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λ -STATISTICAL CONVERGENCE IN FUZZY N NORMED LINEAR SPACES

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Abstract. In this paper, we introduce $\lambda-$ statistical convergence and the condition of being $\lambda-$ statistical Cauchy of real number sequences in fuzzy n normed linear spaces. At the same time, in fuzzy n normed space, we have introduced the concept of (V, λ) summability and $(C, 1)$ summability. Then, we studied the relation between these concepts and $\lambda-$ statistical convergence.

1. Introduction and Background

The concept of statistical convergence of real number sequences was introduced by Fast [9] and Steinhaus [27] and later reintroduced by Schoenberg [26] independently. The concept of statistical convergence has been studied in many branches of mathematics. Examples of these are Fourier analysis, Banach spaces, Number theory, and Measure theory. Many mathematicians, such as, Connor [4], Fridy [11], Et and Cinar [3] ... etc., studied the concept of statistical convergence in summability theory.

Along with these studies, it is seen that the concept of statistical convergence is also used in different studies. Especially the concepts of lacunary statistical convergence and ideal convergence are seen in the works of [1, 5, 12, 25, 32–34]. Their applications in fuzzy normed spaces and double sequences can be seen in the works of [7, 18, 20–24, 30, 31, 35]. In addition, definitions of convergence in different spaces have begun to be given in recent years [8, 14–16, 16, 28, 29]

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive real numbers tending to ∞ such that $\lambda_n \leq \lambda_n + 1$, $\lambda_1 = 1$. Λ will denote the set of all such sequences.

The concept of $\lambda-$ statistical convergence was defined by Mursaleen [19] as follows.

A sequence $x = (x_k)$ is said to be λ- statistically convergent or S_λ- convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : |x_k - L| \geq \varepsilon\}\right| = 0,$$

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where $I_n = [n - \lambda_n + 1, n]$. In this case we write $S_\lambda - \lim x = L$ or $x_k \to L\,(S_\lambda)$ and $S_\lambda = \{x : \exists L\in \mathbb{R}, S_\lambda - \lim x = L\}$.

Fuzzy sets are considered concerning a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [37] a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \to [0, 1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:
i. u is normal, that is, there exists an $x_0\in \mathbb{R}$ such that $u(x_0) = 1$;
ii. u is fuzzy convex, that is, for $x, y\in \mathbb{R}$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

iii. u is upper semicontinuous;
iv. $suppu = cl\{x\in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Let $L(\mathbb{R})$ be a set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a nonnegative fuzzy number. We write $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ iff $u_\alpha^- \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $\widetilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x\in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \ \alpha \in \ \ 0, 1 \\ suppu, & \text{if } \ \ \alpha = 0. \end{cases}$$

A partial order $\preccurlyeq$ on $L(\mathbb{R})$ is defined by $u \preccurlyeq v$ iff $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in [0, 1]$.

Arithmetic operation $\oplus, \ominus, \odot$ and $\oslash$ on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by

$$\begin{aligned} (u \oplus v)(t) &= \sup_{s\in\mathbb{R}} \{u(s) \wedge v(t - s)\} \quad t \in\mathbb{R} \\ (u \ominus v)(t) &= \sup_{s\in\mathbb{R}} \{u(s) \wedge v(s - t)\} \quad t \in\mathbb{R} \\ (u \odot v)(t) &= \sup_{\substack{s\in\mathbb{R} \\ s\neq 0}} \{u(s) \wedge v(t/s)\} \quad t \in\mathbb{R} \\ (u \oslash v)(t) &= \sup_{s\in\mathbb{R}} \{u(st) \wedge v(s)\} \quad t \in\mathbb{R} \end{aligned}$$

For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \widetilde{0}$, $t\in \mathbb{R}$. Some arithmetic operations for $\alpha-$level sets are defined as follows:
$u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in \ \ 0, 1\ \ $. Then

$$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad [u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$$
$$[u \odot v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+] \quad \left[\widetilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right] u_\alpha^- > 0$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u, v) = \sup_{0\leq\alpha\leq 1} \max \left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

D is known as a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D\,(x_k, x_0) < \varepsilon$

for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers convergence to levelwise to x_0, iff $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$ where $[x_k]_\alpha = \left[(x_k)_\alpha^-, (x_k)_\alpha^+\right]$ and $[x_0]_\alpha = \left[(x_0)_\alpha^-, (x_0)_\alpha^+\right]$ for every $\alpha \in (0,1)$. ([2,6,17])

Savas [24] defined the statistical convergence of fuzzy numbers as follows;
A sequence $X = (X_k)$ of fuzzy numbers is said to be $\lambda-$ statistically convergent to fuzzy numbers X_0 if every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |k \in I_n : d(X_k, X_0) \geq \varepsilon| = 0.$$

Later many mathematicians such as Altınok, Altın and ET [1]... etc studied statistical convergence of fuzzy numbers.

Let X be a vector space over $\mathbb{R}$, let $\|.\| : X \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. [10,36],

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space
(briefly $(X, \|.\|)\, FNS$) and $\|.\|$ a fuzzy norm if the following axioms are satisfied

1) $\|x\| = \widetilde{0}$ iff $x = \theta$,
2) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
3) For all $x, y \in X$
$a)$ $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever
$s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,
$b)$ $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever
$s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy n norm $\|.\|$ on X can be obtained

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a\|x\|_C \quad \text{or} \quad t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a} & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1} & \|x\|_C \leq t \leq b\|x\|_C \end{cases} \tag{1.1}$$

where $\|x\|_C$ is the ordinary norm of $x(\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \widetilde{0}$. Hence $(X, \|.\|)$ is a fuzzy normed linear space. [10]

Sencimen [7] was defined convergence in fuzzy normed spaces by taking advantage of Kaleva [13] and Felbin [10] , as follows;

Let $(X, \|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that $(D) - \underset{n\to\infty}{lim} \|x_n\| = \widetilde{0}$ i.e. for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D\left(\|x_n - x\|, \widetilde{0}\right) < \varepsilon$ for all $n \in \mathbb{N}$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$sup_{\alpha\in[0,\ 1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$$

for all $n \geq N(\varepsilon)$.

Let X be a real linear space of dimension d, where $2 \leq d < \infty$. Let $\|\cdot, \cdot, ..., \cdot\| : X^n \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0,1] \times [0,1] \to [0,1]$ be symmetric,nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$ then the quadruple $(X, \|\cdot, \cdot, ..., \cdot\|, L, R)$ is called fuzzy n normed linear space $(briefly\, (X, \|\cdot, \cdot, ..., \cdot\|)\, FnNS)$ and $\|\cdot, \cdot, ..., \cdot\|$ a fuzzy n norm

if the following axioms are satisfied for every $y, x_1, x_2, ..., x_n \in X$ and $s, t \in \mathbb{R}$
$fnN_1 : \|x_1, x_2, ..., x_n\| = \widetilde{0}$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent vectors,
$fnN_2 : \|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,
$fnN_3 : \|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ for all $\alpha \in \mathbb{R}$,
$fnN_4 : \|x_1 + y, x_2, ..., x_n\| (s + t) \geq L (\|x_1, x_2, ..., x_n\| (s), \|y, x_2, ..., x_n\| (t))$
whenever $s \leq \|x_1, x_2, ..., x_n\|_1^-$, $t \leq \|y, x_2, ..., x_n\|_1^-$ and $s+t \leq \|x_1 + y, x_2, ..., x_n\|_1^-$,
$fnN_5 : \|x_1 + y, x_2, ..., x_n\| (s + t) \leq R (\|x_1, x_2, ..., x_n\| (s), \|y, x_2, ..., x_n\| (t))$
whenever $s \geq \|x_1, x_2, ..., x_n\|_1^-$, $t \geq \|y, x_2, ..., x_n\|_1^-$ and $s+t \geq \|x_1 + y, x_2, ..., x_n\|_1^-$,
where $[\|x_1, x_2, ..., x_n\|]_\alpha = \left[\|x_1, x_2, ..., x_n\|_\alpha^- , \|x_1, x_2, ..., x_n\|_\alpha^+\right]$ for $x_1, x_2, ..., x_n \in X$, $0 \leq \alpha \leq 1$ and $\inf_{\alpha \in [0,1]} \|x_1, x_2, ..., x_n\|_\alpha^- > 0$.

Hence the norm $\|\cdot, \cdot, ..., \cdot\|$ is called fuzzy n norm on X and pair $(X, \|\cdot, \cdot, ..., \cdot\|)$ is called fuzzy n normed space.

Let $(X, \|\cdot, \cdot, ..., \cdot\|)$ be fuzzy n normed space. A sequence $\{x_k\}$ in X is said to be convergent to an element $x \in X$ with respect to the fuzzy n norm on X if for every $\varepsilon > 0$ and for every $z_1, z_2, ..., z_{n-1} \neq 0$, $z_1, z_2, ..., z_{n-1} \in X$, $\exists$ a number $N = N(z_1, z_2, ..., z_{n-1}, \varepsilon)$ such that $D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - x\| , \widetilde{0}\right) < \varepsilon$,$\forall k \geq N$ or equivalently $(D) - \lim_{k \to \infty} \|z_1, z_2, ..., z_{n-1}, x_k - x\| = \widetilde{0}$.

Let $(X, \|\cdot, \cdot, ..., \cdot\|)$ be fuzzy n normed space. A sequence $\{x_k\}$ in X is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy n norm on X if for every $\varepsilon > 0$ and for every $z_1, z_2, ..., z_{n-1} \neq 0$, $z_1, z_2, ..., z_{n-1} \in X$, we have

$$\delta\left(\left\{k \in \mathbb{N} : D\left(\|z_1, z_2, ..., z_{n-1}, x_k - x\| , \widetilde{0}\right) \geq \varepsilon\right\}\right) = 0.$$

2. Main Results

In this section, we will define $\lambda-$ statistical convergence sequences, $\lambda-$ statistically Cauchy sequence, strongly $\lambda-$ summability and strongly Cesaro summability in fuzzy n-normed spaces. And then, we are going to study the relation between $\lambda-$ statistically convergence and $\lambda-$ summability in fuzzy n-normed spaces.

Definition 2.1. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an FnNS and $\lambda \in \Lambda$. A sequence $x = (x_k)$ in X is said to be $\lambda-$statistically convergent to $L \in X$ with respect to fuzzy n-norm on X or $FnS_\lambda-$convergent if for each $\varepsilon > 0$

$$\lim_{m \to \infty} \frac{1}{\lambda_m} \left|\{ k \in I_m : D(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\| , \widetilde{0}) \geq \varepsilon \}\right| = 0$$

and we write $x_k \overset{FnS_\lambda}{\to} L$ or $x_k \to L\,(FnS_\lambda)$ or $S_\lambda \overset{FnN}{-} \lim x_k = L$ where $I_m = [m - \lambda_m + 1, m]$.

This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{k \in I_m : \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+ \geq \varepsilon\right\}$$

has natural density zero, namely, for each $\varepsilon > 0$, $\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+ < \varepsilon$ for a.a.k.

In this case we write $S_\lambda \overset{FnN}{-} \lim x_k = L$. The set of all statistically convergent sequences with respect to fuzzy n norm on X will be denoted by FnS_λ .

The element $L \in X$ is the $FnS_\lambda-$ limit of (x_k). In terms of neigborhoods, we have $x_k \overset{FnS_\lambda}{\to} L$ provided that for each $\varepsilon > 0$, $x_k \in \aleph_x(\varepsilon, 0)$ for a.a.k.

A useful interpretation of the above definition is the following:

$$x_k \overset{FnS_\lambda}{\rightarrow} L \iff S_\lambda \overset{FnN}{-} \lim \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+ = 0$$

Note that $S_\lambda \overset{FnN}{-} \lim \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+ = 0$ implies that

$$S_\lambda \overset{FnN}{-} \lim \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_\alpha^- = S_\lambda \overset{FnN}{-} \lim \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_\alpha^+ = 0$$

for each $\alpha \in [0, 1]$ since $0 \leq \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_\alpha^- \leq \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_\alpha^+ \leq \|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+$ holds for every $k \in I_m$ and for each $\alpha \in [0, 1]$.

In this case we write throughout the paper (x_k) $FnS_\lambda-$convergent to $L \in X$ means that (x_k) is $\lambda-$statistically convergent to $L \in X$ w.r.t the fuzzy n norm on X.

Since the natural density of a finite set is zero, every convergent sequence is statistically convergent on FnNS, but the converse is not true in general as can be seen in the following example.

Example 2.2. Let $(\mathbb{R}^p, \|\cdot, \ldots, \cdot\|)$ be an FnNS and $x = (x_1, x_2, ..., x_p) \in \mathbb{R}^p$ be a fixed non-zero vector, where the fuzzy n norm on $\mathbb{R}^p$ is defined as in (1.1) such that $\|x\|_C = \left(x_1^2 + x_2^2 + ... + x_p^2\right)^{1/2}$ and $\lambda = (\lambda_m) = (m) \in \Lambda$.

Now we define a sequence (x_k) in $\mathbb{R}^p$ as

$$x_k = \begin{cases} x & \text{if } k = t^2, \\ \theta & \text{if } k \neq t^2, \end{cases}$$

where $t \in \mathbb{N}$.

Then we see that for any ε satisfying $0 < \varepsilon \leq b\left(x_1^2 + x_2^2 + ... + x_p^2\right)^{1/2}$, we have $K(\varepsilon) = \left\{k \in I_m : \|z_1, z_2, \ldots, z_{n-1}, x_k - \theta\|_0^+ \geq \varepsilon\right\} = \{1, 4, 9, ...\}$ and hence $\delta\left(K(\varepsilon)\right) = 0$. If we choose $\varepsilon > b\left(x_1^2 + x_2^2 + ... + x_p^2\right)^{1/2}$ then we get $K(\varepsilon) = \varnothing$ and hence $\delta(\varnothing) = 0$. Thus, $x_k \overset{FnS_\lambda}{\rightarrow} \theta$ but (x_k) is not convergent since the set $\{1, 4, 9, 16, ...\}$ has infinitely many elements.

Moreover, since the subsequence (x_{k^2}) statistically converges to x, we see that a subsequence of a statistically convergent sequence doesn't need to be statistically convergent to the statistical limit of the sequence in an FnNS.

In the following, some basic properties of the statistical limit are summarized.

Proposition 2.3. *Let (x_k) and (y_k) be sequences in an FnNS $(X, \|\cdot, \ldots, \cdot\|)$ such that $x_k \overset{FnS_\lambda}{\rightarrow} x$ and $y_k \overset{FnS_\lambda}{\rightarrow} y$, where $x, y \in X$. Then we have*

i) $(x_k + y_k) \overset{FnS_\lambda}{\rightarrow} x + y$,

ii) $tx_k \overset{FnS_\lambda}{\rightarrow} tx$ $(t \in \mathbb{R})$,

iii) $S_\lambda \overset{FnN}{-} \lim \|z_1, z_2, \ldots, z_{n-1}, x_k\| = \|z_1, z_2, \ldots, z_{n-1}, x\|$.

Definition 2.4. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an FnNS. A sequence (x_k) in X is λ- statistically Cauchy with respect to the fuzzy n norm on X provided that for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \frac{1}{\lambda_m} \left|\left\{k \in I_m : \|z_1, z_2, \ldots, z_{n-1}, x_k - x_N\|_0^+ \geq \varepsilon\right\}\right| = 0.$$

In the sequel (x_k) is $FnS_\lambda-$Cauchy means that (x_k) is $\lambda-$statistically Cauchy with respect to the fuzzy n norm on X.

Proposition 2.5. *In an FnNS $(X, \|\cdot,\ \ldots, \cdot\|)$ every $FnS_\lambda-$convergent sequence is also an $FnS_\lambda-$Cauchy sequence.*

Proof. Let $x_k \overset{FnS_\lambda}{\to} L$ and $\varepsilon > 0$. Then we have $\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|_0^+ < \varepsilon/2$ for a.a.k. Choose $N\in \mathbb{N}$ such that $\|z_1, z_2, \ldots, z_{n-1}, x_N - x\|_0^+ < \varepsilon/2$. Now $\|\cdot,\ \ldots, \cdot\|_0^+$ being a norm in the usual sense, we get

$$\begin{aligned}\|z_1, z_2, \ldots, z_{n-1}, x_k - x_N\|_0^+ &= \|z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_N)\|_0^+ \\ &\leq \|z_1, z_2, \ldots, z_{n-1}, x_k - x\|_0^+ \\ &+ \|z_1, z_2, \ldots, z_{n-1}, x_N - x\|_0^+ \\ &< \varepsilon/2 + \varepsilon/2 < \varepsilon\end{aligned}$$

for a.a.k. This shows that (x_k) is $FnS_\lambda-$ Cauchy.

In the following, we introduce and study the concepts of strongly $\lambda-$summability w.r.t fuzzy n norm on X and find its relation with $\lambda-$statistically convergent w.r.t fuzzy n norm on X. Before giving the promised relations, we will provide definitions of $\lambda-$ summability concerning fuzzy n norm on X.

Definition 2.6. Let $(X, \|\cdot,\ \ldots, \cdot\|)$ be an FnNS and $\lambda = (\lambda_m)$ be a nondecreasing sequence of positive numbers tending to ∞ and $\lambda_{m+1} \leq \lambda_m + 1$, $\lambda_1 = 1$ and $x = (x_k)$ be a sequence in X. The sequence x is said to be strongly $\lambda -$ *summable* with respect to fuzzy n norm on X if there is a $L \in X$ such that

$$\lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} D(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}) = 0$$

where $I_m = [m - \lambda_m + 1,\ m]$.

In this case we write $[V, \lambda]_{FnN} - \lim x_k = L$. The set of all strongly (V, λ) summable to fuzzy n norm on X denoted by $[V, \lambda]_{FnN}$.

We have said that X is strongly λ-summable to L concerning fuzzy norm on X. If $\lambda_m = m$, then strongly λ-summable reduces to strongly Cesaro summable w.r.t fuzzy n norm on X defined as follows:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} D(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}) = 0$$

In this case we write $[C, 1]_{FnN} - \lim x_k = L$. The set of all strongly $(C, 1)$ summable to fuzzy n norm on X denoted by $[C, 1]_{FnN}$.

So, we write for some L

$$[V,\ \lambda]_{FnN} = \{x = (x_k) : \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) = 0\}$$

$$[C,\ 1]_{FnN} = \{x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) = 0\}$$

Theorem 2.7. *If a sequence $x = (x_k)$ is $[V, \lambda]_{FnN}-$ summable to L , then it is $FnS_\lambda-$ convergent to L .*

Proof. Let $\varepsilon > 0$. Since

$$\begin{aligned} &\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ \geq &\sum_{\substack{k\in I_m \\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon}} D(\|z_1,z_2,\ldots,z_{n-1}\ ,x_k-L\|,\widetilde{0}) \\ \geq &\ \varepsilon\left|\{k\in I_m:\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon\}\right| \end{aligned}$$

This implies that if $[V,\lambda]_{FnN}-$ summable to L, then x is $FnS_\lambda-$ convergent to L .

Theorem 2.8. *If a bounded $x=(x_k)$ is $FnS_\lambda-$ convergent to L, then it is $[V,\lambda]_{FnN}-$ summable to L , and hence x is $[C,1]_{FnN}-$summable to L .*

Proof. Suppose that $x=(x_k)$ is bounded and $FnS_\lambda-$ convergent to L . Since x is bounded we write $D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) < M$ for all k. Given $\varepsilon>0$, we have

$$\begin{aligned} &\frac{1}{\lambda_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ &\frac{1}{\lambda_m}\sum_{\substack{k\in I_m \\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon}} D(\|z_1,z_2,\ldots,z_{n-1}\ ,x_k-L\|,\widetilde{0}) \\ &+\frac{1}{\lambda_m}\sum_{\substack{k\in I_m \\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)<\varepsilon}} D(\|z_1,z_2,\ldots,z_{n-1}\ ,x_k-L\|,\widetilde{0}) \\ \leq\ &\frac{M}{\lambda_m}\left|\left\{k\in I_m:\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|+\varepsilon. \end{aligned}$$

This implies that x is $[V,\lambda]_{FnN}-$summable to L . Further, we have

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ =\ &\frac{1}{m}\sum_{k=1}^{m-\lambda_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ &+\frac{1}{m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ \leq\ &\frac{1}{m}\sum_{k=1}^{m-\lambda_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ &+\frac{1}{\lambda_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\ \leq\ &\frac{2}{\lambda_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right). \end{aligned}$$

Hence x is $[C,1]_{FnN}-$ summable to L since x is $[V,\lambda]_{FnN}-$ summable to L .

Theorem 2.9. *If a sequence $x=(x_k)$ is statistically convergent to L w.r.t fuzzy n norm on X and* $\lim\underset{m}{inf}\ (\frac{\lambda_m}{m})>0$ *then it is $FnS_\lambda-$ convergent to L.*

Proof. For given $\varepsilon>0$, we have

$$\begin{aligned} &\left\{k\leq m:\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\\ \supset\ &\left\{k\in I_m:D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{m}\left|\left\{k\leq m:\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\\ \geq\ &\frac{1}{m}\left|\left\{k\in I_m:D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\\ \geq\ &\frac{\lambda_m}{m}.\frac{1}{\lambda_m}\left|\{k\in I_m:D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right|\end{aligned}$$

Taking limit as $m\to\infty$ and using $\lim\underset{m}{inf}\ (\frac{\lambda_m}{m})>0$, we get that x is $FnS_\lambda-$ convergent to L .

Throughout the paper, unless stated otherwise, by "for all $m\in\mathbb{N}_{m_0}$" we mean "for all $m\in\mathbb{N}$ except finite numbers of positive integers" where $\mathbb{N}_{m_0}=\{m_0\ ,\ m_0+1,\ m_0+2,\ \ldots\}$ for some $m_0\in\mathbb{N}=\{1,\ 2,\ 3,\ \ldots\}$

Theorem 2.10. *Let $\lambda=(\lambda_m)$ and $\mu=(\mu_m)$ be two sequences in Λ such that $\lambda_m\leq\mu_m$ for all $m\in\mathbb{N}_{m_0}$.*

i. If

$$\lim_{m\to\infty} inf\ \frac{\lambda_m}{\mu_m}>0 \tag{2.1}$$

then $FnS_\mu\ \subseteq FnS_\lambda$.

ii. If

$$\lim_{m\to\infty}\frac{\lambda_m}{\mu_m}=1 \tag{2.2}$$

then $FnS_\lambda\ \subseteq FnS_\mu$.

Proof. (i) Suppose that $\lambda_m\leq\mu_m$ for all $m\in\mathbb{N}_{m_0}$ and let (2.1) satisfied. Then $I_m\ \subseteq J_m$ and so that for $\varepsilon>0$ we may write

$$\begin{aligned} &\left|\left\{k\ \in\ J_m\ :D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\\ \geq\ &\left|\left\{k\ \in\ I_m\ :D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\end{aligned}$$

and therefore, we have

$$\begin{aligned} &\frac{1}{\mu_m}\left|\left\{k\ \in\ J_m\ :D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\\ \geq\ &\frac{\lambda_m}{\mu_m}.\frac{1}{\lambda_m}\left|\left\{k\ \in\ I_m\ :D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|\end{aligned}$$

for all $m_0\in\mathbb{N}_{m_0}$, where $J_m=[m-\mu_m+1,m]$.

Now, taking the limit as $m \to \infty$ in the last inequality and using (2.1) we get $FnS_{\mu} \subset FnS_{\lambda}$.

ii. Let $(x_k) \in FnS_{\lambda}$ and (2.2) be satisfied. Since $I_m \subset J_m$, for $\varepsilon > 0$ we may write

$$\begin{aligned} &\frac{1}{\mu_m}\left|\left\{k \in J_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \\ = &\frac{1}{\mu_m}\left|\left\{m - \mu_m + 1 \leq k \leq m - \lambda_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \\ &+\frac{1}{\mu_m}\left|\left\{k \in I_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \\ \leq &\frac{\mu_m - \lambda_m}{\mu_m} + \frac{1}{\lambda_m}\left|\left\{k \in I_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \\ \leq &(1 - \frac{\lambda_m}{\mu_m}) + \frac{1}{\lambda_m}\left|\left\{k \in I_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \end{aligned}$$

for all $m_0 \in \mathbb{N}_{m_0}$. Since $\lim_{m\to\infty} \frac{\lambda_m}{\mu_m} = 1$ by (2.2) the first term and since $x = (x_k) \in FnS_{\lambda}$ and so $FnS_{\lambda} \subseteq FnS_{\mu}$, the second term of right hand side of above inequality has tended to 0 as $m \to \infty$. This implies that

$$\frac{1}{\mu_m}\left|\left\{k \in J_m : D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \geq \varepsilon\right\}\right| \to 0$$

$m \to \infty$. Therefore $x = (x_k) \in FnS_{\mu}$ and so $FnS_{\lambda} \subseteq FnS_{\mu}$.

Corollary 2.11. *Let $\lambda = (\lambda_m)$ and $\mu = (\mu_m)$ be two sequences in Λ such that $\lambda_m \leq \mu_m$ for all $m \in \mathbb{N}_{m_0}$. If (2.2) holds then $FnS_{\lambda} = FnS_{\mu}$.*

If we take $\mu = (\mu_m) = (m)$ in Corollary 2.11 we have the following result.

Corollary 2.12. *Let $\lambda = (\lambda_m) \in \Lambda$. If $\lim_m \frac{\lambda_m}{m} = 1$ then we have $FnS_{\lambda} = FnS$.*

Theorem 2.13. *Let $\lambda = (\lambda_m), \mu = (\mu_m) \in \Lambda$ and suppose that $\lambda_m \leq \mu_m$ for all $m \in \mathbb{N}_{m_0}$.*

i. If (2.1) holds then $[V, \mu]_{FnN} \subseteq [V, \lambda]_{FnN}$,
ii. If (2.2) holds then $l_{\infty} \cap [V, \lambda]_{FnN} \subseteq [V, \mu]_{FnN}$.

Proof. (i) Suppose that $\lambda_m \leq \mu_m$ for all $m \in \mathbb{N}_{m_0}$. Then $I_m \subset J_m$ and so that we may write

$$\begin{aligned} &\frac{1}{\mu_m} \sum_{k \in J_m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \\ \geq &\frac{1}{\mu_m} \sum_{k \in I_m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \end{aligned}$$

for all $m \in \mathbb{N}_{m_0}$. This gives that

$$\begin{aligned} &\frac{1}{\mu_m} \sum_{k \in J_m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \\ \geq &\frac{\lambda_m}{\mu_m}\frac{1}{\lambda_m} \sum_{k \in I_m} D\left(\|z_1, z_2, \ldots, z_{n-1}, x_k - L\|,\ \widetilde{0}\right) \end{aligned}$$

Then, taking the limit as $m \to \infty$ in the last inequality and using (2.1) we obtain $[V,\mu]_{FnN} \subseteq [V,\lambda]_{FnN}$.

(ii) Let $x=(x_k) \in l_\infty \cap [V,\lambda]_{FnN}$ and suppose that (2.2) holds. Since $x=(x_k) \in l_\infty$ then there exist some $M>0$ such that $D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \leq M$ for all k. Now, since $\lambda_m \leq \mu_m$ and so that $\frac{1}{\mu_m} \leq \frac{1}{\lambda_m}$ and $I_m \subset J_m$ for all $m \in \mathbb{N}_{m_0}$, we may write

$$\begin{aligned}
& \frac{1}{\mu_m}\sum_{k\in J_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
= & \frac{1}{\mu_m}\sum_{k\in \frac{J_m}{I_m}} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
& +\frac{1}{\mu_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
\leq & \frac{\mu_m-\lambda_m}{\mu_m}\cdot M+\frac{1}{\mu_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
\leq & (1-\frac{\lambda_m}{\mu_m})\cdot M+\frac{1}{\lambda_m}\sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)
\end{aligned}$$

for every $m \in \mathbb{N}_{m_0}$. Since $\lim_m \frac{\lambda_m}{\mu_m}=1$ by (2.2) the first term and since $x=(x_k) \in [V,\lambda]_{FnN}$ the second term of right hand side of the above inequality tends to 0 as $m \to \infty$ (Note that $1-\frac{\lambda_m}{\mu_m} \geq 0$ for all $m \in \mathbb{N}_{m_0}$).

This implies that $l_\infty \cap [V,\lambda]_{FnN} \subseteq [V,\mu]_{FnN}$ and so that $l_\infty \cap [V,\lambda]_{FnN} \subseteq l_\infty \cap [V,\mu]_{FnN}$.

We have the following result from the theorem 2.13.

Corollary 2.14. *Let $\lambda,\mu \in \Lambda$ such that $\lambda_m \leq \mu_m$ for all $m \in \mathbb{N}_{m_0}$. If (2.2) holds then $l_\infty \cap [V,\lambda]_{FnN} = l_\infty \cap [V,\mu]_{FnN}$*

Theorem 2.15. *Let $\lambda,\mu \in \Lambda$ such that $\lambda_m \leq \mu_m$ for all $m \in \mathbb{N}_{m_0}$.*

i. If (2.1) holds then $x_k \to L\left([V,\mu]_{FnN}\right) \Rightarrow x_k \to L\left(FnS_\lambda\right)$ and the inclusion $[V,\mu]_{FnN} \subset FnS_\lambda$ is strict for some $\lambda,\mu \in \Lambda$.

ii. If $(x_k) \in l_\infty$ and $x_k \to L\left(FnS_\lambda\right)$ then $x_k \to L\left([V,\mu]_{FnN}\right)$, whenever (2.2) holds.

Proof. (i) Let $\varepsilon>0$ be given and let $x_k \to L\left([V,\mu]_{FnN}\right)$. Now for every $\varepsilon>0$ we may write

$$\begin{aligned}
& \sum_{k\in J_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
\geq & \sum_{k\in I_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right) \\
\geq & \sum_{\substack{k\in I_m \\ D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon}} D(\|z_1,z_2,\ldots,z_{n-1}\ ,x_k-L\|,\widetilde{0}) \\
\geq & \varepsilon.\ \left|\left\{k\in I_m : D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|,\ \widetilde{0}\right)\geq\varepsilon\right\}\right|
\end{aligned}$$

and so that

$$\begin{aligned}
&\frac{1}{\mu_m}\sum_{k\in J_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\,\widetilde{0}\right)\\
\geq\ &\frac{\lambda_m}{\mu_m}\frac{1}{\lambda_m}.\varepsilon.\left|\{k\in I_m : D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\,\widetilde{0}\right)\geq\varepsilon\right|
\end{aligned}$$

for all $m\in\mathbb{N}_{m_0}$. Then, taking the limit as $m\to\infty$ in the last inequality and using (2.1), we obtain that $x_k\to L\,(FnS_\lambda)$ whenever $x_k\to L\,([V,\mu]_{FnN})$.

To show that the inclusion $[V,\mu]_{FnN}\subset FnS_\lambda$ is strict for some $\lambda,\mu\in\Lambda$ we take $\lambda_m=\sqrt{m}$, $\mu_m=m$ for all $m\in\mathbb{N}$. Define $x=(x_k)$ as

$$x_k=\begin{cases}k & k=m^2\\ 0 & k\neq m^2\end{cases}$$

Then clearly, $x_k\to 0\quad(FnS_\lambda)$ same as Example 2.2. On the other hand, we know that

$1+2^2+3^2+\cdots+m^2=\frac{m.(m+1).(2m+1)}{6}$ is satisfield for every $m\in\mathbb{N}$. Considering this equality, we can write

$$\begin{aligned}
&\frac{1}{\mu_m}\sum_{j\in J_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-0\|\,,\,\widetilde{0}\right)\\
=\ &\frac{1}{m}\sum_{k=1}^{m}\|z_1,z_2,\ldots,z_{n-1},x_k\|_0^+\\
=\ &\frac{1}{m}(1+0+0+4+\cdot,\cdot,\ \ldots,\cdot+[\sqrt{m}]^2)\\
=\ &\frac{1}{m}.\frac{([\sqrt{m}]).([\sqrt{m}]+1).\ (2[\sqrt{m}]+1)}{6}
\end{aligned}$$

Since we have $\sqrt{m}<[\sqrt{m}]+1$ and $\sqrt{m}<2\,[\sqrt{m}]+1$, we can write $\frac{1}{m}>\frac{1}{([\sqrt{m}]+1).\ (2[\sqrt{m}]+1)}$ and so that

$$\frac{1}{\mu_m}\sum_{j\in J_m} D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-0\|\,,\,\widetilde{0}\right)>\frac{[\sqrt{m}]}{6}\to\infty$$

as $m\to\infty$. Therefore $x\notin[V,\mu]_{FnN}$.

(ii) Suppose that $x_k\to L\,(FnS_\lambda)$ and $x=(x_k)\in l_\infty$. Then there exist some $M>0$ such that $D\left(\|z_1,z_2,\ldots,z_{n-1},x_k-L\|\,,\,\widetilde{0}\right)\leq M$ for all k. Since $\frac{1}{\mu_m}\leq\frac{1}{\lambda_m}$, then for every $\varepsilon>0$ we may write

$$\begin{aligned}
& \frac{1}{\mu_m}\sum_{k\in J_m} D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right) \\
= & \frac{1}{\mu_m}\sum_{k\in \frac{J_m}{I_m}} D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right) \\
& +\frac{1}{\mu_m}\sum_{k\in I_m} D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right) \\
\leq & \frac{\mu_m-\lambda_m}{\mu_m}\cdot M+\frac{1}{\mu_m}\sum_{k\in I_m} D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right) \\
\leq & (1-\frac{\lambda_m}{\mu_m})\cdot M+\frac{1}{\lambda_m}\sum_{k\in I_m} D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right) \\
\leq & \left(1-\frac{\lambda_m}{\mu_m}\right)\cdot M+\frac{1}{\lambda_m}\sum_{\substack{k\in I_m \\ D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right)\geq\varepsilon}} D(\left\|z_1,z_2,\ldots,z_{n-1}\right.,x_k-L\|,\widetilde{0}) \\
& +\frac{1}{\lambda_m}\sum_{\substack{k\in I_m \\ D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right)<\varepsilon}} D(\left\|z_1,z_2,\ldots,z_{n-1}\right.,x_k-L\|,\widetilde{0}) \\
\leq & \left(1-\frac{\lambda_m}{\mu_m}\right)\cdot M \\
& +\frac{M}{\lambda_m}\left|\left\{k\in I_m : D\left(\left\|z_1,z_2,\ldots,z_{n-1},x_k-L\right\|,\ \widetilde{0}\right)\right\}\geq\varepsilon\right|+\varepsilon
\end{aligned}$$

for all $m\in\mathbb{N}_{m_0}$. Using (2.2), we obtain that $x_k \to L\left([V,\mu]_{FnN}\right)$ whenever $x_k \to L\left(FnS_\lambda\right)$. Hence we have $l_\infty \cap FS_\lambda \subseteq [V,\mu]_{FnN}$.

If we take $\mu_m = m$ for all $m\in\mathbb{N}_{m_0}$ in Theorem 2.15 then, we have the following results. As a result of $\lim_{m\to\infty}\frac{\lambda_m}{\mu_m}=1$ implies that $\lim_{m\to\infty} inf\ \frac{\lambda_m}{\mu_m}=1>0$, that is (2.2) $\Rightarrow$ (2.1).

Corollary 2.16. *Let* $\lim_{m\to\infty}\frac{\lambda_m}{\mu_m}=1$ *. Then*

i. If $(x_k)\in l_\infty$ and $x_k \to L\left(FnS_\lambda\right)$ then $x_k \to L\left([C,1]_{FnN}\right)$
ii. If $x_k \to L\left([C,1]_{FnN}\right)$ then $x_k \to L\left(FnS_\lambda\right)$

3. Conclusion

In this study, definitions of FnS_λ-convergent, $[V,\lambda]_{FnN}$ summabilty and $[C,1]_{FnN}$ summability were given in fuzzy n normed spaces. It has been given the relation between $[V,\lambda]_{FnN}$ and $[C,1]_{FnN}$. Also, it has been given statistical convergent equal to λ- statistical convergent for which conditions. Moreover, the requirements for

$$x_k \to L\left([V,\mu]_{FnN}\right) \Rightarrow x_k \to L\left(FnS_\lambda\right)$$

Finally, the following similarities have been shown:
i. Let $\lambda=(\lambda_m)\in\Lambda$. If $\lim_m \frac{\lambda_m}{m}=1$ then we have $FnS_\lambda = FnS$.
ii. If $\lim_{m\to\infty}\frac{\lambda_m}{\mu_m}=1$, $(x_k)\in l_\infty$ and $x_k \to L\left(FnS_\lambda\right)$ then $x_k \to L\left([C,1]_{FnN}\right)$
iii. If $\lim_{m\to\infty}\frac{\lambda_m}{\mu_m}=1$ and $x_k \to L\left([C,1]_{FnN}\right)$ then $x_k \to L\left(FnS_\lambda\right)$

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