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BERNSTEIN WIDTHS OF MULTIPLIER OPERATORS

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Abstract. We study Bernstein widths of multiplier operators $\Lambda = \{\lambda_k,\, k \in \mathbb{N}\}$ on general probability spaces. It is shown that in the case of rapidly decaying sequence λ_k (e.g., $\lambda_k = \exp\left(-\alpha k^r\right),\, \alpha > 0,\, r \geq 1$) and slowly decaying sequence λ_k $\left(\text{e.g., } \lambda_k = \log^{-\beta}(k+1),\, \alpha > 0\right)$ our estimates are sharp in the sense of order. In these cases Bernstein widths of multiplier operators demonstrate essentially different asymptotic behaviour when compared with the Sobolev classes.

1. Introduction

Consider probability space $(\Omega, \mathcal{A}, \nu)$. Let $L_p(\Omega, \mathcal{A}, \nu) = L_p$ be the set of functions $f : \Omega \to \mathbb{R}$ of a finite norm

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p \, d\nu\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup} |f|, & p = \infty \end{cases}.$$

with the unit ball $U_p = \left\{ f \,\middle|\, \|f\|_p \leq 1 \right\}$. Let $(f, g) = \int_\Omega f g d\nu$ be the inner product in L_2. Consider an orthonormal system of functions $\phi_k : \Omega \to \mathbb{R}$, $k \in \mathbb{N}$ which are uniformly bounded, i.e. $\|\phi_k\|_\infty \leq M$ for any $k \in \mathbb{N}$. The formal Fourier series for $\varphi \in L_p$, $1 \leq p \leq \infty$ can be written as

$$\varphi \sim \sum_{k=1}^{\infty} c_k(\varphi)\, \phi_k,\; c_k(\varphi) = \int_\Omega \varphi \phi_k d\nu.$$

The Fourier sum of order m is

$$S_m(\varphi) = \sum_{k=1}^{m} c_k(\varphi)\, \phi_k.$$

Consider a sequence of real numbers $\Lambda = \{\lambda_k,\, k \in \mathbb{N}\}$. We say that Λ is a multiplier operator of (p, q)-type if for any $\varphi \in L_p$ there is a function $f = \Lambda\varphi \in L_q$ such that

$$f = \Lambda\varphi \sim \sum_{k=1}^{\infty} \lambda_k c_k(\varphi)\, \phi_k. \tag{1.1}$$

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If $\varphi \in U_p = \left\{ \varphi \,\middle|\, \|\varphi\|_p \leq 1 \right\}$ in (1.1) then we say that $f \in \Lambda U_p$. Let X be a Banach space with the unit ball $U_X = \{\varphi \,|\|\varphi\|_X \leq 1\}$ and A be a convex origin-symmetric set, $A \subset X$. The Bernstein n-width of A in X is defined as

$$b_n(A, X) = \sup_{X_{n+1} \subset X} \sup_{\epsilon > 0} \{\epsilon U_X \cap X_{n+1} \subset A\},$$

where X_{n+1} is any $(n+1)$-dimensional subspace of X. The Kolmogorov n-width of A in X is defined by

$$d_n(A, X) = \inf_{X_n \subset X} \sup_{f \in A} \inf_{g \subset X_n} \|f - g\|_X,$$

where X_n runs over all subspaces of X of dimension $\leq n$. It is known (see e.g. [14]) that

$$b_n(A, X) \leq d_n(A, X). \tag{1.2}$$

Let W_p^r be the Sobolev class of 2π-periodic functions with fractional smoothness $r > 0$, i.e. $W_p^r = \left\{ f \,\middle|\, \left\|f^{(r)}\right\|_p \leq 1 \right\}$. Bernstein n-widths $b_n\left(W_p^r, L_q\right)$ have been extensively studied in [6], [7] [13]. It is known that in the case of "finite" smoothness

$$b_n\left(W_p^r, L_q\right) \asymp \begin{cases} n^{-r}, & r > \frac{1}{p} - \frac{1}{q},\ 1 < p \leq q < \infty, \\ n^{-r}, & r > 0,\ 1 < q \leq p \leq 2,\ 1 < q = p < \infty, \\ n^{-r+\frac{1}{p}-\frac{1}{q}}, & r > \frac{\frac{1}{q}-\frac{1}{p}}{\frac{p}{2}-1},\ 2 \leq q < p < \infty, \\ n^{-r+\frac{1}{p}-\frac{1}{2}}, & r > \frac{1}{p},\ 1 < q \leq 2 < p < \infty. \end{cases} \tag{1.3}$$

and in the case of "small" smoothness,

$$b_n\left(W_p^r, L_q\right) \asymp n^{-\frac{rp}{2}}, \tag{1.4}$$

where

$$0 < r < \frac{\frac{1}{q} - \frac{1}{p}}{\frac{p}{2} - 1},\ 2 \leq q < p < \infty,\ 1 < q \leq 2 < p < \infty.$$

In this article we demonstrate, in general settings, that asymptotic behaviour of Bernstein widths $b_n(\Lambda U_p, L_q)$ in the case of rapidly decaying multiplier sequence λ_k is essentially different from (1.3) and (1.4). In particular, if $\Lambda = \{\lambda_k,\, k \in \mathbb{N}\}$, where $\lambda_k = \exp(-\alpha k^r)$, $\alpha > 0$, $r \geq 1$ then

$$b_n(\Lambda U_p, L_q) \asymp \exp\left(-\alpha (n+1)^r\right) \tag{1.5}$$

for any $1 \leq p, q \leq \infty$. Consider the set ΛU_p of 2π-periodic functions f representable in the form

$$f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K(x - t)\, \varphi(t)\, dt,$$

where

$$K(x) = \sum_{k=1}^{\infty} \exp(-\alpha k^r) \cos kx,\ \alpha > 0,\ r \geq 1.$$

In this case ΛU_p is the set of analytic (if $r = 1$) or entire (if $r > 1$) functions f representable in the convolution form

$$f(x) = (K * \varphi)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K(x - t)\, \varphi(t)\, dt,$$

where $\varphi \in U_p$. It follows from [5] that

$$b_{2n}(K * U_p, L_q) \asymp \exp\left(-\alpha (n+1)^r\right),$$

where $\alpha > 0$, $r \geq 1$, $1 < p, q < \infty$, while the estimate (1.5) remains valid for any $1 \leq p, q \leq \infty$. Observe that (2.6) is order sharp in the case of slowly decaying sequence $\Lambda = \{\lambda_k \, | k \in \mathbb{N} \cup \{0\}\}$, where $\lambda_k = \log^{-\beta}(k+1)$, $\beta > 0$,

$$b_n(\Lambda U_p, L_q) \asymp \log^{-\beta} n, \; n \to \infty, \; 1 \leq q \leq 2 \leq p < \infty.$$

This asymptotic behaviour is different when compared with the Sobolev case (1.3) and (1.4). Remark n-widths of multiplier operarors have been studied in [6]-[11], [1], [2].

2. Results

Let $\mathbf{e}_1, \cdots, \mathbf{e}_m$ be the canonical basis in $\mathbb{R}^m$. For any $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m$ and $\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{R}^m$ we put $\langle \alpha, \beta \rangle = \sum_{k=1}^m \alpha_k \; \beta_k$. Let $\Phi(m) = \operatorname{lin}\{\phi_k, \, 1 \leq k \leq m\}$. Consider the coordinate isomorphism

$$\begin{array}{ccc} \mathrm{J}_m : \mathbb{R}^m & \longrightarrow & \Phi(m) \\ \alpha & \longmapsto & t_m^\alpha = \sum_{k=1}^m \alpha_k \phi_k. \end{array}$$

Let $\|\alpha\|_{(\mathrm{J}_m,2)} = \langle \alpha, \alpha \rangle^{\frac{1}{2}}$ be the Euclidean norm on $\mathbb{R}^m$and V be a convex origin-symmetric body in $\mathbb{R}^m$. Fix a norm on $\mathbb{R}^m$ and denote by E the Banach space $E = (\mathbb{R}^m, \|\cdot\|_V)$ with the unit ball V. Define the norm on $\mathbb{R}^m$ as

$$\|\alpha\|_{(\mathrm{J}_m,p)} = \|(\alpha_1, \cdots, \alpha_m)\|_{(\mathrm{J}_m,p)} := \|t^\alpha\|_p \, , \, 1 \leq p \leq \infty.$$

The Banach-Mazur distance between isomorphic Banach spaces X and Y with the unit balls B_X and B_Y respectively is defined as

$$d(X, Y) = \inf \|T\| \left\|T^{-1}\right\|,$$

where inf is taken over all isomorphisms from X onto Y. We will need the following statements.

Lemma 2.1. [4] *p.61 (2.10) Let X_m be an m-dimensional Banach space with the norm $\|\cdot\|_{X_m}$ and $a\|x\|_{(\mathrm{J}_m,2)} \leq \|x\|_{X_m} \leq b\|x\|_{(\mathrm{J}_m,2)}$, $\frac{b}{a} \leq m^{\frac{1}{2}}$ for any $x \in X_m$. Then there are absolute constants C_1,C_2,C_3,> 0 and subspace $Y_n \subset X_m$,*

$$\dim Y_n = n \geq C_1 m \left(\frac{a}{b}\right)^2$$

such that $C_2\|x\|_{(\mathrm{J}_m,2)} \leq \|x\|_{X_m} \leq C_3\|x\|_{(\mathrm{J}_m,2)}$ for all $x \in Y_n$.

Lemma 2.2. *Let $\{\phi_k \, | k \in \mathbb{N}\}$ be an orthonormal and uniformly bounded system, $\|\phi_k\|_\infty \leq M$, $k \in \mathbb{N}$ on a probability space $(\Omega, \mathcal{A}, \nu)$. Then*

$$\|t_m\|_2 \leq \|t_m\|_p \leq M^{1-\frac{2}{p}} m^{\frac{1}{2}-\frac{1}{p}} \|t_m\|_2 \, , \, 2 \leq p \leq \infty \tag{2.1}$$

and

$$M^{1-\frac{2}{p}} m^{\frac{1}{2}-\frac{1}{p}} \|t_m\|_2 \leq \|t_m\|_p \leq \|t_m\|_2 \, , \, 1 \leq p \leq 2 \tag{2.2}$$

for any $t_m \in \Phi(m) = \operatorname{lin}\{\phi_k, \, 1 \leq k \leq m\}$ and $m \in \mathbb{N}$.

Proof. We show just (2.1). The proof of (2.2) is similar. Clearly,

$$\|S_m \, | L_2 \to L_2\| = 1. \tag{2.3}$$

Observe that

$$S_m(f, x) = \int_\Omega K_m(x, y) \, f(y) \, dy,$$

where

$$K_m(x,y)=\sum_{k=1}^{m}\phi_k(x)\,\phi_k(y).$$

Hence,

$$\begin{aligned}\|S_m\,|L_2\to L_\infty\|&=\sup\{\|S_mf\|_\infty\,|\|f\|_2\le 1\}\\ &\sup\left\{\left\|\int_\Omega K_m(x,y)\,f(y)\,dy\right\|_\infty\,|\|f\|_2\le 1\right\}\\ &\le\sup\{\|K_m(x,\cdot)\|_2\,|x\in\Omega\}\\ &=\sup\left\{\left\|\sum_{k=1}^{m}\phi_k(x)\,\phi_k(\cdot)\right\|_2\,|x\in\Omega\right\}\\ &=\sup\left\{\left(\sum_{k=1}^{m}\phi_k^2(x)\right)^{\frac12}\,|x\in\Omega\right\}\le Mm^{\frac12}.\end{aligned}\tag{2.4}$$

Applying (2.3), (2.4) and Riesz-Thorin Theorem [3] we find

$$\|t_m\|_p\le M^{1-\frac{2}{p}}m^{\frac12-\frac1p}\,\|t_m\|_2\,,\ 2\le p\le\infty$$

for any $t_m\in\Phi(m)$. Finally, by embedding, we get $\|t_m\|_2\le\|t_m\|_p$, $2\le p\le\infty$. □

Theorem 2.3. *Let $\{\phi_k\,|k\in\mathbb{N}\}$ be an orthonormal and uniformly bounded system, $\|\phi_k\|_\infty\le M$, $k\in\mathbb{N}$ on a probability space $(\Omega,\mathcal{A},\nu)$. Let $\Lambda=\{\lambda_k\,|k\in\mathbb{N}\}$ be a multiplier, where $|\lambda_1|\ge|\lambda_2|\ge\cdots>0$, $k\in\mathbb{N}$ then*

$$M^{-1}\left(\sum_{k=1}^{m+1}|\lambda_k|^{-1}\right)^{-1}\le b_m(\Lambda U_p,L_q)\le M^2\sum_{k=m+1}^{\infty}|\lambda_k|\tag{2.5}$$

for any $m\in\mathbb{N}$ and $1\le p,q\le\infty$. Let $\left(0<\frac1p\le\frac12\le\frac1q\right)\cap\left(\frac1p-\frac1q<\frac12\right)$ then

$$C_4\,|\lambda_s|\le b_{s-1}(\Lambda U_p,L_q)\le|\lambda_m|\,,\tag{2.6}$$

where $s=\left[C_5^2M^{4\left(\frac1p-\frac2q\right)}m^{1+2\left(\frac1p-\frac1q\right)}\right]$ and C_4,$C_5>0$ are some absolute constants.

Proof. It is easy to check that

$$\begin{aligned}\sup\{\|f-S_m(f)\|_\infty\,|f\in\Lambda U_1\}&\le M\sum_{k=m+1}^{\infty}|\lambda_k|\,|c_k(\varphi)|\\ &=M\sum_{k=m+1}^{\infty}|\lambda_k|\left|\int_\Omega\varphi\phi_k d\nu\right|\\ &\le M^2\sum_{k=m+1}^{\infty}|\lambda_k|\,\|\varphi\|_1\le M^2\sum_{k=m+1}^{\infty}|\lambda_k|\,.\end{aligned}\tag{2.7}$$

Let

$$B_{(\mathrm{J}_m,p)}=\left\{\alpha\in\mathbb{R}^m\,\middle|\,\|\alpha\|_{(\mathrm{J}_m,p)}\le1\right\}.$$

For any $\alpha^{(1)}=\left(\alpha_1^{(1)},\cdots,\alpha_m^{(1)}\right)\in B_{(\mathrm{J}_m,p)}$, $\alpha^{(2)}=\left(\alpha_1^{(2)},\cdots,\alpha_m^{(2)}\right)\in B_{(\mathrm{J}_m,p)}$ and $0\le t\le1$ we have

$$\left\|t\alpha^{(1)}+(1-t)\,\alpha^{(2)}\right\|_{(\mathrm{J}_m,p)}\le t\left\|\alpha^{(1)}\right\|_{(\mathrm{J}_m,p)}+(1-t)\left\|\alpha^{(2)}\right\|_{(\mathrm{J}_m,p)}\le1.$$

Observe that for any $\alpha = (\alpha_1, \cdots, \alpha_m) \in B_{(\mathrm{J}_m,p)}$ we have

$$\|-\alpha\|_{(\mathrm{J}_m,p)} = \left\|t_m^{-\alpha}\right\|_p = \left\|\sum_{k=1}^m (-\alpha_k)\,\phi_k\right\|_p$$

$$= \left\|\sum_{k=1}^m \alpha_k \phi_k\right\|_p = \|\alpha\|_{(\mathrm{J}_m,p)}.$$

Hence, $B_{(\mathrm{J}_m,\infty)}$ is a convex origin-symmetric body in $\mathbb{R}^m$ and

$$B_{(\mathrm{J}_m,\infty)} = \mathrm{J}^{-1}\left(\Phi(m) \cap U_\infty\right).$$

Let $\Lambda_1 = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$ then

$$\mathrm{J}_m^{-1}\left(\Lambda U_\infty \cap \Phi(m)\right) = \Lambda_1 B_{(\mathrm{J}_m,\infty)},$$

where $\Lambda_1 = \mathrm{diag}\,(\lambda_k, 1 \le k \le n)$. Since $\|\phi_k\|_\infty \le M$ for any $k \in \mathbb{N}$ then $M^{-1}\phi_k \in B_{(\mathrm{J}_m,\infty)}$. Cosequently, since $B_{(\mathrm{J}_m,\infty)}$ is convex, then $\Lambda_1 B_{(\mathrm{J}_m,\infty)}$ is also convex and

$$\Lambda_1 B_{(\mathrm{J}_m,\infty)} \supset M^{-1}\mathrm{conv}\left(\pm|\lambda_1|\,\mathbf{e}_1, \cdots, \pm|\lambda_n|\,\mathbf{e}_m\right),$$

where $\mathbf{e}_1, \cdots, \mathbf{e}_m$ is the standard basis in $\mathbb{R}^m$. Let

$$Q_m = \{\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m \,||\alpha_k| \le 1,\, 1 \le k \le m\} \subset \mathbb{R}^m$$

be the unit cube in $\mathbb{R}^m$ and

$$\varrho_m := \min\left\{\mu > 0 \,\middle|\, \mu Q_n \cap M^{-1}\mathrm{conv}\left(\pm|\lambda_1|\,\mathbf{e}_1, \cdots, \pm|\lambda_m|\,\mathbf{e}_m\right) \ne \varnothing\right\}.$$

Consider the simplex $\mathrm{conv}\left(|\lambda_1|\,\mathbf{e}_1, \cdots, |\lambda_m|\,\mathbf{e}_m\right)$. Since $\lambda_k \ne 0$, $1 \le k \le m$ then the point of intersection of $\varrho_m Q_m$ with the plain $\sum_{k=1}^m |\lambda_k|^{-1} x_k = 1$ has components $(\rho_m, \cdots, \rho_m)$, where

$$\varrho_m = \left(\sum_{k=1}^m |\lambda_k|^{-1}\right)^{-1}.$$

Hence,

$$\varrho_m Q_m \subset M^{-1}\mathrm{conv}\left(\pm|\lambda_1|\,\mathbf{e}_1, \cdots, \pm|\lambda_m|\,\mathbf{e}_m\right) \subset \Lambda_1 B_{(\mathrm{J}_m,\infty)}. \tag{2.8}$$

Assume that $\|t_m^\alpha\|_1 \le 1$. Since ϕ_k, $k \in \mathbb{N}$ are orthonormal and $\|\phi_k\|_\infty \le M$ for any $k \in \mathbb{N}$ then

$$\|t_m^\alpha\|_1 = \sup\left\{\int_\Omega t_m^\alpha \xi d\nu \,|\, \|\xi\|_\infty \le 1\right\}.$$

$$\ge \max\left\{\left|\int_\Omega M^{-1}\phi_k\left(\sum_{s=1}^m \alpha_s\phi_s\right)\right| \,|\, 1 \le k \le m\right\}$$

$$= M^{-1}\max\{|\alpha_k|\,|\,1 \le k \le m\}.$$

This means that $B_{(\mathrm{J}_m,1)} \subset MQ_m$. Therefore, using (2.8) we get

$$\varrho_m B_{(\mathrm{J}_m,1)} \subset M\varrho_m Q_m \subset M\Lambda_1 B_{(\mathrm{J}_m,\infty)},$$

or

$$M^{-1}\varrho_m B_{(\mathrm{J}_m,1)} \subset \Lambda_1 B_{(\mathrm{J}_m,\infty)}.$$

By the definition of Bernstein width we get

$$b_{m-1}\left(\Lambda U_\infty, L_1\right) \ge M^{-1}\varrho_m. \tag{2.9}$$

Comparing (2.7) and (2.9) we get (2.5).

We prove (2.6) now. Clearly, if $\{|\lambda_k|,\, k\in\mathbb{N}\}$ is a decreasing sequence then

$$\sup\left\{\|f-S_n(f)\|_2 \,|f\in\Lambda U_2\right\}=|\lambda_{n+1}|. \tag{2.10}$$

Hence,

$$b_n\left((\Lambda U_2,L_2)\right)\le\sup\left\{\|f-S_n(f)\|_2\,|f\in\Lambda U_2\right\}\le|\lambda_{n+1}|$$

by (1.2) and (2.10). Therefore, by embedding,

$$b_n\left((\Lambda U_p,L_q)\right)\le|\lambda_{n+1}|,\ 1\le q\le 2\le p\le\infty.$$

To get the respective lower bounds observe that

$$b_{n-1}\left((\Lambda U_p,L_q)\right)\ge b_{n-1}\left((\Lambda U_p\cap\Phi(m),L_q)\right)$$

Applying Lemma 2.1 we get

$$\|x\|_{(\mathrm{J}_m,p)}\le C_3\|x\|_{(\mathrm{J}_m,2)}$$

for any $x\in Y_n$, where by (2.1),

$$\begin{aligned}\dim Y_n=n&\ge C_1m\left(\frac{a}{b}\right)^2\\&\ge C_1m\left(M^{1-\frac{2}{p}}m^{\frac12-\frac1p}\right)^{-2}\\&\ge C_1M^{-2+\frac4p}m^{\frac2p}.\end{aligned}\tag{2.11}$$

Hence,

$$B_{(\mathrm{J}_m,2)}\cap Y_n\subset C_3B_{(\mathrm{J}_m,p)}\cap Y_n. \tag{2.12}$$

Recall that $|\lambda_k|$ is a decreasing sequence, $\lambda_k\neq0$. Let

$$\Lambda_{1,m}^{-1}=\begin{cases}\lambda_k^{-1}, & 1\le k\le m,\\ 0, & k\ge m+1.\end{cases}$$

It is easy to check that

$$\left\|\Lambda_{1,m}^{-1}\mathrm{J}_m^{-1}t_m\right\|_{(\mathrm{J}_m,2)}\le|\lambda_m|^{-1}\left\|\mathrm{J}_m^{-1}t_m\right\|_{(\mathrm{J}_m,2)},$$

for any $t_m\in\Phi(m)$. This means that

$$\Lambda_{1,m}B_{(\mathrm{J}_m,2)}\supset|\lambda_m|\,B_{(\mathrm{J},m,2)}. \tag{2.13}$$

By (2.11) Lemma 2.1 and Lemma 2.2 there is subspace $Z_s\subset Y_n\subset X_m$ such that $C_2\|x\|_{(\mathrm{J}_m,2)}\le\|x\|_{(\mathrm{J}_m,q)}$ for any $x\in Z_s$,

$$\begin{aligned}s=\dim Z_s&\ge C_1n\left(\frac{a}{b}\right)^2\\&\ge C_1^2\left(M^{-2+\frac4p}m^{\frac2p}\right)\left(M^{1-\frac2q}m^{\frac12-\frac1q}\right)^2\\&=C_1^2M^{4\left(\frac1p-\frac2q\right)}m^{1+2\left(\frac1p-\frac1q\right)}.\end{aligned}$$

Note that $1+2\left(\frac1p-\frac1q\right)>0$ if $\frac1q-\frac1p<\frac12$. Comparing (2.12) and (2.13) we obtain

$$\Lambda_{1,m}B_{(\mathrm{J}_m,p)}\cap Z_s\supset C_2C_3\,|\lambda_m|\,B_{(\mathrm{J}_m,q)}.$$

Consequently,

$$b_{[s]-1}\left(\Lambda U_p,L_q\right)\ge C\,|\lambda_m|,$$

where $C=C_2C_3$. □

Observe that lower bound (2.5) remains valid in a more general case. Let X be any Banach space with the unit ball U_X and Y be a Banach space with the unit ball U_Y such that

$$\Phi(m) \cap U_Y \subset Q_m, m \in \mathbb{N}$$

then

$$b_m(\Lambda U_X, Y) \geq M^{-1} \left(\sum_{k=1}^{m+1} |\lambda_k|^{-1} \right)^{-1}.$$

We say that an orthonormal system $\{\xi_k,\, k \in \mathbb{N}\} \subset L_\infty$ is essentially bounded [11] if for any $0 < E < 1$ and any $n \in \mathbb{N}$ there exists a set of orthonormal functions

$$\{\phi_k, 1 \leq k \leq s\} \subset \operatorname{lin}\{\xi_k, 1 \leq k \leq n\}, s \geq En$$

such that

$$\max\{\|\phi_k\|_\infty, 1 \leq k \leq s\} \leq M(E).$$

The class of essentially bounded orthonormal systems is sufficiently large. For example, the set of orthonormal harmonics on any two-point homogeneous Riemannian manifold $\mathbb{M}^d$, i.e. a real sphere $\mathbb{S}^d$, a real $\mathrm{P}^d(\mathbb{R})$, complex $\mathrm{P}^d(\mathbb{C})$ and quaternion $\mathrm{P}^d(\mathbb{H})$ projective spaces or a Cayley elliptic plane $\mathrm{P}^{16}(\mathrm{Cay})$ is essentially bounded [12]. A direct consequence of Theorem 2.3 is the following statement.

Corollary 2.4. *Let $\{\xi_k \,|k \in \mathbb{N}\}$ be essentially bounded on a probability space $(\Omega, \mathcal{A}, \nu)$. Let $\Lambda = \{\lambda_k \,|k \in \mathbb{N}\}$ be a multiplier, where $|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$, $k \in \mathbb{N}$ and $\left(0 < \frac{1}{p} \leq \frac{1}{2} \leq \frac{1}{q}\right) \cap \left(\frac{1}{p} - \frac{1}{q} < \frac{1}{2}\right)$ then*

$$C|\lambda_s| \leq b_{s-1}(\Lambda U_p, L_q) \leq |\lambda_m|, \tag{2.14}$$

where $s = \left[C_1^2 M(E)^{4\left(\frac{1}{p}-\frac{2}{q}\right)} m^{1+2\left(\frac{1}{p}-\frac{1}{q}\right)}\right]$ and C,$C_1 > 0$ are some absolute constants.

Consider applications of Corollary 2.4. Let Ω be a two-point homogeneous Riemannian manifold $\mathbb{M}^d$, ν its normalised volume element and Δ its Laplace-Beltrami operator. Consider here just the case $\mathbb{M}^d = \mathbb{S}^d$,$\mathrm{P}^d(\mathbb{C})$,$\mathrm{P}^d(\mathbb{H})$,$\mathrm{P}^{16}(\mathrm{Cay})$ The case $\mathbb{M}^d = \mathrm{P}^d(\mathbb{R})$ can be treated similarly. It is well-known that the eigenspaces H_k, $k \geq 0$ of Δ are finite dimensional, orthonormal with respect to the scalar product $[f,g] = \int_{\mathbb{M}^d} fg d\nu$ and $L_2 = \oplus_{k=0}^{\infty} \mathrm{H}_k$. Fix an orthonormal basis $\left\{Y_m^k\right\}_{m=1}^{d_k}$ in H_k, $\dim \mathrm{H}_k = d_k$. Let $\varphi \in L_p$, $1 \leq p \leq \infty$ with the formal Fourier series

$$\varphi \sim \sum_{k=0}^{\infty} \sum_{m=0}^{d_k} c_{k,m}(\varphi) Y_m^k,\ c_{k,m}(\varphi) = [\varphi, Y_m^k]$$

and $\Lambda = \{\lambda_k,\, k \in \mathbb{N} \cup \{0\}\}$. If for any $\varphi \in L_p$ there is a function $f = \Lambda\varphi \in L_q$ such that

$$f \sim \sum_{k=0}^{\infty} \sum_{m=0}^{d_k} \lambda_k c_{k,m}(\varphi) Y_m^k$$

then we say that the multiplier Λ is of (p,q)-type. Let, in particular, $\lambda_k = \log^{-\beta}(k+1)$, $\beta > 0$ then using (2.14) we get

$$\lambda_s = \log^{-\beta}(s+1) = \log^{-\beta}\left[C_1^2 M^{4\left(\frac{1}{p}-\frac{2}{q}\right)} m^{1+2\left(\frac{1}{p}-\frac{1}{q}\right)}\right]$$

$$\geq C_4 \left(1 + 2\left(\frac{1}{p} - \frac{1}{q}\right)\right)^{-\beta} \log^{-\beta} m.$$

Hence, by Corollary 2.4 we find

$$C_4 \left(1 + 2\left(\frac{1}{p} - \frac{1}{q}\right)\right)^{-\beta} \log^{-\beta}(m+1) \leq b_{s-1}(\Lambda U_p, L_q) \leq \log^{-\beta}(m+1).$$

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