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ON UPPER BOUNDS OF $H_{2,1}(f)$ AND $H_{2,2}(f)$ HANKEL DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we give upper bounds of the Hankel determinants $H_{2,1}(f)$ and $H_{2,2}(f)$ for the classes $S^*_{(\lambda,n)}$, where f is analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ and normalized so that $f(z) = z + a_2z^2 + a_3z^3 + \ldots$

1. INTRODUCTION

Let A be the collection of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ and let S denote the subclass of A consisting of functions which are univalent in Δ.

With a view to recalling the principal of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic functions in Δ. Then we say that the function $f(z)$ is subordinate to $g(z)$ in Δ, if there exits a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad (z \in \Delta)$$

such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$.

If g is a univalent function in Δ, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

The famous coefficient conjucture Beiberbach conjucture for the functions $f \in S$ of the form (1.1) was first presented by Beiberbach [25] in 1916 and proven by

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de-Branges [26] in 1985. In between the years 1916 and 1985, many mathematicians worked to prove Beiberbach's conjucture. Consequently, they defined several subclasses of S connected with different image domains.

Among these, the families S^*, C and K of starlike functions and convex functions respectively, are the most fundamental subclasses of S and have a nice geometric interpretation. These families are defined as follows:

$$S^* = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, (z \in E) \right\}$$
$$C = \left\{ f \in S : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, (z \in E) \right\}$$

A function $f \in$ A is said to be starlike of order $\alpha, 0 \leq \alpha < 1$, if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \Delta).$$

We denote this class by $S^*(\alpha)$. If $\alpha = 0$, then $S^*(0) = S^*$ is the well-known class of starlike functions.

By $C(\alpha), -\frac{1}{2} \leq \alpha < 1$, we denote the class Ozaki close-to-convex of functions $f \in$ A for

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta).$$

The special case of this class, when $\alpha = -1/2$ was introduced by Ozaki in 1941 [1] and it is a subclass of the class of close-to-convex functions. This, general form of the class, was introduced [2] by Kargar and Ebadian. We note that for $\alpha = 0$ we have the class of convex functions.

Similarly, by $\wp(\alpha), 0 < \alpha \leq 1$, we denote the class of functions $f \in$ A for which

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \quad (z \in \Delta)$$

Ozaki [1] introduced the class $\wp(1)$ and proved that functions in $\wp(1)$ are univalent in the unit disk. Later, Umezawa [3], Sakaguchi [4] and R.Singh and S.Singh [5] showed, respectively, that functions in $\wp(1)$ are convex in one direction, close-to-convex and starlike.

In the 1960s Pommerenke [6],[7] defined the Hankel determinant $H_{q,n}(f)$ for a given f of the form (1.1) f as follows

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

where $n, q \in \mathbb{N} = \{1, 2, 3, \ldots\}$. In particular,

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{is} \quad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

The studies on Hankel determinants are concentrated on estimating $H_{2,2}(f)$ and $H_{3,1}(f)$ for different subclasses of S. The absolute sharp bounds of the functional $H_{2,2}(f)$ were found in [8],[9] for each of the families S^*, C. In [9], Janteng et al. proved that $|H_{2,2}(f)| \leq 1$ for S^* and $|H_{2,2}(f)| \leq \frac{1}{8}$ for K, where S^* and K are very well known classes of starlike and convex functions. The estimation of the determinant $|H_{3,1}(f)|$ is very hard as compared to deriving the bound of $|H_{2,2}(f)|$. The paper on $|H_{3,1}(f)|$ was given in 2010 by Babalola[10], in which he obtained the upper bound of $H_{3,1}(f)$ for the families of S^*, C. Later on, many researchers published their work regarding $|H_{3,1}(f)|$ for different subclasses of univalent functions. For additional details see[11],[15]. In 2017, Zaprawa [16] improved the results of Babalola. In 2018, Kowalczyk et al.[17] and Lecko et al.[18] obtained the sharp inequalities:

$$|H_{3,1}(f)| \leq \frac{4}{35} \quad \text{and} \quad |H_{3,1}(f)| \leq \frac{1}{9}$$

for the recognizable families K and $S^*\left(\frac{1}{2}\right)$, respectively, where the symbol $S^*\left(\frac{1}{2}\right)$ stands for the family of starlike functions of order $\frac{1}{2}$. Arif M. et al.[19] obtained the upper bound of $|H_{3,1}(f)|$ for the subclasses $S^*_{\sin}$, $C_{\sin}$ and $R_{\sin}$ in in 2019 . In 2019, Shi L. et al.[20] investigated the estimate of $|H_{3,1}(f)|$ for the subclasses S^*_{car}, C_{car} and R_{car} of analytic functions connected with the cardioid domain. In 2019, Zaprawa [21] studied the Hankel determinant for univalent functions related to the exponential function. Additionally, in recent years, S.Verma et al. [28] and D.Breaz et al. [29] have worked on the upper bounds of Hankel determinants.

For $f \in A, n \in \mathbb{N} = \{0, 1, 2, , 3, \dots\}$, the operator $D^n f$ is defined by $D^n : A \to A$ [22]

$$D^0 f(z) = f(z) \quad D^{n+1} f(z) = z\left[D^n f(z)\right]', z \in E.$$

If $f \in A, f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then $\quad D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad z \in E.$

Let $n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\lambda \geq 0$. We let D^n_λ, as denoted in [23], be the operator defined by;

$$D^n_\lambda : A \to A.$$

$$D^0_\lambda f(z) = f(z).$$

$$D^1_\lambda f(z) = (1-\lambda) D^0_\lambda f(z) + \lambda z \left(D^0_\lambda f(z)\right)' = (1-\lambda) f(z) + \lambda z f'(z).$$

$$\dots$$

$$D^{n+1}_\lambda f(z) = (1-\lambda) D^n_\lambda f(z) + \lambda z \left(D^n_\lambda f(z)\right)'.$$

We observe that D^n_λ is a linear operator and for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, we have [24]

$$D^n_\lambda f(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k$$

Now, we define a subclass of analytic functions as follows:

Definition 1.1. *A function $f \in A$ is said to be starlike of order α, for $0 \le \alpha < 1$, if and only if*

$$Re\left[\frac{z\left(D_\lambda^n f(z)\right)'}{D_\lambda^n f(z)}\right] > \alpha \quad (z \in \Delta).$$

We denote this class by $S^*_{(\lambda,n)}(\alpha)$. If $n, \lambda = 0$, then $S^*_{(0,0)}(0) = S^*(\alpha)$ is the well -known class of starlike functions.

2. SOME INEQUALITIES AND MAIN RESULTS

For the function $\psi(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots.$ (with $|\psi(z)| < 1, z \in \Delta$) the next relations is valid (for example [7], expression (13) on page 128):

$$\begin{aligned} |c_1| &\le 1. & (2.1)\\ |c_2| &\le 1 - |c_1|^2. & (2.2)\\ \left|c_3\left(1-|c_1|^2\right) + \overline{c_1}c_2^2\right| &\le \left(1-|c_1|^2\right)^2 - |c_2|^2. & (2.3)\end{aligned}$$

From (2.3), we can write

$$\begin{aligned} \left|c_3\left(1-|c_1|^2\right) + \overline{c_1}c_2^2\right| \le \left(1-|c_1|^2\right)^2 - |c_2|^2 \Rightarrow \\ \left|c_3\left(1-|c_1|^2\right)\right| - \left|\bar{c}_1 c_2^2\right| \le \left|c_3\left(1-|c_1|^2\right) + \overline{c_1}c_2^2\right| \le \left(1-|c_1|^2\right)^2 - |c_2|^2 \Rightarrow \\ \left|c_3\left(1-|c_1|^2\right)\right| - \left|\overline{c_1}c_2^2\right| \le \left(1-|c_1|^2\right)^2 - |c_2|^2 \Rightarrow \\ \left|c_3\left(1-|c_1|^2\right)\right| \le \left(1-|c_1|^2\right)^2 - |c_2|^2 + |\overline{c_1}|\left|c_2^2\right| \Rightarrow \\ |c_3| \le \frac{\left(1-|c_1|^2\right)^2 - |c_2|^2 + |\overline{c_1}|\left|c_2^2\right|}{\left|\left(1-|c_1|^2\right)\right|} \Rightarrow \end{aligned}$$

$$|c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{(1+|c_1|)} \tag{2.4}$$

Let $0 \le c_1 \le 1$. In this case, since $0 \le c_2 \le 1$, from (2.4) is written

$$|c_3| \le 1 - c_1^2 - \frac{c_2^2}{1+c_1}. \tag{2.5}$$

Let's take $c_1 = x$ and $c_2 = y$ to get the maksimum value of the right side of (2.5). Let's take

$$\varphi(x,y) = 1 - x^2 - \frac{y^2}{1+x}.$$

Let's calculate the maximum value of the bivariate function

$$\left.\begin{array}{r}\varphi_x(x,y)=-2x+\frac{y^2}{(1+x)^2}\\ \varphi_y(x,y)=-\frac{2y}{(1+x)}\end{array}\right\}\Rightarrow x=0,y=0$$

$$\varphi_{xx}(x,y)=-2-\frac{2y^2}{(1+x)^3},\quad \varphi_{yy}(x,y)=-\frac{2}{(1+x)},\quad \varphi_{xy}(x,y)=\frac{2y}{(1+x)^2}$$

According to the values of the second-order partial derivatives at the point $(0,0)$, the following inequalities are written as

$$[\varphi_{xy}(0,0)]^2-\varphi_{xx}(0,0)\varphi_{yy}(0,0)=-4<0 \quad \text{and} \quad \varphi_{xx}(0,0)=-2<0$$

then the point $(0,0)$ is a maximum of $\varphi(x,y)$ and

$$\max\varphi(x,y)=1.$$

So,

$$|c_3|\le 1 \tag{2.6}$$

is obtained.

Theorem 2.1. *Let $f(z)=z+a_2z^2+a_3z^3+\dots$ belongs to the class $S^*_{(\lambda,n)}(\alpha), 0\le\alpha<1$. Then we have the coefficients estimation as follows.*

$$|a_2|\le\frac{2(1-\alpha)}{(1+\lambda)^n},\quad |a_3|\le\frac{(1-\alpha)(3-2\alpha)}{(1+2\lambda)^n},\quad |a_4|\le\frac{2(1-\alpha)\left(2\alpha^2-7\alpha+7\right)}{3(1+3\lambda)^n}$$

Proof. From the definition of the class $S^*_{(\lambda,n)}$, we have

$$\frac{z\left(D^n_\lambda f(z)\right)'}{D^n_\lambda f(z)}=\alpha+(1-\alpha)\frac{1+\psi(z)}{1-\psi(z)}=1+2(1-\alpha)\left\{\psi(z)+\psi^2(z)+\psi^3(z)+\dots\right\} \tag{2.7}$$

where ψ is analytic in Δ with $\psi(0)=0$ and $|\psi(z)|<1, z\in\Delta$.
Let $\psi(z)=c_1z+c_2z^2+c_3z^3+\dots$. From (2.7), we have

$$\begin{aligned}\frac{z\left(D^n_\lambda f(z)\right)'}{D^n_\lambda f(z)} &= \frac{z+2(1+\lambda)^na_2z^2+3(1+2\lambda)^na_3z^3+4(1+3\lambda)^na_4z^4+\dots}{z+(1+\lambda)^na_2z^2+(1+2\lambda)^na_3z^3+(1+3\lambda)^na_4z^4+\dots}\\ &= 1+(1+\lambda)^na_2z+\left[2(1+2\lambda)^na_3-(1+\lambda)^{2n}a_2^2\right]z^2\\ &+ \left[3(1+3\lambda)^na_4-3(1+2\lambda)^n(1+\lambda)^na_2a_3+(1+\lambda)^{3n}a_2^3\right]z^3\\ &+ \cdots\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&1+2(1-\alpha)\left\{\psi(z)+\psi^2(z)+\psi^3(z)+\dots\right\}\\ &\quad=1+2(1-\alpha)\left\{c_1z+\left(c_2+c_1^2\right)z^2+\left(c_3+2c_1c_2+c_1^3\right)z^3+\dots\right\}.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), comparing the coefficients on z, z^2, z^3 in (2.7) and doing necessary calculations, finally we obtain

$$a_2 = \frac{2(1-\alpha)c_1}{(1+\lambda)^n} \tag{2.10}$$

$$a_3 = \frac{(1-\alpha)\left\{c_2 + (3-2\alpha)c_1^2\right\}}{(1+2\lambda)^n} \tag{2.11}$$

$$a_4 = \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{c_3 + (5-3\alpha)c_1c_2 + \left(2\alpha^2 - 7\alpha + 6\right)c_1^3\right\}. \tag{2.12}$$

If (2.1) is substituted in (2.10), we write

$$|a_2| = \left|\frac{2(1-\alpha)c_1}{(1+\lambda)^n}\right| = \frac{2(1-\alpha)}{(1+\lambda)^n}|c_1| \leq \frac{2(1-\alpha)}{(1+\lambda)^n}. \tag{2.13}$$

and if (2.2) is substituted is in (2.11), we also write

$$\begin{aligned}
|a_3| &= \left|\frac{(1-\alpha)\left\{c_2 + (3-2\alpha)c_1^2\right\}}{(1+2\lambda)^n}\right| = \frac{(1-\alpha)}{(1+2\lambda)^n}\left|\left\{c_2 + (3-2\alpha)c_1^2\right\}\right| \\
&\leq \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{|c_2| + (3-2\alpha)\left|c_1^2\right|\right\} \\
&\leq \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{1 - |c_1|^2 + (3-2\alpha)|c_1|^2\right\} \\
&= \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{1 + 2(1-\alpha)|c_1|^2\right\} \\
&= \frac{(1-\alpha)(3-2\alpha)}{(1+2\lambda)^n}.
\end{aligned}$$

By means of the similar operations we can obtain an upper bound for $|a_4|$ as follows,

$$\begin{aligned}
a_4 &= \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{c_3 + (5-3\alpha)c_1c_2 + \left(2\alpha^2 - 7\alpha + 6\right)c_1^3\right\} \Rightarrow \\
|a_4| &= \left|\frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{c_3 + (5-3\alpha)c_1c_2 + \left(2\alpha^2 - 7\alpha + 6\right)c_1^3\right\}\right| \Rightarrow \\
|a_4| &= \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left|c_3 + (5-3\alpha)c_1c_2 + \left(2\alpha^2 - 7\alpha + 6\right)c_1^3\right| \Rightarrow \\
|a_4| &\leq \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{|c_3| + (5-3\alpha)|c_1||c_2| + \left(2\alpha^2 - 7\alpha + 6\right)\left|c_1^3\right|\right\} \\
&\leq \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{|c_3| + (5-3\alpha)c_1\left(1 - c_1^2\right) + \left(2\alpha^2 - 7\alpha + 6\right)c_1^3\right\} \Rightarrow \\
&\leq \frac{2(1-\alpha)}{3(1+3\lambda)^n}\left\{1 + (5-3\alpha) + \left(2\alpha^2 - 4\alpha + 1\right)\right\} \Rightarrow \\
|a_4| &\leq \frac{2(1-\alpha)\left(2\alpha^2 - 7\alpha + 7\right)}{3(1+3\lambda)^n}.
\end{aligned}$$

□

Theorem 2.2. *Let $f(z) = z + a_2z^2 + a_3z^3 + \dots$ belongs to the class $S^*_{(\lambda,n)}(\alpha), 0 \leq \alpha < 1$. Then we have next sharp estimation:*

$$|H_{2,1}(f)| \leq (1-\alpha)\left\{\frac{3}{(1+2\lambda)^n} + \frac{4(1-\alpha)}{(1+\lambda)^{2n}}\right\} \leq \frac{(1-\alpha)(7-6\alpha)}{(1+2\lambda)^n}$$

Proof. From the definition of Hankel determinant has the form of $H_{2,1}(f) = a_3 - a_2^2$. In this definition by using (2.10) and (2.11) and taking module $H_{2,1}(f)$ is written as

$$H_{2,1}(f) = a_3 - a_2^2 = \frac{(1-\alpha)\left\{c_2 + (3-2\alpha)c_1^2\right\}}{(1+2\lambda)^n} - \left\{\frac{2(1-\alpha)c_1}{(1+\lambda)^n}\right\}^2 \Rightarrow$$

$$|H_{2,1}(f)| = \left|a_3 - a_2^2\right| \leq \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{|c_2| + (3-2\alpha)\left|c_1^2\right|\right\} + \frac{4(1-\alpha)^2}{(1+\lambda)^{2n}}\left|c_1^2\right| \Rightarrow$$
$$|H_{2,1}(f)| \leq \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{1 - \left|c_1^2\right| + (3-2\alpha)\left|c_1^2\right|\right\} + \frac{4(1-\alpha)^2}{(1+\lambda)^{2n}}\left|c_1^2\right| \Rightarrow$$
$$|H_{2,1}(f)| \leq \frac{(1-\alpha)}{(1+2\lambda)^n}\left\{1 + 2(1-\alpha)\left|c_1^2\right|\right\} + \frac{4(1-\alpha)^2}{(1+\lambda)^{2n}}\left|c_1^2\right|.$$

Thus, from the (2.1)

$$|H_{2,1}(f)| \leq \frac{(1-\alpha)(3-2\alpha)}{(1+2\lambda)^n} + \frac{4(1-\alpha)^2}{(1+\lambda)^{2n}}$$

is obtained. Taking into account the following inequality

$$(1+\lambda)^{2n} \geq (1+2\lambda)^n \Rightarrow \frac{1}{(1+2\lambda)^n} \geq \frac{1}{(1+\lambda)^{2n}}$$

finally, it also can be written as

$$|H_{2,1}(f)| \leq \frac{(1-\alpha)(3-2\alpha)}{(1+2\lambda)^n} + \frac{4(1-\alpha)^2}{(1+2\lambda)^n} = \frac{(1-\alpha)(7-6\alpha)}{(1+2\lambda)^n}$$

□

Theorem 2.3. *Let $f(z) = z + a_2z^2 + a_3z^3 + \dots$ belongs to the class $S^*_{(x,n)}(\alpha), 0 \leq \alpha < 1$. Then we have next sharp estimation:*

$$|H_{2,2}(f)| \leq \frac{(1-\alpha)^2\left(20\alpha^2 - 64\alpha + 51\right)}{3(1+\lambda)^{2n}}.$$

Proof. From the definition of Hankel determinant has the form of $H_{2,2}(f) = a_2a_4 - a_3^2$. In this definition by using (2.10), (2.11) and (2.12) and taking module $H_{2,2}(f)$ is written as

$$H_{2,2}(f)=\left\{\frac{2(1-\alpha)c_1}{(1+\lambda)^n}\right\}\frac{2(1-\alpha)}{3(1+3\lambda)^n}$$

$$\left\{c_3+(5-3\alpha)c_1c_2+\left(2\alpha^2-7\alpha+6\right)c_1^3\right\}-\left[\frac{(1-\alpha)\left\{c_2+(3-2\alpha)c_1^2\right\}}{(1+2\lambda)^n}\right]^2$$

$$=\frac{4(1-\alpha)^2}{3(1+\lambda)^n(1+3\lambda)^n}\left\{c_3c_1+(5-3\alpha)c_1^2c_2+\left(2\alpha^2-7\alpha+6\right)c_1^4\right\}$$

$$-\frac{(1-\alpha)^2}{(1+2\lambda)^{2n}}\left\{c_2+(3-2\alpha)c_1^2\right\}^2\Rightarrow$$

$$|H_{2,2}(f)|=\frac{4(1-\alpha)^2}{3(1+\lambda)^n(1+3\lambda)^n}$$

$$\left|c_3c_1+(5-3\alpha)c_1^2c_2+\left(2\alpha^2-7\alpha+6\right)c_1^4\right|+\frac{(1-\alpha)^2}{(1+2\lambda)^{2n}}\left|\left\{c_2+(3-2\alpha)c_1^2\right\}^2\right|$$

Taking into account the following inequality

$$1+\lambda\le 1+2\lambda\le 1+3\lambda\Rightarrow\frac{1}{(1+3\lambda)}\le\frac{1}{(1+2\lambda)}\le\frac{1}{(1+\lambda)}$$

finally, it also can be written as

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{4\left|c_3c_1+(5-3\alpha)c_1^2c_2+\left(2\alpha^2-7\alpha+6\right)c_1^4\right|+3\left|\left[c_2+(3-2\alpha)c_1^2\right]^2\right|\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4|c_3||c_1|+4(5-3\alpha)\left|c_1^2\right||c_2|+4\left(2\alpha^2-7\alpha+6\right)\left|c_1^4\right|+\\ 3\left|c_2^2+2(3-2\alpha)c_2c_1^2+(3-2\alpha)^2c_1^4\right|\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4|c_3||c_1|+4(5-3\alpha)\left|c_1^2\right||c_2|+4\left(2\alpha^2-7\alpha+6\right)\left|c_1^4\right|+3\left|c_2^2\right|+\\ (18-12\alpha)\left|c_1^2\right||c_2|+3(3-2\alpha)^2\left|c_1^4\right|\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{4|c_3||c_1|+(38-24\alpha)\left|c_1^2\right||c_2|+\left(20\alpha^2-64\alpha+51\right)\left|c_1^4\right|+3\left|c_2^2\right|\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4c_1\left\{1-c_1^2-\frac{|c_2|^2}{1+c_1}\right\}+(38-24\alpha)\left(1-c_1^2\right)c_1^2+\\ \left(20\alpha^2-64\alpha+51\right)c_1^4+3\left|c_2^2\right|\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4c_1\left(1-c_1^2\right)+\left(\frac{3-c_1}{1+c_1}\right)|c_2|^2+(38-24\alpha)\left(1-c_1^2\right)c_1^2+\\ \left(20\alpha^2-64\alpha+51\right)c_1^4\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4c_1\left(1-c_1^2\right)+\left(\frac{3-c_1}{1+c_1}\right)\left(1-c_1^2\right)^2+\\ (38-24\alpha)c_1^2-(38-24\alpha)c_1^4+\left(20\alpha^2-64\alpha+51\right)c_1^4\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\begin{array}{l}4c_1\left(1-c_1^2\right)+\left(\frac{3-c_1}{1+c_1}\right)\left(1-c_1^2\right)(1+c_1)(1-c_1)+\\ (38-24\alpha)c_1^2+\left(20\alpha^2-40\alpha+13\right)c_1^4\end{array}\right\}\Rightarrow$$

$$|H_{2,2}(f)|\le\frac{(1-\alpha)^2}{3(1+\lambda)^{2n}}\left\{\left(20\alpha^2-40\alpha+12\right)c_1^4+(36-24\alpha)c_1^2+3\right\}$$

C_1		$-\sqrt{-\frac{B}{2A}}$	0	1	$\sqrt{-\frac{B}{2A}}$
C_1	$-$	$-$	$+$	$+$	$++$
$2AC_1^2+B$ $A<0$	$-$	$+$	$+$	$+$	$--$
$2AC_1^2+B$ $A>0$	$+$	$+$	$+$	$+$	$++$
$C_1(2AC_1^2+B)$ $A<0$	$+$	$+$	$+$	$+$	$--$
$C_1(2AC_1^2+B)$ $A>0$	$-$	$-$	$+$	$+$	$++$
	////////////////////////////////		/////////////////////////////////		

Let $\psi(c_1) = \left(20\alpha^2 - 40\alpha + 12\right) c_1^4 + (36 - 24\alpha)c_1^2 + 3$. For convenience, saying $A = 20\alpha^2 - 40\alpha + 12$ and $B = 36 - 24\alpha$ then $\psi(c_1)$ takes the form $\psi(c_1) = Ac_1^4 + Bc_1^2 + 3$.

If the derivative is taken and set to zero, the followings are obtained.

$$\psi'(c_1) = 4Ac_1^3 + 2Bc_1 \Rightarrow 2c_1\left(2Ac_1^2 + B\right) = 0; c_1 = 0 \text{ and } 2Ac_1^2 + B = 0 \Rightarrow \quad c_1^2 = -\frac{B}{2A}$$

Since, $0 \le \alpha < 1$ and $B = 36 - 24\alpha > 0$ then $-B < 0$. From the inequality $c_1^2 = -\frac{B}{2A}$, having two distinct real roots is possible for the case $A = 20\alpha^2 - 40\alpha + 12 < 0$. That is, since $A < 0$ while $1 - \frac{\sqrt{10}}{5} < \alpha < 1$, there exist two distinct real roots such that $c_1 = \mp\sqrt{-\frac{B}{2A}}$. Otherwise, that is since $A > 0$ while $0 \le \alpha < 1 - \frac{\sqrt{10}}{5}$ there is no real number satisfying the condition $c_1^2 = -\frac{B}{2A}$. Accordingly, the following table can be organized.

Considering the table, the maximum value of the $\psi(c_1)$

$$\psi(c_1) = \left(20\alpha^2 - 40\alpha + 12\right) c_1^4 + (36 - 24\alpha)c_1^2 + 3.$$

will be

$$\psi(1) = 20\alpha^2 - 64\alpha + 51$$

Thus, $|H_{2,2}(f)|$ is obtained as follows

$$|H_{2,2}(f)| \le \frac{(1-\alpha)^2\left(20\alpha^2 - 64\alpha + 51\right)}{3(1+\lambda)^{2n}}.$$

□

CONCLUSIONS: A new subclass of analytic functions of denoted by $S^*_{(\lambda,n)}$ has been introduced by means D^n_λ is a linear operator . we give upper bounds of the

$H_{2,1}(f)$ and $H_{2,2}(f)$ Hankel determinants created from the coefficients of functions belonging to class $S^*_{(\lambda,n)}$

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