$ is called a* ***new fuzzy Banach-type contraction*** *if there exists a constant $\lambda \in (0,1)$ such that for all $\varkappa, y \in S$ and all $\ell > 0$:*

$$\digamma_M(T\varkappa, Ty, \ell) \geq 1 - \lambda\,(1 - \digamma_M(\varkappa, y, \ell))\,.$$

The inequality $\digamma_M(T\varkappa, Ty, \ell) \geq 1 - \lambda\,(1 - \digamma_M(\varkappa, y, \ell))$, contracts the "fuzzy distance deficit" $1 - \digamma_M(\varkappa, y, \ell)$ directly by factor λ.

Example Illustrating the Purely Fuzzy Nature of the Contraction:

Before presenting the example, we emphasize that the following construction works only in a *fuzzy metric space*. In particular, the fuzzy metric used below is not generated by any classical metric d. Consequently, the example cannot be interpreted in the framework of classical (crisp) metric spaces and hence it does not fall under the scope of the classical Banach contraction principle.

Indeed, if a fuzzy metric $\digamma_M$ is generated by a metric d, then it could be of a standard form such as $\digamma_M(\varkappa, y, \ell) = \frac{\ell}{\ell + d(\varkappa, y)}$ or $\digamma_M(\varkappa, y, \ell) = e^{-d(\varkappa, y)/\ell}$. In the example below, the fuzzy metric does not depend on ℓ, and therefore no metric d can generate it. Hence the contraction property arises purely from the fuzzy structure and not from any underlying crisp distance.

Example 2.2. *Let*

$$X = \{a, b, c\},$$

and define a fuzzy metric

$$\digamma_M : X \times X \times (0, \infty) \longrightarrow (0, 1]$$

by

$$\digamma_M(\varkappa, y, \ell) = \begin{cases} 1, & \varkappa = y, \\ 0.7, & \{\varkappa, y\} = \{a, b\}, \\ 0.7, & \{\varkappa, y\} = \{a, c\}, \\ 0.9, & \{\varkappa, y\} = \{b, c\}, \end{cases}$$

for every $\ell > 0$, and let the continuous ℓ-norm be the product

$$a \odot b = ab.$$

Since $\digamma_M$ is positive, symmetric, satisfies

$$\digamma_M(\varkappa, y, \ell) = 1 \iff \varkappa = y,$$

and is independent of the parameter ℓ, the left continuity condition is immediate.

It remains to verify the fuzzy triangle inequality

$$\digamma_M(\varkappa, z, \ell + s) \geq \digamma_M(\varkappa, y, \ell) \odot \digamma_M(y, z, s) = \digamma_M(\varkappa, y, \ell)\digamma_M(y, z, s),$$

for all $\varkappa, y, z \in X$ and $\ell, s > 0$.

Since X contains only three points, it is sufficient to consider the distinct non-trivial triples.

$(\varkappa, y, z)$	$\digamma_M(\varkappa, z, \ell + s)$	$\digamma_M(\varkappa, y, \ell)\digamma_M(y, z, s)$
(a, b, c)	0.7	$0.7 \times 0.9 = 0.63$
(a, c, b)	0.7	$0.7 \times 0.9 = 0.63$
(b, a, c)	0.9	$0.7 \times 0.7 = 0.49$
(b, c, a)	0.7	$0.9 \times 0.7 = 0.63$
(c, a, b)	0.9	$0.7 \times 0.7 = 0.49$
(c, b, a)	0.7	$0.9 \times 0.7 = 0.63$

Each inequality is satisfied, each entry in the second column is greater than or equal to corresponding entry. The remaining cases involve repeated points, for which equality holds trivially. Therefore,

$$\digamma_M(\varkappa, z, \ell + s) \geq \digamma_M(\varkappa, y, \ell)\digamma_M(y, z, s)$$

for all $\varkappa, y, z \in X$ *and* $\ell, s > 0$.

Moreover, since X *is finite, every Cauchy sequence in* $(X, \digamma_M, \odot)$ *is eventually constant. Hence every Cauchy sequence converges, and therefore* $(X, \digamma_M, \odot)$ *is a complete fuzzy metric space.*

Now define the mapping

$$T : X \longrightarrow X$$

by

$$T(\varkappa) = \begin{cases} b, & \text{if } \varkappa = a \\ c, & \text{if } \varkappa = b \\ c, & \text{if } \varkappa = c. \end{cases}$$

We verify that T *satisfies the fuzzy Banach-type contraction*

$$\digamma_M(T\varkappa, Ty, \ell) \geq 1 - \lambda\left[1 - \digamma_M(\varkappa, y, \ell)\right],$$

with

$$\lambda = 0.8.$$

Case 1. $\varkappa = a,\ y = b.$

Since

$$\digamma_M(Ta, Tb, \ell) = \digamma_M(b, c, \ell) = 0.9,$$

and

$$1 - \lambda\left[1 - \digamma_M(a, b, \ell)\right] = 1 - 0.8(1 - 0.7) = 0.76,$$

we obtain

$$0.9 \geq 0.76.$$

Case 2. $\varkappa = a,\ y = c.$

Similarly,

$$\digamma_M(Ta, Tc, \ell) = \digamma_M(b, c, \ell) = 0.9,$$

while

$$1 - 0.8(1 - 0.7) = 0.76,$$

and therefore

$$0.9 \geq 0.76.$$

Case 3. $\varkappa = b,\ y = c.$

Here

$$\digamma_M(Tb, Tc, \ell) = \digamma_M(c, c, \ell) = 1,$$

whereas

$$1 - 0.8(1 - 0.9) = 1 - 0.08 = 0.92.$$

Thus,

$$1 \geq 0.92.$$

The remaining distinct pairs follow immediately by symmetry of $\digamma_M$*, while for* $\varkappa = y$ *we have*

$$\digamma_M(T\varkappa, T\varkappa, \ell) = 1,$$

so the inequality holds trivially. Hence

$$\digamma_M(T\varkappa, Ty, \ell) \geq 1 - \lambda\,[1 - \digamma_M(\varkappa, y, \ell)]$$

for every $\varkappa, y \in X$*.*

Finally, this fuzzy metric cannot be generated by the standard metric-induced fuzzy metric $\digamma_{Md}$*,*

$$\digamma_{Md}(\varkappa, y, \ell) = \frac{\ell}{\ell + d(\varkappa, y)}.$$

Indeed, if such a metric d *existed, then*

$$0.9 = \digamma_M(b, c, \ell) = \frac{\ell}{\ell + d(b, c)}$$

for every $\ell > 0$*, which implies*

$$d(b, c) = \frac{\ell}{9}.$$

Since the right-hand side depends on ℓ*, this contradicts the definition of a metric, where the distance between two points must be independent of the parameter* ℓ*. Therefore* $(X, \digamma_M, \odot)$ *is a genuine fuzzy metric space that cannot be induced by any classical metric.*

Remark. *The classical fuzzy contraction*

$$\digamma_M(T\varkappa, Ty, k\ell) \geq \digamma_M(\varkappa, y, \ell), \qquad 0 < k < 1,$$

is closely related to the Banach contraction principle through the standard metric-induced fuzzy metric

$$\digamma_{Md}(\varkappa, y, \ell) = \frac{\ell}{\ell + d(\varkappa, y)}.$$

Indeed, every Banach contraction on a metric space induces a fuzzy contraction of this form. However, Example 1 shows that the fuzzy metric considered here cannot be generated by any standard metric-induced fuzzy metric. Consequently, the present contraction principle is established directly within the framework of fuzzy metric spaces and is not merely a reformulation of the classical Banach contraction via an induced fuzzy metric.

The following Lemma will be crucial for the proof of main results.

Lemma 2.3. [17] *Let* $\odot$ *be a continuous t-norm on* $[0, 1]$*. Then, for every* $a, b \in [0, 1]$*,*

$$a \odot b \;\geq\; \max\{0,\, a + b - 1\}.$$

By the Mostert–Shields Representation Theorem (see Klement, Mesiar and Pap [17]*), every continuous t-norm is isomorphic to an ordinal sum of Archimedean components, each of which is isomorphic either to the product t-norm or to the Lukasiewicz t-norm. Outside these components, the operation* $\odot$ *coincides with the minimum t-norm.*

Proof. Let $a, b \in [0,1]$. We distinguish the possible cases.
Case 1. Suppose a and b belong to the same Lukasiewicz component. Then

$$a \odot b = \max\{0,\, a+b-1\},$$

and therefore

$$a \odot b \geq \max\{0,\, a+b-1\}$$

holds with equality.
Case 2. Suppose a and b belong to the same product component. Then

$$a \odot b = ab.$$

Since

$$(1-a)(1-b) \geq 0,$$

we obtain

$$ab - a - b + 1 \geq 0,$$

that is,

$$ab \geq a + b - 1.$$

Also,

$$ab \geq 0.$$

Hence

$$ab \geq \max\{0,\, a+b-1\},$$

and therefore

$$a \odot b \geq \max\{0,\, a+b-1\}.$$

Case 3. Suppose a and b do not belong to the same Archimedean component. By the construction of the ordinal sum of continuous t-norms (see [17, Chapter 3]), the operation $\odot$ coincides with the minimum t-norm whenever the two arguments belong to different components. Hence

$$a \odot b = \min\{a, b\}.$$

Assume without loss of generality that $a \leq b$. Then

$$a \odot b = a.$$

Since $b \leq 1$,

$$a + b - 1 \leq a,$$

and clearly

$$a \geq 0.$$

Therefore

$$a = \min\{a, b\} \geq \max\{0,\, a+b-1\},$$

and consequently

$$a \odot b \geq \max\{0,\, a+b-1\}.$$

Since every pair (a, b) falls under exactly one of the above cases, the desired inequality

$$a \odot b \geq \max\{0,\, a+b-1\}$$

holds for all $a, b \in [0,1]$. □

The following Lemma is for finitely many application of result used in above lemma.

Lemma 2.4 (k-fold extension of the t-norm bound)**.** *Let $*$ be any continuous ℓ-norm. Then for all $a_1, \ldots, a_k \in [0,1]$,*

$$a_1 * a_2 * \cdots * a_k \;\geq\; \max\left\{0,\; \sum_{i=1}^{k} a_i - (k-1)\right\} \;\geq\; \sum_{i=1}^{k} a_i - (k-1).$$

Proof. We induct on k. For $k = 1$ the statement is trivial (equality). For $k = 2$ it is the pairwise bound $a * b \geq \max\{0, a + b - 1\}$, valid for every continuous ℓ-norm (by above Lemma).
Assume the claim holds for k factors, and set $S = \sum_{i=1}^{k} a_i - (k-1)$, so that $\varkappa := a_1 * \cdots * a_k \geq \max\{0, S\}$ by hypothesis. Since every ℓ-norm is nondecreasing in each argument,

$$a_1 * \cdots * a_k * a_{k+1} = \varkappa * a_{k+1} \;\geq\; \max\{0, S\} * a_{k+1} \;\geq\; \max\left\{0,\; \max\{0, S\} + a_{k+1} - 1\right\}.$$

Case $S \geq 0$. The right-hand side equals $\max\{0,\; S + a_{k+1} - 1\} = \max\left\{0,\; \sum_{i=1}^{k+1} a_i - k\right\}$, which is exactly the $(k+1)$-term formula.
Case $S < 0$. Then $\sum_{i=1}^{k} a_i < k - 1$, so $\sum_{i=1}^{k+1} a_i < (k-1) + a_{k+1} \leq k$, hence $\max\{0, \sum_{i=1}^{k+1} a_i - k\} = 0$. The left-hand side $a_1 * \cdots * a_{k+1}$ is nonnegative (every ℓ-norm value lies in $[0,1]$), so the bound ≥ 0 holds trivially.
In either case $a_1 * \cdots * a_{k+1} \geq \max\{0, \sum_{i=1}^{k+1} a_i - k\}$, completing the induction. □

Next we prove a that every **new fuzzy Banach-type contraction** with a complete fuzzy metric space has a unique fixed point.

Theorem 2.5 (Fixed Point for New Fuzzy Banach-type Contractions)**.** *Let $(S, \mathcal{F}_M, \odot)$ be a complete fuzzy metric space in the sense of George and Veeramani, where $\odot$ is a continuous ℓ-norm. Suppose that $T : S \to S$ satisfies*

$$\mathcal{F}_M(T\varkappa, Ty, \ell) \geq 1 - \lambda\left[1 - \mathcal{F}_M(\varkappa, y, \ell)\right],$$

for all $\varkappa, y \in S$, $\ell > 0$, and for some $\lambda \in (0,1)$. Then T possesses a unique fixed point $u \in S$. Moreover, for every $\varkappa_0 \in S$, the Picard sequence

$$\varkappa_{n+1} = T\varkappa_n, \qquad n \geq 0,$$

converges to u.

Proof. Let $\varkappa_0 \in S$ be arbitrary and define the Picard sequence by

$$\varkappa_{n+1} = T\varkappa_n, \qquad n \geq 0.$$

Applying the contractive condition with $\varkappa = \varkappa_n$ and $y = \varkappa_{n+1}$ gives

$$\mathcal{F}_M(\varkappa_{n+1}, \varkappa_{n+2}, \ell) \geq 1 - \lambda\left[1 - \mathcal{F}_M(\varkappa_n, \varkappa_{n+1}, \ell)\right].$$

Hence,

$$1 - \mathcal{F}_M(\varkappa_{n+1}, \varkappa_{n+2}, \ell) \leq \lambda\left[1 - \mathcal{F}_M(\varkappa_n, \varkappa_{n+1}, \ell)\right].$$

By induction,

$$1 - \mathcal{F}_M(\varkappa_n, \varkappa_{n+1}, \ell) \leq \lambda^n\left[1 - \mathcal{F}_M(\varkappa_0, \varkappa_1, \ell)\right].$$

Since

$$0 \leq 1 - \mathcal{F}_M(\varkappa_0, \varkappa_1, \ell) \leq 1,$$

it follows that

$$\mathcal{F}_M(\varkappa_n, \varkappa_{n+1}, \ell) \geq 1 - \lambda^n,$$

for every $n \geq 0$ and $\ell > 0$.

Consequently,

$$\lim_{n\to\infty} F_M(\varkappa_n, \varkappa_{n+1}, \ell) = 1,$$

for every $\ell > 0$.

We now prove that $\{\varkappa_n\}$ is a Cauchy sequence.
For every $\varepsilon > 0$ there exists N such that $F_M(\varkappa_n, \varkappa_m, \ell) \geq 1 - \varepsilon$ for all $m > n \geq N$ and all $\ell > 0$.
Set $p = m - n$ and split $[\varkappa_n, \varkappa_m]$ into p steps using the fuzzy triangle inequality with equal time increments ℓ/p:

$$F_M(\varkappa_n, \varkappa_m, \ell) \;\geq\; F_M(\varkappa_n, \varkappa_{n+1}, \ell/p) * F_M(\varkappa_{n+1}, \varkappa_{n+2}, \ell/p) * \cdots * F_M(\varkappa_{m-1}, \varkappa_m, \ell/p).$$

Writing $a_i := F_M(\varkappa_i, \varkappa_{i+1}, \ell/p) = 1 - d_i(\ell/p) \geq 1 - \lambda^i$ for $i = n, \ldots, m-1$, monotonicity of $*$ in each argument gives

$$F_M(\varkappa_n, \varkappa_m, \ell) \;\geq\; a_n * a_{n+1} * \cdots * a_{m-1}.$$

Applying Lemma 2, with $k = m - n$ factors,

$$a_n * \cdots * a_{m-1} \;\geq\; \sum_{i=n}^{m-1} a_i - (m - n - 1) \;\geq\; \sum_{i=n}^{m-1} (1 - \lambda^i) - (m - n - 1).$$

Expanding, $\sum_{i=n}^{m-1}(1 - \lambda^i) = (m - n) - \sum_{i=n}^{m-1} \lambda^i$, so

$$\sum_{i=n}^{m-1} (1 - \lambda^i) - (m - n - 1) = (m - n) - \sum_{i=n}^{m-1} \lambda^i - (m - n - 1) = 1 - \sum_{i=n}^{m-1} \lambda^i.$$

This gives $F_M(\varkappa_n, \varkappa_m, \ell) \geq 1 - \sum_{i=n}^{m-1} \lambda^i$. Now since $0 < \lambda < 1$,

$$\sum_{i=n}^{m-1} \lambda^i \;\leq\; \sum_{i=n}^{\infty} \lambda^i = \frac{\lambda^n}{1 - \lambda},$$

which tends to 0 as $n \to \infty$. Given $\varepsilon > 0$, choose N with $\lambda^N/(1 - \lambda) < \varepsilon$. Then for all $m > n \geq N$,

$$F_M(\varkappa_n, \varkappa_m, \ell) \;\geq\; 1 - \sum_{i=n}^{m-1} \lambda^i \;\geq\; 1 - \sum_{i=N}^{\infty} \lambda^i \;>\; 1 - \varepsilon,$$

for every $\ell > 0$, as claimed.

Hence,

$$\lim_{m,n\to\infty} F_M(\varkappa_n, \varkappa_m, \ell) = 1,$$

which shows that $\{\varkappa_n\}$ is a Cauchy sequence.

Since $(S, F_M, \odot)$ is complete, there exists $u \in S$ such that

$$\lim_{n\to\infty} F_M(\varkappa_n, u, \ell) = 1,$$

for every $\ell > 0$.

Applying the contractive condition with $\varkappa = \varkappa_n$ and $y = u$, we obtain

$$F_M(\varkappa_{n+1}, Tu, \ell) \geq 1 - \lambda \left[1 - F_M(\varkappa_n, u, \ell)\right].$$

Passing to the limit yields

$$\lim_{n\to\infty} F_M(\varkappa_{n+1}, Tu, \ell) = 1.$$

For any $s > 0$, the fuzzy triangle inequality gives

$$\digamma_M(u, Tu, s) \geq \digamma_M\left(u, \varkappa_{n+1}, \frac{s}{2}\right) \odot \digamma_M\left(\varkappa_{n+1}, Tu, \frac{s}{2}\right).$$

Letting $n \to \infty$, both factors tend to 1. Therefore,

$$\digamma_M(u, Tu, s) = 1,$$

for every $s > 0$. Hence,

$$Tu = u.$$

Finally, let u and υ be two fixed points of T. Then

$$\digamma_M(u, \upsilon, \ell) = \digamma_M(Tu, T\upsilon, \ell) \geq 1 - \lambda\left[1 - \digamma_M(u, \upsilon, \ell)\right].$$

Rearranging,

$$(1 - \lambda)\left[1 - \digamma_M(u, \upsilon, \ell)\right] \leq 0.$$

Since

$$1 - \lambda > 0$$

and

$$1 - \digamma_M(u, \upsilon, \ell) \geq 0,$$

we conclude that

$$\digamma_M(u, \upsilon, \ell) = 1,$$

for every $\ell > 0$. Hence,

$$u = \upsilon.$$

Therefore, the fixed point is unique. □

3. Fuzzy Hardy-Rogers Interpolative Contractions Under Fuzzy Partial Orders

Before going into the details of what we are presenting in this section, we emphasize the importance of ordered structures and topological properties. The partial order $\preccurlyeq$ on a set X endowed with a metric d forms a topological space that has very important properties in the mappings and contractive conditions. For example a self mapping $T : X \to X$ satisfying

$$d(T\varkappa, Ty) \leq kd(\varkappa, y) \text{ for all } \varkappa \preccurlyeq y$$

is not continuous on X, while without partial order, the contractive condition forces the mapping to be continuous. Same is the case for non expansive operators. It means the applications in differential equations or matrix equations with partially ordered domains, the discontinuous operator can have a solution, a wonderful application only due to ordered structure. Now we provide details about this section. We generalized the ordered structures from partial ordered to fuzzy partial ordered, and the space from a metric space to fuzzy metric space. We also have shown above the under consideration fuzzy contractions are suitable for fuzzy spaces only.

Definition 3.1 (Fuzzy Partial Order)**.** [36]*Let S be a nonempty set and let $\odot$ be a continuous t-norm. A fuzzy relation $\pounds : S \times S \to [0, 1]$ is called a* weak fuzzy partial order *on S if, for every $\varkappa, y, z \in S$,*

(i) $\pounds(\varkappa, \varkappa) = 1$;
(ii) $\pounds(\varkappa, z) \geq \pounds(\varkappa, y) \odot \pounds(y, z)$.

A mapping $T : S \to S$ is said to be *$\pounds$-non decreasing* if

$$\pounds(\varkappa, y) > 0 \implies \pounds(T\varkappa, Ty) > 0.$$

Definition 3.2 (Ordered Fuzzy Hardy–Rogers Interpolative Contraction)**.** *Let $(S, \digamma_M, \odot)$ be a fuzzy metric space equipped with a weak fuzzy partial order $\pounds$. A mapping $T : S \to S$ is called an* ordered fuzzy Hardy–Rogers interpolative contraction *if*

(i) *T is $\pounds$-nondecreasing;*
there exist constants

$$\lambda \in (0,1), \qquad \alpha, \beta, \gamma, \delta > 0, \qquad \alpha + \beta + \gamma + \delta = 1,$$

such that, for every $\varkappa, y \in S$ with $\pounds(\varkappa, y) > 0$ and every $\ell > 0$,

$$\begin{aligned} \digamma_M(T\varkappa, Ty, \ell) \quad \geq \quad & 1 - \lambda \left[1 - \digamma_M(\varkappa, y, \ell)\right]^{\alpha} \left[1 - \digamma_M(\varkappa, T\varkappa, \ell)\right]^{\beta} \left[1 - \digamma_M(y, Ty, \ell)\right]^{\gamma} \\ & \left[\tfrac{1}{2}\left[1 - \digamma_M(\varkappa, Ty, \ell)\right] + \tfrac{1}{2}\left[1 - \digamma_M(y, T\varkappa, \ell)\right]\right]^{\delta}. \end{aligned} \tag{3.1}$$

Remark. *The powers $\alpha, \beta, \ldots$ are the ususal powers of reals and multiplication like $\left[1 - \digamma_M(\varkappa, y, \ell)\right]^{\alpha} \left[1 - \digamma_M(\varkappa, T\varkappa, \ell)\right]^{\beta}$ is also the usual mutiplication of real numbers.*

Theorem 3.3. *Let $(S, \digamma_M, \odot)$ be a complete fuzzy metric space equipped with a weak fuzzy partial order $\pounds$. Suppose $T : S \to S$ is an ordered fuzzy Hardy–Rogers interpolative contraction, and suppose:*

(i) *there exists $\varkappa_0 \in S$ such that $\pounds(\varkappa_0, T\varkappa_0) > 0$;*
(ii) *(*Order continuity*) whenever $\varkappa_n \to u$ in $(S, \digamma_M, \odot)$ and $\pounds(\varkappa_n, \varkappa_{n+1}) > 0$ for all n, then $\pounds(\varkappa_n, u) > 0$ for all sufficiently large n;*
(iii) *every two fixed points of T are $\pounds$-comparable.*

Then T possesses a unique fixed point $u \in S$. Moreover, for every admissible initial point $\varkappa_0$, the Picard iteration $\varkappa_{n+1} = T\varkappa_n$ converges to u.

Proof. Let $\varkappa_{n+1} = T\varkappa_n$, $n \geq 0$. Since $\pounds(\varkappa_0, \varkappa_1) = \pounds(\varkappa_0, T\varkappa_0) > 0$ and T is $\pounds$-non decreasing,

$$\pounds(\varkappa_n, \varkappa_{n+1}) > 0 \qquad \text{for all } n \geq 0.$$

Hence every pair $(\varkappa_n, \varkappa_{n+1})$ (and $(\varkappa_n, \varkappa_{n-1})$) is comparable, and (3.0) applies throughout. For $\ell > 0$ define $d_n(\ell) := 1 - \digamma_M(\varkappa_n, \varkappa_{n+1}, \ell) \in [0, 1]$. We prove the theorem in 5 steps.

Step 1: Monotonicity of $\{d_n(\ell)\}$. Fix $\ell > 0$, write $d_n := d_n(\ell)$. Applying (3.0) to $(\varkappa_n, \varkappa_{n-1})$,

$$\begin{aligned} \digamma_M(\varkappa_{n+1}, \varkappa_n, \ell) \quad & \geq \quad 1 - \lambda\, d_{n-1}^{\alpha}\, d_n^{\beta}\, d_{n-1}^{\gamma} \left[\tfrac{1}{2} e_{n-1}\right]^{\delta}, \\ \text{set } e_{n-1} \quad & : \quad = 1 - \digamma_M(\varkappa_{n-1}, \varkappa_{n+1}, \ell), \end{aligned}$$

using $\digamma_M(\varkappa_n, \varkappa_n, \ell) = 1$ to remove the first bracket term. So we have

$$d_n \;\leq\; \lambda\, d_{n-1}^{\alpha+\gamma}\, d_n^{\beta} \left[\tfrac{1}{2} e_{n-1}\right]^{\delta}. \tag{3.2}$$

By the fuzzy triangle inequality and $1 - (a \odot b) \leq (1 - a) + (1 - b)$ (valid for every continuous t-norm),

$$e_{n-1} \;\leq\; d_{n-1} + d_n \qquad \Longrightarrow \qquad \tfrac{1}{2} e_{n-1} \;\leq\; \max\{d_{n-1}, d_n\}. \tag{3.3}$$

Suppose $d_n > d_{n-1}$ for some $n \geq 1$. Then (3.2) gives $\frac{1}{2}e_{n-1} \leq d_n$, and (3.1) becomes

$$d_n \;\leq\; \lambda\, d_{n-1}^{\alpha+\gamma}\, d_n^{\beta+\delta} \;<\; \lambda\, d_n^{\alpha+\gamma}\, d_n^{\beta+\delta} = \lambda\, d_n^{\alpha+\beta+\gamma+\delta} = \lambda d_n,$$

using $d_{n-1} < d_n$. This gives $d_n(1-\lambda) < 0$, impossible since $d_n \geq 0$, $\lambda < 1$. Hence

$$d_n \;\leq\; d_{n-1} \qquad \text{for all } n \geq 1. \tag{3.4}$$

Step 2: Geometric decay. By (3.3), $\max\{d_{n-1}, d_n\} = d_{n-1}$, so (3.2) gives $\frac{1}{2}e_{n-1} \leq d_{n-1}$, and since $d_n \leq d_{n-1}$ also $d_n^{\beta} \leq d_{n-1}^{\beta}$. Substituting into (3.1),

$$d_n \;\leq\; \lambda\, d_{n-1}^{\alpha+\gamma}\, d_{n-1}^{\beta}\, d_{n-1}^{\delta} = \lambda\, d_{n-1}^{\alpha+\beta+\gamma+\delta} = \lambda\, d_{n-1}.$$

By induction,

$$d_n(\ell) \;\leq\; \lambda^n d_0(\ell) \;\leq\; \lambda^n \qquad \text{for all } n \geq 0,\ \ell > 0. \tag{3.5}$$

Step 3: $\{\varkappa_n\}$ is Cauchy. Fix $\varepsilon > 0$, $\ell > 0$; choose N with $\lambda^N/(1-\lambda) < \varepsilon$. For $m > n \geq N$, $m = n + p$, the fuzzy triangle inequality (applied p times with equal time splitting) and $1 - (u_1 \odot \cdots \odot u_p) \leq \sum_{i=1}^{p}(1-u_i)$ give

$$1 - \digamma_M(\varkappa_n, \varkappa_m, \ell) \;\leq\; \sum_{j=n}^{m-1} d_j\Big(\tfrac{\ell}{p}\Big) \;\leq\; \sum_{j=n}^{\infty} \lambda^j = \frac{\lambda^n}{1-\lambda} < \varepsilon.$$

So $\digamma_M(\varkappa_n, \varkappa_m, \ell) > 1 - \varepsilon$; $\{\varkappa_n\}$ is Cauchy. By completeness there exists $u \in S$ with

$$\digamma_M(\varkappa_n, u, \ell) \;\to\; 1 \qquad \text{for every } \ell > 0. \tag{3.6}$$

Step 4: u is a fixed point. By order continuity (ii) and Step 1, $\pounds(\varkappa_n, u) > 0$ for all sufficiently large n, so (3.0) applies to $(\varkappa_n, u)$:

$$\digamma_M(\varkappa_{n+1}, Tu, \ell) \;\geq\; 1 - A_n,$$

where

$$\begin{aligned} A_n \;\;:\;\; &= [1 - \digamma_M(\varkappa_n, u, \ell)]^{\alpha}\, [1 - \digamma_M(\varkappa_n, \varkappa_{n+1}, \ell)]^{\beta}\, [1 - \digamma_M(u, Tu, \ell)]^{\gamma} \\ &\quad \cdot \left[\tfrac{1}{2}\,[1 - \digamma_M(\varkappa_n, Tu, \ell)] + \tfrac{1}{2}\,[1 - \digamma_M(u, \varkappa_{n+1}, \ell)]\right]^{\delta}. \end{aligned}$$

Since $0 \leq [1 - \digamma_M(u, Tu, \ell)]^{\gamma} \leq 1$ and the bracketed factor also lies in $[0,1]$ (each term inside is in $[0,1]$, so their average is too), bounding these two factors by 1 gives

$$0 \;\leq\; A_n \;\leq\; [1 - \digamma_M(\varkappa_n, u, \ell)]^{\alpha}\; [1 - \digamma_M(\varkappa_n, \varkappa_{n+1}, \ell)]^{\beta}. \tag{3.7}$$

By (3.5), $\digamma_M(\varkappa_n, u, \ell) \to 1$, so $1 - \digamma_M(\varkappa_n, u, \ell) \to 0$; and by (3.4), $1 - \digamma_M(\varkappa_n, \varkappa_{n+1}, \ell) = d_n(\ell) \to 0$. Since $\alpha, \beta > 0$ are fixed constants, a fixed positive power of a null sequence is null, so both factors on the right of (3.6) tend to 0, and hence

$$A_n \;\to\; 0.$$

Consequently $0 \leq 1 - \digamma_M(\varkappa_{n+1}, Tu, \ell) \leq A_n \to 0$, so

$$\digamma_M(\varkappa_{n+1}, Tu, \ell) \;\to\; 1 \qquad \text{for every } \ell > 0. \tag{3.8}$$

Now fix $s > 0$. By the fuzzy triangle inequality,

$$\digamma_M(u, Tu, s) \;\geq\; \digamma_M\big(u, \varkappa_{n+1}, \tfrac{s}{2}\big) \odot \digamma_M\big(\varkappa_{n+1}, Tu, \tfrac{s}{2}\big).$$

Letting $n \to \infty$, using (3.5), (3.7), and continuity of $\odot$ (in particular at $(1,1)$), the right side tends to $1 \odot 1 = 1$. Since always $\digamma_M(u, Tu, s) \leq 1$, we conclude $\digamma_M(u, Tu, s) = 1$ for every $s > 0$, i.e.

$$Tu = u.$$

Step 5: Uniqueness. Let u, v be fixed points of T. By hypothesis (iii), $\pounds(u, v) > 0$, so (3.0) applies to (u, v):

$$\begin{aligned}\digamma_M(u, v, \ell) &= \digamma_M(Tu, Tv, \ell) \geq 1 - \lambda\left[1 - \digamma_M(u, v, \ell)\right]^{\alpha}\left[1 - \digamma_M(u, u, \ell)\right]^{\beta} \\ &\quad \left[1 - \digamma_M(v, v, \ell)\right]^{\gamma}\left[\tfrac{1}{2}\left[1 - \digamma_M(u, Tv, \ell)\right] + \tfrac{1}{2}\left[1 - \digamma_M(v, Tu, \ell)\right]\right]^{\delta}.\end{aligned}$$

Since $\digamma_M(u, u, \ell) = \digamma_M(v, v, \ell) = 1$, both $\left[1 - \digamma_M(u, u, \ell)\right]^{\beta} = 0$ and $\left[1 - \digamma_M(v, v, \ell)\right]^{\gamma} = 0$ (valid since $\beta, \gamma > 0$), so the entire product vanishes and $\digamma_M(u, v, \ell) \geq 1$. As $\digamma_M(u, v, \ell) \leq 1$ always, so $\digamma_M(u, v, \ell) = 1$ for every $\ell > 0$, which implies, $u = v$. The fixed point is unique. □

4. Applications to Fuzzy Caputo Fractional Differential Equations

In this section first we recall some well know definitions and results from the literature [29, 6]. Let $\mathbb{R}_{\pounds}$ denote the space of fuzzy numbers $u : \mathbb{R} \to [0, 1]$ whose r-level sets

$$[u]^r = \{\varkappa \in \mathbb{R} : u(\varkappa) \geq r\} = [u_L(r), u_U(r)], \qquad r \in [0, 1],$$

are nonempty, compact, and convex for every $r \in [0, 1]$. The Hausdorff distance on $\mathbb{R}_{\pounds}$ is

$$D(u, v) = \sup_{r \in [0,1]} \max\left[|u_L(r) - v_L(r)|,\, |u_U(r) - v_U(r)|\right],$$

and $(\mathbb{R}_{\pounds}, D)$ is a complete metric space. For $u, v \in C([0, 1]; \mathbb{R}_{\pounds})$ we use the supremum metric

$$D_{\infty}(u, v) = \sup_{\ell \in [0,1]} D\left(u(\ell), v(\ell)\right), \tag{4.0}$$

under which $(C([0, 1]; \mathbb{R}_{\pounds}), D_{\infty})$ is likewise complete. Scalar multiplication and addition on $\mathbb{R}_{\pounds}$ are performed level-wise:

$$\begin{aligned}[\lambda v]^r &= [\lambda v_L(r), \lambda v_U(r)] \text{ for } \lambda \geq 0 \\ [\lambda v]^r &= [\lambda v_U(r), \lambda v_L(r)] \text{ for } \lambda < 0\end{aligned}$$

and

$$[u + v]^r = [u_L(r) + v_L(r), u_U(r) + v_U(r)].$$

For a continuous map $u : [0, 1] \to \mathbb{R}_{\pounds}$ with continuous level functions $u_L(\cdot, r)$,and $u_U(\cdot, r)$, a given continuous nonnegative kernel $k(\ell, s)$, the Aumann-type fractional integral is the fuzzy number defined level-wise by

$$\left[\int_0^{\ell} k(\ell, s)\, u(s)\, ds\right]^r = \left[\int_0^{\ell} k(\ell, s) u_L(s, r)\, ds,\ \int_0^{\ell} k(\ell, s) u_U(s, r)\, ds\right], \qquad r \in [0, 1]. \tag{4.1}$$

For the Caputo case, $k(\ell, s) = (\ell - s)^{\alpha - 1}/\Gamma(\alpha)$, $\alpha \in (0, 1)$, and (4.0) defines the fuzzy Riemann–Liouville fractional integral $I^{\alpha} u$.

4.1. **Problem Statement.** Consider the fuzzy Caputo fractional initial value problem

$$ {}^{C}D^{\alpha}u(\ell) = -\lambda\, u(\ell), \qquad \ell \in [0,1], \qquad u(0) = \widetilde{A}, \qquad \alpha \in (0,1),\ \lambda > 0, \tag{4.2}$$

where $\widetilde{A} = (1,2,3)$ is the triangular fuzzy number with $[\widetilde{A}]^r = [1+r,\ 3-r]$, $r \in [0,1]$.

We will prove the existence of the FFBVP (4.2) using our main result.

4.2. **Fixed-Point Formulation.** Using (4.1) with $k(\ell,s) = (\ell-s)^{\alpha-1}/\Gamma(\alpha)$, problem (4.2) is equivalent to

$$u(\ell) = \widetilde{A} - \frac{\lambda}{\Gamma(\alpha)} \int_0^{\ell} (\ell - s)^{\alpha-1} u(s)\, ds,$$

so that u solves (4.2) if and only if u is a fixed point of the operator $T : C([0,1];\mathbb{R}_{\mathcal{L}}) \to C([0,1];\mathbb{R}_{\mathcal{L}})$, defined as;

$$(Tu)(\ell) = \widetilde{A} - \frac{\lambda}{\Gamma(\alpha)} \int_0^{\ell} (\ell - s)^{\alpha-1} u(s)\, ds, \tag{4.3}$$

Level-wise,

$$[(Tu)(\ell)]^r = \left[(1+r) - \tfrac{\lambda}{\Gamma(\alpha)} \int_0^{\ell} (\ell-s)^{\alpha-1} u_U(s,r)\, ds,\ (3-r) - \tfrac{\lambda}{\Gamma(\alpha)} \int_0^{\ell} (\ell-s)^{\alpha-1} u_L(s,r)\, ds \right].$$

Let $\mu > \lambda$ and set $w(\ell) = E_{\alpha}(\mu \ell^{\alpha})$, where $E_{\alpha}(z) = \sum_{k=0}^{\infty} z^k/\Gamma(\alpha k + 1)$ is the Mittag Leffler function.

Lemma 4.1. *The inequality,* $\frac{1}{\Gamma(\alpha)} \int_0^{\ell} (\ell - s)^{\alpha-1} w(s)\, ds \ \leq\ \frac{1}{\mu}\, w(\ell)$ *holds for all* $\ell \geq 0$.

Proof. Since $I^{\alpha} \ell^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} \ell^{\beta+\alpha}$ for $\beta > -1$, term-by-term integration of $w(\ell) = \sum\limits_{k \geq 0} \frac{\mu^k \ell^{\alpha k}}{\Gamma(\alpha k+1)}$ gives

$$I^{\alpha} w(\ell) = \sum_{k=0}^{\infty} \mu^k \frac{\ell^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)} = \frac{1}{\mu} \left[w(\ell) - 1 \right] \leq \frac{1}{\mu} w(\ell).$$

□

Define

$$D_{\infty}^{w}(u,v) = \sup_{\ell \in [0,1]} \frac{D(u(\ell), v(\ell))}{w(\ell)}. \tag{4.4}$$

Since $1 \leq w(\ell) \leq E_{\alpha}(\mu)$ on $[0,1]$, D_{∞}^{w} is equivalent to D_{∞}, and $(C([0,1];\mathbb{R}_{\mathcal{L}}), D_{\infty}^{w})$ is complete.

Proposition 4.2. *Let* $S_R = \{u \in C([0,1];\mathbb{R}_{\mathcal{L}}) : D_{\infty}^{w}(u, \widetilde{A}) \leq R\}$, *let* $K > 2R \geq \operatorname{diam}_w(S_R)$, *and define*

$$\mathcal{F}_M(u,v,\ell) = 1 - \frac{D_{\infty}^{w}(u,v)}{K}, \qquad \ell > 0, \qquad u, v \in S_R, \tag{4.5}$$

together with the Lukasiewicz ℓ*-norm* $a * b = \max\{0, a+b-1\}$. *Then* $(S_R, \mathcal{F}_M, *)$ *is a complete fuzzy metric space (in the sense of George and Veeramani).*

Proof. The space $(C([0,1];\mathbb{R}_{\mathcal{L}}), D_\infty^w)$ is complete, since it is metrically equivalent to $(C([0,1];\mathbb{R}_{\mathcal{L}}), D_\infty)$, which is complete because $(\mathbb{R}_{\mathcal{L}}, D)$ is complete and a uniform limit of continuous fuzzy-number-valued functions is continuous. The map $u \mapsto D_\infty^w(u, \widetilde{A})$ is continuous, so S_R, being the preimage of the closed set $[0, R]$, is closed in this complete space. Hence

$$(S_R, D_\infty^w) \text{ is a complete metric space.} \tag{4.6}$$

Also, by definition of S_R and the triangle inequality, for $u, \upsilon \in S_R$,

$$D_\infty^w(u, \upsilon) \le \operatorname{diam}_w(S_R) \le 2R < K. \tag{4.7}$$

Claim 1: $\mathcal{F}_M$ is a fuzzy metric on S_R.
(GV1) Positivity. By (4.7), $D_\infty^w(u,\upsilon)/K < 1$, so $\mathcal{F}_M(u,\upsilon,\ell) = 1 - D_\infty^w(u,\upsilon)/K > 0$ for all $\ell > 0$.
(GV2) $\mathcal{F}_M(u,\upsilon,\ell) = 1 \iff u = \upsilon$. $\mathcal{F}_M(u,\upsilon,\ell) = 1 \iff D_\infty^w(u,\upsilon) = 0$. Since D_∞^w is a metric (equivalent to D_∞^w, which separates points of $C([0,1];\mathbb{R}_{\mathcal{L}})$), therefore $D_\infty^w(u,\upsilon) = 0 \iff u = \upsilon$.
(GV3) Symmetry. $D_\infty^w(u,\upsilon) = D_\infty^w(\upsilon,u)$ since the Hausdorff distance D is symmetric, hence $\mathcal{F}_M(u,\upsilon,\ell) = \mathcal{F}_M(\upsilon,u,\ell)$.
(GV4) ℓ-norm triangle inequality. For $u, \upsilon, w \in S_R$ and $\ell, s > 0$,

$$\begin{aligned}\mathcal{F}_M(u,\upsilon,\ell) * \mathcal{F}_M(\upsilon,w,s) &= \max\left\{0,\ 1 - \frac{D_\infty^w(u,\upsilon)}{K} + 1 - \frac{D_\infty^w(\upsilon,w)}{K} - 1\right\} \\ &= \max\left\{0,\ 1 - \frac{D_\infty^w(u,\upsilon) + D_\infty^w(\upsilon,w)}{K}\right\}.\end{aligned}$$

By the triangle inequality for D_∞^w, $D_\infty^w(u,w) \le D_\infty^w(u,\upsilon) + D_\infty^w(\upsilon,w)$, so

$$1 - \frac{D_\infty^w(u,\upsilon) + D_\infty^w(\upsilon,w)}{K} \ \le\ 1 - \frac{D_\infty^w(u,w)}{K} = \mathcal{F}_M(u,w,\ell+s).$$

Also $\mathcal{F}_M(u,w,\ell+s) > 0$ by (GV1). Hence both terms in the maximum on the left-hand side are $\le \mathcal{F}_M(u,w,\ell+s)$, so

$$\mathcal{F}_M(u,\upsilon,\ell) * \mathcal{F}_M(\upsilon,w,s) \ \le\ \mathcal{F}_M(u,w,\ell+s).$$

(GV5) Continuity in ℓ. $\mathcal{F}_M(u,\upsilon,\cdot)$ does not depend on ℓ; a constant function is (trivially) continuous on $(0,\infty)$.Hence $(S_R, \mathcal{F}_M, *)$ is a fuzzy metric space.

Claim 2: Completeness.
For $\varepsilon \in (0,1)$, set $\delta = \varepsilon K > 0$. Then for $u, \upsilon \in S_R$,

$$\mathcal{F}_M(u,\upsilon,\ell) > 1-\varepsilon \iff 1 - \frac{D_\infty^w(u,\upsilon)}{K} > 1-\varepsilon \iff D_\infty^w(u,\upsilon) < \varepsilon K = \delta.$$

So a sequence $\{u_n\} \subset S_R$ satisfies the George–Veeramani Cauchy condition for any $\varepsilon \in (0,1)$, and $\ell > 0$,

$$\text{there is } N, \text{ such that } \mathcal{F}_M(u_n, u_m, \ell) > 1-\varepsilon \quad (\text{ for all } n, m \ge N)$$

if and only if it is Cauchy for D_∞^w: for all $\delta > 0$, there is N, such that $D_\infty^w(u_n, u_m) < \delta$ (for all $n, m \ge N$). The identical computation with u_n, u in place of u_n, u_m shows

$$\mathcal{F}_M(u_n, u, \ell) \to 1 \ \ (\text{for all } \ell > 0) \quad iff \quad D_\infty^w(u_n, u) \to 0,$$

i.e. $\mathcal{F}_M$-convergence in S_R coincides with D_∞^w-convergence.
Conclusion. Let $\{u_n\} \subset S_R$ be $\mathcal{F}_M$-Cauchy. By the equivalence above it is D_∞^w-

Cauchy, so by (4.6) it converges to some $u^* \in S_R$ in D^w_∞. By the equivalence again, $u_n \to u^*$ in $\digamma_M$ as well. Hence every $\digamma_M$-Cauchy sequence in S_R converges (in $\digamma_M$) to a point of S_R, i.e. $(S_R, \digamma_M, *)$ is complete. □

4.3. **Existence and Uniqueness.** Now we state and prove our existence theorem.

Theorem 4.3. *Fix $\mu = 2\lambda$. The Problem (4.2) has a unique fuzzy solution $u^* \in C([0,1]; \mathbb{R}_{\mathcal{L}})$, and the Picard sequence $u_{n+1} = Tu_n$ converges to u^* in the fuzzy metric $\digamma_M$ of (4.6), for every $u_0 \in S_R$.*

Proof. For $u, v \in C([0,1]; \mathbb{R}_{\mathcal{L}})$ and $\ell \in [0,1]$,

$$D\left((Tu)(\ell), (Tv)(\ell)\right) \le \frac{\lambda}{\Gamma(\alpha)} \int_0^\ell (\ell - s)^{\alpha-1} D(u(s), v(s))\, ds \le \lambda D^w_\infty(u, v) \cdot \frac{1}{\Gamma(\alpha)} \int_0^\ell (\ell - s)^{\alpha-1} w(s)\, ds.$$

By Lemma 4.1, the last integral is bounded by $w(\ell)/\mu$, hence, dividing by $w(\ell)$ and taking the supremum,

$$D^w_\infty(Tu, Tv) \;\le\; L\, D^w_\infty(u, v), \qquad L = \frac{\lambda}{\mu} = \frac{1}{2} \in (0,1).$$

Let $S_R = \{u \in C([0,1]; \mathbb{R}_{\mathcal{L}}) : D^w_\infty(u, \widetilde{A}) \le R\}$. Set $C_0 := D^w_\infty(T\widetilde{A}, \widetilde{A}) < \infty$,

$$D^w_\infty(Tu, \widetilde{A}) \le D^w_\infty(Tu, T\widetilde{A}) + D^w_\infty(T\widetilde{A}, \widetilde{A}) \le LR + C_0$$

for $u \in S_R$, so $T(S_R) \subset S_R$ whenever $R \ge C_0/(1-L) = 2C_0$. Thus S_R is a nonempty, D^w_∞-closed, T-invariant subset of the complete space $(C([0,1]; \mathbb{R}_{\mathcal{L}}), D^w_\infty)$, with $\operatorname{diam}_w(S_R) \le 2R$.
Fix $K > 2R$ and define the fuzzy metric, with the Lukasiewicz ℓ-norm,

$$\digamma_M(u, v, \ell) = 1 - \frac{D^w_\infty(u, v)}{K}, \qquad \ell > 0, \qquad u, v \in S_R.$$

Then $(S_R, \digamma_M, *)$ is a complete fuzzy metric space by above Proposition, consider

$$1 - \digamma_M(Tu, Tv, \ell) = \frac{D^w_\infty(Tu, Tv)}{K} \le L \frac{D^w_\infty(u, v)}{K} = L\left[1 - \digamma_M(u, v, \ell)\right],$$

i.e.

$$\digamma_M(Tu, Tv, \ell) \;\ge\; 1 - L\left[1 - \digamma_M(u, v, \ell)\right], \qquad L = \tfrac{1}{2},$$

for all $u, v \in S_R$, $\ell > 0$.
$T : S_R \to S_R$ satisfies the hypothesis of Theorem 2.5, on the complete fuzzy metric space $(S_R, \digamma_M, *)$. Hence T has a unique fixed point $u^* \in S_R$, the unique fuzzy solution of (4.2), and $u_n = T^n u_0 \to u^*$ in $\digamma_M$ for every $u_0 \in S_R$. □

The following nontrivial example explains the above existence result.

Example 4.4. *Take $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $\widetilde{A} = (1, 2, 3)$ in (4.2), to get the following special case;*

$${}^C D^{1/2} u(\ell) = -\tfrac{1}{2} u(\ell), \qquad \ell \in [0,1], \qquad u(0) = \widetilde{A}.$$

The exact fuzzy solution is

$$u^*(\ell) = \widetilde{A} \odot E_{1/2}\left(-\tfrac{1}{2}\ell^{1/2}\right),$$

$$[u^*(\ell)]^r = \left[(1+r)E_{1/2}\left(-\tfrac{1}{2}\ell^{1/2}\right),\ (3-r)E_{1/2}\left(-\tfrac{1}{2}\ell^{1/2}\right)\right] \tag{4.8}$$

which is well defined and bounded since $E_{1/2}(-\frac{1}{2}\ell^{1/2}) \in (0,1]$ for $\ell \geq 0$. Where $\odot$ denotes scalar multiplication of a fuzzy number by a crisp (real) number. Now we iterate our solutions terms generated by Picard operator.

With $u_0(\ell) = \widetilde{A}$, (4.3) gives

$$u_n(\ell) = \widetilde{A} \odot \sum_{k=0}^{n} \frac{(-\frac{1}{2})^k \, \ell^{k/2}}{\Gamma(\frac{k}{2}+1)}.$$

Using $\Gamma(3/2) = 0.886227\ldots$ and $\Gamma(2) = 1$,

$$u_1(\ell) = \widetilde{A} \odot \left(1 - 0.564190\,\ell^{1/2}\right), \tag{4.9}$$

$$u_2(\ell) = \widetilde{A} \odot \left(1 - 0.564190\,\ell^{1/2} + 0.25\,\ell\right). \tag{4.10}$$

Level-wise, e.g. for u_2:

$$u_{2,L}(\ell, r) = (1+r)\left(1 - 0.564190\ell^{1/2} + 0.25\ell\right),$$

$$u_{2,U}(\ell, r) = (3-r)\left(1 - 0.564190\ell^{1/2} + 0.25\ell\right).$$

for u_3:

$$u_{3,L}(\ell, r) = (1+r)\left(1 - 0.564190\ell^{1/2} + 0.25\ell - 0.094031\ell^{3/2}\right),$$

$$u_{3,U}(\ell, r) = (3-r)\left(1 - 0.564190\ell^{1/2} + 0.25\ell - 0.094031\ell^{3/2}\right).$$

Recall the Picard iterates of ${}^{C}D^{1/2}u(\ell) = -\frac{1}{2}u(\ell)$, $u(0) = \widetilde{A} = (1,2,3)$, are

$$u_n(\ell) = \widetilde{A} \odot \sum_{k=0}^{n} \frac{(-\frac{1}{2})^k \, \ell^{k/2}}{\Gamma(\frac{k}{2}+1)}.$$

Individual terms, $k = 0,1,2,3,4,5$, using $\Gamma(1) = 1$, $\Gamma(3/2) = 0.8862269255\ldots$, $\Gamma(2) = 1$, $\Gamma(5/2) = 1.3293403882\ldots$, $\Gamma(3) = 2$, $\Gamma(7/2) = 3.323350970$, are

$$k = 0: \quad \frac{1}{\Gamma(1)}\,\ell^0 = 1$$

$$k = 1: \quad \frac{-\frac{1}{2}}{\Gamma(3/2)}\,\ell^{1/2} = -0.5641895835\,\ell^{1/2}$$

$$k = 2: \quad \frac{\frac{1}{4}}{\Gamma(2)}\,\ell^{1} = 0.25\,\ell$$

$$k = 3: \quad \frac{-\frac{1}{8}}{\Gamma(5/2)}\,\ell^{3/2} = -0.0940315973\,\ell^{3/2}$$

$$k = 4: \quad \frac{\frac{1}{16}}{\Gamma(3)}\,\ell^{2} = 0.03125\,\ell^{2}$$

$$k = 5: \quad \frac{-\frac{1}{32}}{\Gamma(7/2)}\,\ell^{5/2} = -0.0094031597\,\ell^{5/2}$$

Cumulative Picard partial sums are

$$u_1(\ell) = \widetilde{A} \odot \left(1 - 0.5641895835\,\ell^{1/2}\right)$$

$$u_2(\ell) = \widetilde{A} \odot \left(1 - 0.5641895835\,\ell^{1/2} + 0.25\,\ell\right)$$

$$u_3(\ell) = \widetilde{A} \odot \left(1 - 0.5641895835\,\ell^{1/2} + 0.25\,\ell - 0.0940315973\,\ell^{3/2}\right)$$

$$u_4(\ell) = \widetilde{A} \odot \left(1 - 0.5641895835\,\ell^{1/2} + 0.25\,\ell - 0.0940315973\,\ell^{3/2} + 0.03125\,\ell^2\right)$$

$$u_5(\ell) = \widetilde{A} \odot \left(1 - 0.5641895835\,\ell^{1/2} + 0.25\,\ell - 0.0940315973\,\ell^{3/2} + 0.03125\,\ell^2 - 0.0094031597\,\ell^{5/2}\right)$$

Exact solution as the limit

$$u^*(\ell) = \widetilde{A} \odot E_{1/2}\left(-\tfrac{1}{2}\ell^{1/2}\right) = \widetilde{A} \odot \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k\,\ell^{k/2}}{\Gamma(\frac{k}{2}+1)} = \widetilde{A} \odot \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-\frac{1}{2})^k\,\ell^{k/2}}{\Gamma(\frac{k}{2}+1)}.$$

Each $u_n(\ell)$, for each $n \in \{1,2,3,4,5\}$, is exactly the n-th partial sum of the series defining $E_{1/2}(-\frac{1}{2}\ell^{1/2})$: the Picard iterates are not merely numerically close to u^*, they are truncations of the very same series.

Numerical Illustration

Table-1, lists the core-level ($r = 1$) values of u_1, u_2 and u^* at several points of $[0, 1]$;

ℓ	u_1	u_2	u_3	u^*	$\lvert u_3 - u^*\rvert$
0.25	1.4358	1.5608	1.5373	1.5407	0.0034
0.5	1.2021	1.4521	1.3856	1.3985	0.0129
0.75	1.0228	1.3978	1.2756	1.3034	0.0278
1.00	0.8716	1.3716	1.1836	1.2314	0.0478

(Table-1)

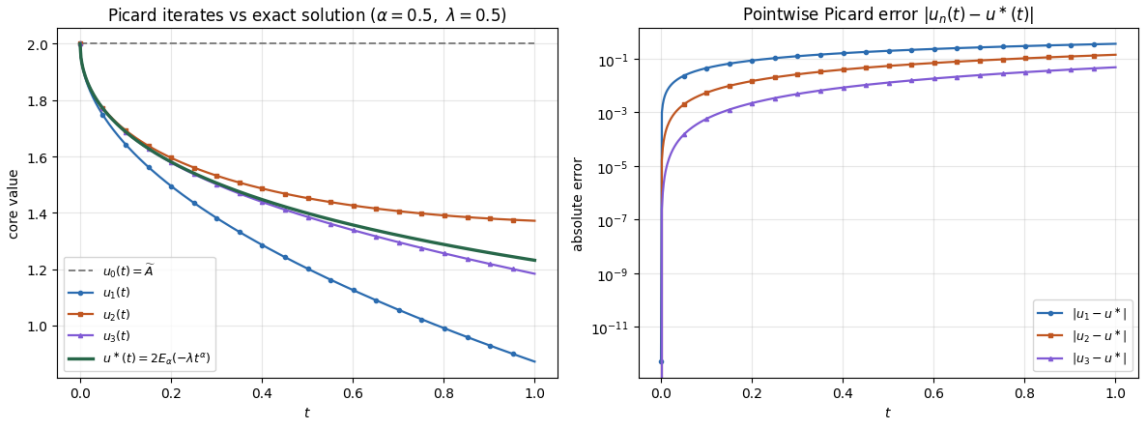

FIGURE 1. The graphs are showing the comparisons of 3 iterations with exact solution and their respective errors. Figure 1, plots the corresponding iterates and exact slution, together with the pointwise Picard error $|u_n(\ell) - u^*(\ell)|$, $n = 1, 2, 3$.

Conclusion

We introduced a fuzzy Banach-type contraction acting directly on the distance deficit $1 - F_M(\varkappa, y, \ell)$, proved a fixed point theorem for it, and extended it to an ordered fuzzy Hardy-Rogers interpolative contraction (Theorem 2) on complete fuzzy metric spaces.

As an application, we built a Mittag-Leffler-weighted fuzzy metric on a closed ball of $C([0,1];\mathbb{R}_{\mathcal{L}})$ and used it to prove existence and uniqueness of the fuzzy solution to a Caputo fractional initial value problem, for every $\alpha \in (0,1)$ and $\lambda > 0$ – a genuinely global result, unlike the naive supremum-metric estimate. A worked example ($\alpha = \lambda = 1/2$) confirmed geometric convergence of the Picard iterates to the exact solution, both analytically and numerically.

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