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APPROXIMATION BY AN INTEGRAL TYPE APOSTOL-GENOCCHI OPERATORS

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Abstract. The goal of the current paper is to present an integral type Favard-Szasz operators including Apostol-Genocchi poynomials. With the help of the moments, we investigate the order of convergence in terms of the first and the second order modulus of continuity and Peetres K-functional. We also examine the convergence in the weighted spaces of functions by means of weighted Korovkin type theorem.

1. Introduction

Towards the end of the 19^{th} century, Weierstrass established a very significant theorem on the approximation of continuous functions. He proved that every continuous function on a closed and finite interval $[a, b]$ can be approximated uniformly by polynomial sequences. In the very early 1900's, Bernstein proved Weierstrass' theorem in a much more comprehensible way by giving not only the existence but also the notation of the polynomial sequence. In 1953, Korovkin [12] presented a very important theorem on the uniform convergence of the linear positive operators to functions continuous on a closed interval $[a, b]$. In this theorem, known as Korovkin's theorem in the literature, only three conditions are checked in order to guarantee the uniform convergence of the linear positive operators to the continuous functions f in $[a, b]$. The ease of application of Korovkin's theorem has allowed several authors to define new linear positive operators and study their approximation properties. Over the years, various generalizations of previously defined operators have been constructed and approximation properties of these new operators have been studied. The well known Szász-Mirakjan operators, defined by Otto Szász [20] in 1950, is a generalization of Bernstein operators to an infinite interval $[0, \infty)$. These operators are of the form

$$\left(S_n f\right)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{1.1}$$

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In order to obtain approximation process in the space of Lebesque integrable functions, Mazhar and Totik [15] modified the Szász operators and gave the following Szász-Durrmeyer type linear positive operators as

$$(S_n^* f)(x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt. \tag{1.2}$$

Durrmeyer variants of several operators have been constructed and still continue to be studied today by many authors. The operators given in (1.2) is an example of an integral generalization of the operators. Now we will mention some linear positive operators which are constructed with the help of generating functions. Jakimovski and Leviatan [10] were the first to use Appell polynomials and their generating functions in constructing a generalization of Szász operators. They defined the operators as

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.3}$$

with $f \in \tilde{E}[0,\infty)$. Here $\tilde{E}[0,\infty)$ indicates the set of functions satisfying the property $|f(x)| \leq e^{Ax}$ for each $x \geq 0$ and some finite number A. Here $p_k(nx)$ are the Appell polynomials whose generating function is given by

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x) u^k,$$

where $g(u) = \sum\limits_{n=0}^{\infty} a_n u^n$ is an analytic function in the disc $|u| < r$, $(r > 1)$ and $g(1) \neq 0$.

In 1995, Ciupa [2] modified the Jakimovski-Leviatan operators for the functions f, Lebesque integrable in $[0,\infty)$. The operators are given by

$$(P_n^* f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt, \tag{1.4}$$

and this study of Ciupa became a pioneer for defining new operators using the Gamma function. For some recent studies on linear positive operators including Appell polynomials or operators constructed with the help of Gamma and beta functions, we refer to papers [1, 3, 9, 11, 14, 19, 21].

Recently, Parakash et. al. [18] have proposed a new sequence of operators with the help of Apostol-Genocchi polynomials. For $f \in C[0,\infty)$, these operators are constructed as

$$M_n^{\alpha,\beta}(f;x) = e^{-nx} \left(\frac{1+e\beta}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{G_k^{(\alpha)}(nx;\beta)}{k!} f(k/n) \tag{1.5}$$

where $G_k^{(\alpha)}(x;\beta)$ is generalized Apostol-Genocchi polynomials having the generating function of the form

$$\left(\frac{2z}{\beta e^z + 1}\right)^{\alpha} e^{xz} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x;\beta) \frac{z^k}{k!}. \qquad (\alpha, \beta \in \mathbb{C}, |t| < \pi) \tag{1.6}$$

With the help of the Gaussian hypergeometric function

$$ {}_2F_1(a,b;c;t)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{t^k}{k!}, $$

where, $(\eta)_0=1,(\eta)_n=\eta(\eta+1)...(\eta+n-1)=\frac{\Gamma(n+\eta)}{\Gamma(\eta)}, n\geq 1$, the explicit formula for the Apostol-Genocchi polynomials is given in [13] by Luo and Srivastava as,

$$ \begin{aligned} G_k^{(\alpha)}(x;\alpha)&=2^{\alpha}(\alpha!)\binom{k}{\alpha}\sum_{n=0}^{k-\alpha}\binom{k-\alpha}{n}\binom{\alpha+n-1}{n}\frac{\beta^n}{(1+\beta)^{\beta+n}}\\ &\times\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n(x+j)^{k-n-\alpha}\ {}_2F_1(\alpha+n-k,n;n+1;\frac{j}{x+j}). \end{aligned} $$

Immediately after the aforementioned study of Prakash et. al. [18], various generalizations of the operators started to appear. N. Deo et. al. [4, 5, 6] introduced Durrmeyer type generalizations of the operators (1.5) in several papers. For example, in [4] and [5], they used Jain and Baskakov operators, respectively, in order to modify the operators (1.5). Similarly in [6], they studied the approximation properties of the Beta-Apostol-Genocchi operators. M.M. Yilmaz [22] gave a generalization of (1.5), by adding new parameters to the operators. And lastly, in a very recent paper Mishra and Deo [16] constructed again Durrmeyer variant of the operators, this time using the Paltanea basis.

Inspired by the studies summarized above, especially from [2] and [18] we introduce a Durrmeyer form generalization of Apostol-Genocchi operators as

$$ D_n^{(\alpha)}(f;x)=n\sum_{k=0}^{\infty}v_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\int_0^{\infty}e^{-ns}(ns)^{\gamma+k}f(s)ds, x\geq 0, \tag{1.7} $$

where the parameter $\gamma\geq 0$, Γ is a Gamma function and

$$ v_{n,k}^{\alpha}=e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha}\frac{G_k^{(\alpha)}(nx;\beta)}{k!}. \tag{1.8} $$

The article will be proceed as follows: we construct the operators with the help of the Gamma function and it will be followed by the calculation of moments. We establish approximation results using the Korovkin's theorem and estimate the rate of convergence with the help of first and second modulus of continuity and Peetre's K-functional. We also discuss weighted approximation results for these operators.

2. Approximation Properties of the Operators

The Lemma given below is relevant for studying the convergence of the operator $D_n^{(\alpha)}(f)$ to the function f. Remember that $C_B[0,\infty)$ denotes the space of all real-valued uniformly continuous and bounded functions on the positive real axis $\mathbb{R}^+$ endowed with the norm $\|g\|:=\sup_{x\in[0,\infty)}|g(x)|$.

Lemma 2.1. *Letting $e_s = t^s$, for all $x \geq 0$, we have*

$$D_n^{(\alpha)}(e_0;x) = 1, \tag{2.1}$$

$$D_n^{(\alpha)}(e_1;x) = x + \frac{1}{n}\left((\gamma+1) + \frac{\alpha}{1+\beta e}\right), \tag{2.2}$$

$$D_n^{(\alpha)}(e_2;x) = x^2 + \frac{x}{n}\left((2\gamma+3) + \frac{1+2\alpha+\beta e}{1+\beta e}\right), \tag{2.3}$$

$$+ \frac{1}{n^2}\left((\gamma+1)(\gamma+2) + (2\gamma+3)\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{n^2(1+\beta e)^2}\right).$$

Proof. The proof of the theorem is obvious from the identities

$$\sum_{k=0}^{\infty} \frac{G_k^{(\alpha)}(nx;\beta)}{k!}k = e^{nx}\left(\frac{2}{1+\beta e}\right)^{\alpha}[nx + \frac{\alpha}{1+\beta e}]$$

$$\sum_{k=0}^{\infty} \frac{G_k^{(\alpha)}(nx;\beta)}{k!}k^2 = e^{nx}\left(\frac{2}{1+\beta e}\right)^{\alpha}\left[n^2x^2 + \frac{nx(1+2\alpha+\beta e)}{1+\beta e} + \frac{\alpha^2 - 2\alpha\beta e - \alpha\beta^2 e^2}{(1+\beta e)^2}\right]$$

previously obtained in [18]. □

By making use of the above Lemma, we can give the following remark.

Remark. *Letting $\upsilon_n^{(i)}(x) = D_n^{(\alpha)}((e_1 - x)^i;x)$ and using the above Lemma, the first two central moments of the operators $D_n^{(\alpha)}(f)$ can be obtained as:*

$$\upsilon_n^{(1)}(x) = \frac{1}{n}\left((\gamma+1) + \frac{\alpha}{1+\beta e}\right)$$

$$\upsilon_n^{(2)}(x) = \frac{1}{n^2}(\gamma+1)(\gamma+2) + \frac{2x}{n} + \frac{(2\gamma+3)}{n^2}\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{n^2(1+\beta e)^2}.$$

Lemma 2.2. *For the operators $D_n^{(\alpha)}(f)$, the following holds for all $g \in C_B[0,\infty)$*

$$|D_n^{(\alpha)}(g;x)| \leq \|g\|.$$

Proof.

$$\begin{aligned} |D_n^{(\alpha)}(g;x)| &= n\left|\sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\int_0^{\infty} e^{-ns}(ns)^{\gamma+k}g(s)ds\right| \\ &\leq \|g\|\sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x)\frac{n}{\Gamma(\gamma+k+1)}\int_0^{\infty} e^{-ns}(ns)^{\gamma+k}ds \\ &= \|g\|D_n^{(\alpha)}(1;x). \end{aligned}$$

The proof is completed from Lemma 2.1. □

Theorem 2.3. *Let $f \in C[0,\infty)\bigcap E$, where E defines the class*

$$E := \left\{f : x \in C[0,\infty), |f(x)| \leq Me^{Ax} \text{for some positive constant } M,A\right\}.$$

Then $D_n^{(\alpha)}(f)$ converges uniformly on every compact subset of $[0,\infty)$.

Proof. According to Lemma 2.1, we obtain

$$\lim_{n\to\infty} D_n^{(\alpha)}(t^i;x) = x^i, \quad \text{for} \quad i = 0,1,2.$$

Hence, the proof is obvious from the well-known Korovkin's Theorem. □

We now recall the first and the second modulus of continuity of a function $f \in C_B[0,\infty)$. They are defined as,

$$w(f;\delta) = \sup_{0\leq h\leq\delta}\ \sup_{x\in[0,\infty]} |f(x+h) - f(x)|$$

and

$$w_2(f;\delta) = \sup_{0\leq h\leq\delta}\ \sup_{x\in[0,\infty]} |f(x+2h) - 2f(x+h) + f(x)|,$$

respectively. The Peetre's K-functional of the function $f \in C_B[0,\infty)$ is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \left\{ \|f-g\|_{C_B[0,\infty)} + \delta \|g\|_{C_B^2[0,\infty)} \right\}. \tag{2.4}$$

Here $C_B^2[0,\infty)$ is the space of functions f such that $f, f', f'' \in C_B[0,\infty)$. i.e.

$$C_B^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\}.$$

The norm on C_B^2 is defined as

$$\|g\|_{C_B^2} = \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}. \tag{2.5}$$

It is already known that the following inequality holds for all $\delta \geq 0$:

$$K(f;\delta) \leq C\left\{ w_2(f;\sqrt{\delta}) + \min\{1,\delta\}\|f\| \right\}, \tag{2.6}$$

where C is a positive constant [2].

We also need the following Lemma for the proof of an approximation theorem given below.

Lemma 2.4. *([8]) Let $u \in C^2[0,\infty)$ and R_n, $n \geq 0$ be a sequence of positive linear operators with the property $R_n(1;x) = 1$. Then*

$$|R_n(u;x) - u(x)| \leq \|u'\|\sqrt{R_n((t-x)^2;x)} + \frac{1}{2}\|u''\|R_n((t-x)^2;x).$$

Now we can give the following approximation results concerning the first and the second modulus of continuity.

Theorem 2.5. *Let $f \in C_B[0,\infty)$. For the operators $D_n^{(\alpha)}(f)$, and for $x \in [0,\infty)$ the following inequality holds:*

$$\left|D_n^{(\alpha)}(f;x) - f(x)\right| \leq$$
$$\left[1 + \sqrt{2x + \frac{1}{n}\left((\gamma+1)(\gamma+2) + (2\gamma+3)\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{n^2(1+\beta e)^2}\right)}\right] w(f;\frac{1}{\sqrt{n}}).$$

Proof. From the definition of the operator $D_n^{(\alpha)}(f)$ and the modulus of continuity function, we can write

$$\begin{aligned}
\left|D_n^{(\alpha)}(f;x) - f(x)\right| &\leq n\sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\int_0^{\infty} e^{-ns}(ns)^{\gamma+k}|f(s) - f(x)|\,ds \\
&\leq n\sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\int_0^{\infty} e^{-ns}(ns)^{\gamma+k}\left(1 + \frac{1}{\delta}|s-x|\right)w(f;\delta)ds \\
&\leq \sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\left(1 + \frac{n}{\delta}\frac{1}{\Gamma(\gamma+k+1)}\int_0^{\infty} e^{-ns}(ns)^{\gamma+k}|s-x|\,ds\right)w(f;\delta).
\end{aligned}$$

Applying Cauchy-Schwarz inequality to the integral term yields

$$\int_0^\infty e^{-ns}(ns)^{\gamma+k}\,|s-x|\,ds \le \frac{1}{n}\Gamma(\gamma+k+1)\sqrt{x^2-\frac{2x}{n}(\gamma+k+1)+\frac{(\gamma+k+1)(\gamma+k+2)}{n^2}}$$

from which we get

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le w(f;\delta)\left(1+\frac{1}{\delta}\sum_{k=0}^{\infty}v_{n,k}^{\alpha}(x)\sqrt{x^2-\frac{2x}{n}(\gamma+k+1)+\frac{(\gamma+k+1)(\gamma+k+2)}{n^2}}\right).$$

Again the application of Cauchy-Schwarz inequality to the sum term gives us

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le \left[1+\frac{1}{\delta}\sqrt{\sum_{k=0}^{\infty}v_{n,k}^{\alpha}(x)\left(x^2-\frac{2x}{n}(\gamma+k+1)+\frac{(\gamma+k+1)(\gamma+k+2)}{n^2}\right)}\right]w(f;\delta)$$

and from the identities given in the proof of Lemma 2.1, we can rewrite the above inequality as

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le$$
$$\left[1+\frac{1}{\delta}\frac{1}{\sqrt{n}}\sqrt{2x+\frac{1}{n}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+\beta e)^2}\right)}\right]w(f;\delta)$$

and finally choosing $\delta=\frac{1}{\sqrt{n}}$, we get the desired result. □

Theorem 2.6. *If $f\in C[0,a]$, then for any $x\in[0,a]$, we have*

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le \frac{2}{a}\|f\|\,h^2+\frac{3}{4}(2+a+h^2)w_2(f;h)$$

where $h=\sqrt[4]{D_n^{(\alpha)}((e_1-x)^2;x)}=\sqrt[4]{\frac{2x}{n}+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+\beta e)^2}\right)}.$

Proof. Let f_h be the second-order Stieklov function attached to the function f.

Since $D_n^{(\alpha)}(1;x)=1$, we can write

$$\begin{aligned}\left|D_n^{(\alpha)}(f;x)-f(x)\right| &\le \left|D_n^{(\alpha)}(f-f_h;x)\right|+\left|D_n^{(\alpha)}(f_h;x)-f_h(x)\right|+|f_h(x)-f(x)|\\ &\le 2\|f-f_h\|+\left|D_n^{(\alpha)}(f_h;x)-f_h(x)\right|. \qquad (2.7)\end{aligned}$$

Considering the fact that $f_h\in C^2[0,a]$ and using Lemma 2.4, we obtain

$$\left|D_n^{(\alpha)}(f_h;x)-f_h(x)\right| \le \|f_h'\|\sqrt{D_n^{(\alpha)}((t-x)^2;x)}+\frac{1}{2}\|f_h''\|D_n^{(\alpha)}((t-x)^2;x). \quad (2.8)$$

The result obtained by Zhuk [23] states that, if $f\in C[a,b]$ and $h\in(0,\frac{b-a}{2})$, then the following inequalities are achieved:

$$\|f-f_h\|\le\frac{3}{4}w_2(f;h) \quad\text{and}\quad \|f_h''\|\le\frac{3}{2}\frac{1}{h^2}w_2(f;h). \qquad (2.9)$$

The Landau's inequality

$$\|f_h'\|\le\frac{2}{a}\|f_h\|+\frac{a}{2}\|f_h''\|$$

and the inequalities in (2.9) implies,

$$\|f_h'\|\le\frac{2}{a}\|f\|+\frac{3a}{4}\frac{1}{h^2}w_2(f,h).$$

Using the above inequality in (2.8) and then choosing $h = \sqrt[4]{D_n^{(\alpha)}((t-x)^2;x)}$, we obtain

$$\left|D_n^{(\alpha)}(f_h;x) - f_h(x)\right| \leq \left(\frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}w_2(f,h)\right)h^2 + \frac{1}{2}\frac{3}{2}\frac{1}{h^2}w_2(f;h)h^4. \quad (2.10)$$

Substituting the last inequality into (2.7), we get

$$\begin{aligned}\left|D_n^{(\alpha)}(f;x) - f(x)\right| &\leq 2\|f - f_h\| + \left|D_n^{(\alpha)}(f_h;x) - f_h(x)\right| \\ &\leq 2\frac{3}{4}w_2(f,h) + \frac{2}{a}\|f\|h^2 + \frac{3a}{4}w_2(f,h) + \frac{3}{4}w_2(f;h)h^2, \quad (2.11)\end{aligned}$$

which gives the desired result as the terms arranged. □

An estimation for the smooth functions is given in the below theorem:

Theorem 2.7. *For functions $f \in C_B^2[0,\infty)$, we have*

$$\left|D_n^{(\alpha)}(f;x) - f(x)\right| \leq \eta_n(x)\,\|f\|_{C_B^2[0,\infty)} \quad (2.12)$$

where $\eta_n(x) = \frac{1}{n}\left\{x + \frac{1}{2}\left\{(\gamma+1)(\gamma+2) + (2\gamma+3)\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{(1+\beta e)^2}\right\}\right\}$.

Proof. By using Taylor's expansion of $f \in C_B^2[0,\infty)$ and the linearity of f, we can write

$$D_n^{(\alpha)}(f;x) - f(x) = f'(x)D_n^{(\alpha)}((t-x);x) + \frac{1}{2}f''(\xi)D_n^{(\alpha)}((t-x)^2;x), \quad \xi \in (t,x).$$

Substituting the central moments of the operators given in Remark into the above equation, we get

$$\begin{aligned}\left|D_n^{(\alpha)}(f;x) - f(x)\right| &\leq \frac{1}{n}\left((\gamma+1) + \frac{\alpha}{1+\beta e}\right)\|f'\|_{C_B} \\ &+ \frac{1}{2}\left\{\frac{1}{n^2}(\gamma+1)(\gamma+2) + \frac{2x}{n} + \frac{(2\gamma+3)}{n^2}\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{n^2(1+\beta e)^2}\right\}\|f''\|_{C_B}.\end{aligned}$$

For sufficiently large n, we have

$$\begin{aligned}&\left|D_n^{(\alpha)}(f;x) - f(x)\right| \leq \frac{1}{n}\left((\gamma+1) + \frac{\alpha}{1+\beta e}\right)\|f'\|_{C_B} \\ &+ \frac{1}{n}\left\{x + \frac{1}{2}\left\{(\gamma+1)(\gamma+2) + (2\gamma+3)\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{(1+\beta e)^2}\right\}\right\}\|f''\|_{C_B} \\ &\leq \frac{1}{n}\left\{x + \frac{1}{2}\left\{(\gamma+1)(\gamma+2) + (2\gamma+3)\frac{\alpha}{1+\beta e} + \frac{\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2}{(1+\beta e)^2}\right\}\right\}(\|f'\|_{C_B} + \|f''\|_{C_B})\end{aligned}$$

from which the proof is completed by considering the equality in (2.5). □

Now we give an estimate with the use of Peetre's K-functional.

Theorem 2.8. *If $f \in C_B[0,\infty)$, then we have*

$$\left|D_n^{(\alpha)}(f;x) - f(x)\right| \leq 2M\left(w_2(f;\gamma) + \nu_n(x)\,\|f\|_{C_B}\right)$$

where $\gamma := \gamma_n(x) = \frac{\eta_n(x)}{2}$ and $\eta_n(x)$ is given in Theorem 2.7. Here $\nu_n(x) = \min\{1, \gamma_n^2(x)\}$ and M is a constant.

Proof. In the proof of this theorem we will use the previous theorem and the definition of Peetre's K-functional. For $f \in C_B[0,\infty)$ and $u \in C_B^2[0,\infty)$, we have

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le \left|D_n^{(\alpha)}(f;x)-D_n^{(\alpha)}(u;x)\right| + \left|D_n^{(\alpha)}(u;x)-u(x)\right| + |u(x)-f(x)|$$

$$\le 2\|f-u\|_{C_B} + \frac{1}{n}\left\{x+\frac{1}{2}\left\{(\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+\beta e)^2}\right\}\right\}\|u\|_{C_B^2}.$$

Taking the infimum of both sides of the above inequality and since the left hand side is independent of the function u, we get

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le 2\inf_{u\in C_B^2[0,\infty)}\left\{\|f-u\|_{C_B}+\gamma_n(x)\|u\|_{C_B^2}\right\} \tag{2.13}$$

where $\gamma_n(x)=\frac{\eta_n(x)}{2}$ and $\eta_n(x)$ is already given in Theorem (2.7).

The definition of K-functional yields,

$$\left|D_n^{(\alpha)}(f;x)-f(x)\right| \le 2M\left\{w_2(f,\sqrt{\gamma_n})+\min\{1,\gamma_n\}\|f\|_{C_B}\right\}. \tag{2.14}$$

Letting $\nu_n(x)=\min\{1,\gamma_n\}$, we get the desired result. □

Now consider the following Lipschitz-type space [17]

$$Lip_M^*(r) := \left\{f\in C_B[0,\infty) : |f(s)-f(x)| \le M\frac{|s-x|^r}{(s+x)^{r/2}},\quad s,x\in(0,\infty)\right\}$$

where M is a positive constant and $r\in(0,1]$.

We first give the following Lemma which will be used in the proof of the next theorem.

Lemma 2.9. *For all $x \ge 0$ and $n\in\mathbb{N}$, we have*

$$D_n^{(\alpha)}(|s-x|;x) \le \sqrt{\upsilon_n^{(2)}(x)} \tag{2.15}$$

where $\upsilon_n^{(2)}(x) = D_n^{(\alpha)}((s-x)^2;x)$ given in Remark.

Proof.

$$D_n^{(\alpha)}(|s-x|;x) = n\sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\int_0^\infty e^{-ns}(ns)^{\gamma+k}|s-x|ds. \tag{2.16}$$

Applying Cauchy-Schwarz inequality to the series above, we get

$$D_n^{(\alpha)}(|s-x|;x) \le n\left\{\sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\left(\frac{1}{\Gamma(\gamma+k+1)}\int_0^\infty e^{-ns}(ns)^{\gamma+k}|s-x|ds\right)^2\right\}^{\frac{1}{2}}. \tag{2.17}$$

Applying Cauchy-Schwarz inequality once more, this time to the integral term above, we obtain

$$\int_0^\infty e^{-ns}(ns)^{\gamma+k}|s-x|ds \le \sqrt{\frac{\Gamma(\gamma+k+1)}{n}}\left(\int_0^\infty e^{-ns}(ns)^{\gamma+k}(s-x)^2ds\right)^{\frac{1}{2}}.$$

Substituting the above inequality into (2.17) and arranging the terms, we finally get

$$D_n^{(\alpha)}(|s-x|;x) \le \left\{n\sum_{k=0}^{\infty} v_{n,k}^{\alpha}(x)\frac{1}{\Gamma(\gamma+k+1)}\left(\int_0^\infty e^{-ns}(ns)^{\gamma+k}(s-x)^2ds\right)\right\}^{\frac{1}{2}} = \sqrt{\upsilon_n^{(2)}(x)}$$

and the proof is completed. □

Theorem 2.10. *Let $f \in Lip_M^*(r)$. Then for all $x > 0$ and $n \in \mathbb{N}$, we have*

$$|D_n^{(\alpha)}(f;x) - f(x)| \leq M \left(\frac{\upsilon_n^{(2)}(x)}{x} \right)^{\frac{r}{2}}. \tag{2.18}$$

where $\upsilon_n^{(2)}(x)$ is given in Remark.

Proof. Assume $r = 1$, i.e., $f \in Lip_M^*(1)$. Hence we can write

$$\begin{aligned} |D_n^{(\alpha)}(f;x) - f(x)| &\leq n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |f(s) - f(x)| ds \\ &\leq n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \left(\frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} \frac{|s-x|}{\sqrt{s+x}} ds \right). \end{aligned} \tag{2.19}$$

By taking the advantage of the fact $\frac{1}{\sqrt{s+x}} < \frac{1}{\sqrt{x}}$ and Lemma 2.9, from (2.19) we get

$$\begin{aligned} |D_n^{(\alpha)}(f;x) - f(x)| &\leq \frac{M}{\sqrt{x}} n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \left(\frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |s-x| ds \right) \\ &= \frac{M}{\sqrt{x}} D_n^{(\alpha)}(|s-x|;x) \\ &\leq M \sqrt{\frac{\upsilon_n^{(2)}(x)}{x}} \end{aligned} \tag{2.20}$$

which confirms the claimed result for $r = 1$.

Now let $r \in (0,1)$. By the application of Hlder's inequality twice by taking $p = \frac{1}{r}$ and $p = \frac{1}{1-r}$, we get

$$\begin{aligned} |D_n^{(\alpha)}(f;x) - f(x)| &\leq n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |f(s) - f(x)| ds \\ &\leq n \left\{ \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \left(\frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |f(s) - f(x)| ds \right)^{\frac{1}{r}} \right\}^{r} \\ &\leq \left\{ n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |f(s) - f(x)|^{\frac{1}{r}} ds \right\}^{r}. \end{aligned}$$

Since $f \in Lip_M^*(r), r \in (0,1)$, we have

$$\begin{aligned} |D_n^{(\alpha)}(f;x) - f(x)| &\leq M \left\{ n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} \frac{|s-x|}{\sqrt{s+x}} ds \right\}^{r} \\ &\leq M \left(\frac{1}{\sqrt{x}} \right)^{r} \left\{ n \sum_{k=0}^{\infty} \upsilon_{n,k}^{\alpha}(x) \frac{1}{\Gamma(\gamma+k+1)} \int_0^{\infty} e^{-ns}(ns)^{\gamma+k} |s-x| ds \right\}^{r} \\ &= M \left(\frac{1}{\sqrt{x}} \right)^{r} \left\{ D_n^{(\alpha)}(|s-x|;x) \right\}^{r}. \end{aligned}$$

By using Lemma (2.9), we get

$$|D_n^{(\alpha)}(f;x)-f(x)| \leq M\left(\frac{1}{\sqrt{x}}\right)^r\left(\sqrt{\upsilon_n^{(2)}(x)}\right)^r = M\left(\frac{\upsilon_n^{(2)}(x)}{x}\right)^{\frac{r}{2}} \tag{2.21}$$

which is the desired result. □

3. Approximation properties in weighted spaces

Let $\rho(x)=1+x^2$ be the weighted function and M_f is a constant depending only on f. $B_\rho\left(\mathbb{R}_0^+\right)$ is the set of all functions defined on $[0,\infty)$ satisfying $|f(x)| \leq M_f\rho(x)$. $B_\rho\left(\mathbb{R}_0^+\right)$ is a normed space with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$$

We also have the following subspaces of $B_\rho\left(\mathbb{R}_0^+\right)$:

$$\begin{aligned} C_\rho\left(\mathbb{R}_0^+\right) &= \left\{f\in B_\rho\left(\mathbb{R}_0^+\right) : f \text{ is continuous on } [0,\infty)\right\} \\ C_\rho^*\left(\mathbb{R}_0^+\right) &= \left\{f\in C_\rho\left(\mathbb{R}_0^+\right) : \lim_{x\to\infty}\frac{|f(x)|}{\rho(x)} = K_f < \infty\right\} \end{aligned}$$

It is obvious that $C_\rho^*\left(\mathbb{R}_0^+\right)\subset C_\rho\left(\mathbb{R}_0^+\right)\subset B_\rho\left(\mathbb{R}_0^+\right)$ [7].

Theorem 3.1. *(See [7]) The sequence of positive linear operators $(L_n)_{n\geq 1}$ act from $C_\rho\left(\mathbb{R}_0^+\right)$ to $B_\rho\left(\mathbb{R}_0^+\right)$ if and only if there exists a positive constant k such that $\|L_n(\rho)\|_\rho \leq k$.*

Theorem 3.2. *(See [7])*

i) There exists a sequence of linear positive operators A_n acting from $C_\rho\left(\mathbb{R}_0^+\right)$ to $B_\rho\left(\mathbb{R}_0^+\right)$ such that

$$\lim_{n\to\infty}\left\|A_n\left(\phi^k;.\right)-\phi^k(.)\right\|_\rho = 0 \quad (k=0,1,2) \tag{3.1}$$

and a function $f^*\in C_\rho\backslash C_\rho^*$ with $\lim\limits_{n\to\infty}\|A_n(f^*;.)-f^*(.)\|_\rho \geq 1$.

ii) If a sequence of linear positive operators A_n acting from $C_\rho\left(\mathbb{R}_0^+\right)$ to $B_\rho\left(\mathbb{R}_0^+\right)$ satisfies the conditions in (3.1), then

$$\lim_{n\to\infty}\|A_n(f;.)-f(.)\|_\rho = 0$$

for every $f\in C_\rho^*\left(\mathbb{R}_0^+\right)$.

Lemma 3.3. *Let $\rho(x)=1+x^2$ be a weight function. If $f\in C_\rho\left(\mathbb{R}_0^+\right)$, then there exists a positive constant k such that*

$$\left\|D_n^{(\alpha)}(\rho;.)\right\|_\rho \leq k.$$

Proof. Lemma (2.1) implies,

$$D_n^{(\alpha)}(1+t^2;x) = 1+x^2+\frac{x}{n}\left((2\gamma+3)+\frac{1+2\alpha+\beta e}{1+\beta e}\right) \tag{3.2}$$

$$+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+\beta e)^2}\right) \tag{3.3}$$

from which we have,

$$\begin{aligned}\left\|D_n^{(\alpha)}(\rho;.)\right\|_\rho &\le \sup_{x\ge 0}\frac{1}{1+x^2}\left\{1+x^2+\frac{x}{n}\left((2\gamma+3)+\frac{1+2\alpha+\beta e}{1+\beta e}\right)\right.\\ &\left.+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+\beta e)^2}\right)\right\}\\ &\le 1+\frac{1}{n}\left((2\gamma+3)+\frac{1+2\alpha+\beta e}{1+\beta e}\right)\\ &+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{(1+\beta e)^2}\right).\end{aligned}$$

Since $\frac{1}{n}\to 0$, there exists a positive constant K such that

$$\left\|D_n^{(\alpha)}(\rho;.)\right\|_\rho \le 1+K$$

which implies that the proof is completed. □

Lemma (2.4) implies that the operators $D_n^{(\alpha)}(f;x)$ defined by (1.7) maps $C_\rho\left(\mathbb{R}_0^+\right)$ into $B_\rho\left(\mathbb{R}_0^+\right)$.

Theorem 3.4. *Let the sequence of operators $D_n^{(\alpha)}(f;x)$ defined by (1.7). For any $f\in C_\rho^*\left(\mathbb{R}_0^+\right)$, one gets*

$$\lim_{n\to\infty}\left\|D_n^{(\alpha)}(f;.)-f\right\|_\rho=0.$$

Proof. It is sufficient to show that the sequence of operators $D_n^{(\alpha)}(f;x)$ satisfies three criterions of the weighted Korovkin Theorem. Keeping the identity $\left\|D_n^{(\alpha)}\left(t^k;.\right)-x^k\right\|_\rho=\sup\limits_{x\ge 0}\frac{\left|D_n^{(\alpha)}\left(t^k;.\right)-x^k\right|}{1+x^2}$ in mind, we have the following calculations:

For $k=0$, Lemma (2.1) implies

$$\lim_{n\to\infty}\left\|D_n^{(\alpha)}(1;.)-1\right\|_\rho=\lim_{n\to\infty}\sup_{x\ge 0}\frac{\left|D_n^{(\alpha)}(1;x)-1\right|}{1+x^2}=0. \tag{3.4}$$

For $k=1$,

$$\sup_{x\ge 0}\frac{\left|D_n^{(\alpha)}(t;x)-x\right|}{1+x^2}=\sup_{x\ge 0}\frac{1}{1+x^2}\left((\gamma+1)+\frac{\alpha}{1+\beta e}\right)\frac{1}{n}$$

from which we have

$$\lim_{n\to\infty}\left\|D_n^{(\alpha)}(t;.)-x\right\|_\rho=0 \tag{3.5}$$

Lastly for $k=2$ we have,

$$\begin{aligned}\sup_{x\geq 0}\frac{\left|D_n^{(\alpha)}\left(t^2;x\right)-x^2\right|}{1+x^2} &= \frac{1}{n}\left((2\gamma+3)+\frac{1+2\alpha+\beta e}{1+\beta e}\right)\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+\beta e)^2}\right)\sup_{x\geq 0}\frac{1}{1+x^2}\\ &\leq \frac{1}{n}\left((2\gamma+3)+\frac{1+2\alpha+\beta e}{1+\beta e}\right)\\ &+\frac{1}{n^2}\left((\gamma+1)(\gamma+2)+(2\gamma+3)\frac{\alpha}{1+\beta e}+\frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+\beta e)^2}\right)\end{aligned}$$

which implies $\lim_{n\to\infty}\left\|D_n^{(\alpha)}\left(t^2;.\right)-x^2\right\|_\rho=0$. From this, the proof of the theorem is completed. □

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