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ON INFINITE MATRIX AND $\sigma-$ CONVERGENCE

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Abstract. The goal of this paper is to present and study a new $\sigma-$ sequence space which is defined using the φ- function and infinite matrix A. Further we also prove some inclusion theorems.

1. Introduction and Background

By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $||x|| = \sup_n |x_n|$, respectively.

Let $\theta = (k_r)$ be the sequence of positive integers such that

$i)$ $k_0 = 0$ and $0 < k_r < k_{r+1}$

$ii)$ $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I = (k_r - k_{r-1}]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [2] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - le|) = 0, \text{ for some } l \right\}.$$

There is a strong connection between N_θ and w, the space of strongly Cesàro summable sequences which is defined by

$$w = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=0}^{n} |x_k - le|) = 0, \text{ for some } l \right\}.$$

If $\theta = (2^r)$, we write $N_\theta = w$.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_n \geq 0$ for all n;
(2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

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For certain class of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$.

Consequently, $c \subset V_\sigma$ where V_σ is the subset of all bounded sequences whose σ-means are equal.

If $x = (x_k)$, set $Tx = (Tx_k) = (x_\sigma(k))$, it can be shown that (see, Schaefer [9])

$$V_\sigma = \left\{x \in l_\infty : \lim_m t_{m,n}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x\right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m+1},\ t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$.

Just as the concept of almost convergence lead naturally to the concept of strong almost convergence, σ- convergence leads naturally to the concept of strong σ-convergence. The space $[V_\sigma]$ is strongly σ-convergent sequence was introduced by Mursaleen [5] as follows: A sequence $x = (x_k)$ is said to be strongly σ-convergent if there exists a number L such that

$$\frac{1}{n}\sum_{k=1}^{n} |x_{\sigma^k(m)} - L| \to 0 \tag{1.1}$$

as $n \to \infty$ uniformly in m. We will denote $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1.1) holds we write $[V_\sigma] - \lim x = L$. If we take $\sigma(m) = m + 1$, then $[V_\sigma] = [\hat{c}]$, the set of all strongly almost convergent sequences which is defined by Maddox in [3]. Quite recently, the concept of lacunary σ-convergent was introduced and studied by Savas [8] as a generalization of the idea of lacunary almost convergence which is presented Das and Mishra [1]. Savas[8] defined the following sequence space,

$$V_\sigma^\theta = \left\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} (x_{\sigma^k(m)} - L) = 0, \text{for some} L,\ \text{uniformly in } m\right\}.$$

Note that for $\sigma(m) = m + 1$, the space V_σ^θ reduces to AC_θ.

Recently E. Savas [7] generalized the concept of strong almost convergence by using a modulus f and examined some properties of the corresponding new sequence spaces.

Following Ruckle [6], a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(x)$ for all $x, y \geq 0$,
(iii) f increasing,
(iv) f is continuous from the right at zero.

By a φ-function, we understood a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0, \varphi(u) > 0$, for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$, (see, [10]).

In the present paper, we introduce and study some properties of the following sequence space which is defined using the φ- function and infinite real matrix.

2. Main Results

Let φ and f be given φ-function and modulus function, respectively. Further, let $A = (a_{jk})$ be the infinite real matrix and a lacunary sequence θ be given. Then we have,

$$N_\theta^0(A,\varphi,\sigma,f) = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{j\in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|) \Big| \Big) = 0, \text{ uniformly in } m \right\}.$$

If $x \in N_\theta^0(A,\varphi,\sigma,f)$, the sequence x is said to be lacunary strong (A,φ,σ)-convergent to zero with respect to a modulus f. When $\varphi(x) = x$ for all x, we obtain,

$$N_\theta^0(A,\sigma,f) = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{j\in I_r} f\Big(\Big| \sum_{k=1}^{\infty} a_{jk}(|x_{\sigma^k(m)}|) \Big| \Big) = 0, \text{ uniformly in } m \right\}.$$

If we take $f(x) = x$, we write

$$N_\theta^0(A,\varphi,\sigma) = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{j\in I_r} \Big| \sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|) \Big| = 0, \text{ uniformly in } m \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then we have,

$$N_\theta^0(\sigma,f) = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} f(|x_{\sigma^k(m)}|) = 0, \text{ uniformly in } m \right\}.$$

Theorem 2.1. *Let f be a any modulus function and let φ-function φ, infinite real matrix A be given. If*

$$w(A,\varphi,\sigma,f) = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{j=1}^{n} f\Big(\Big| \sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|) \Big| \Big) = 0, \textit{ uniformly in } m \right\}.$$

then the following relations are true :
(a) If $\liminf_r q_r > 1$ then we have $w(A,\varphi,\sigma,f) \subseteq N_\theta^0(A,\varphi,\sigma,f)$,
(b) If $\sup_r q_r < \infty$, then we have $N_\theta^0(A,\varphi,\sigma,f) \subseteq w(A,\varphi,\sigma,f)$,
(c) $1 < \liminf_r q_r \le \limsup_r q_r < \infty$,then we have $N_\theta^0(A,\varphi,\sigma,f) = w(A,\varphi,\sigma,f)$.

Proof. (a) Let us suppose that $x \in w(A,\varphi,\sigma,f)$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for all $r \ge 1$ and we have $h_r/k_r \ge \delta/(1+\delta)$ for sufficiently large r. Then, for all m,

$$\begin{aligned}
&\frac{1}{k_r}\sum_{j=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) \\
&\qquad\qquad \geq \frac{1}{k_r}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) \\
&\qquad\qquad = \frac{h_r}{k_r}\frac{1}{h_r}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) \\
&\qquad\qquad \geq \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big).
\end{aligned}$$

Hence, $x \in N_\theta^0(\mathbf{A}, \varphi, \sigma, f)$.

(b) If $\limsup_r q_r < \infty$ then there exist $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $x \in N_\theta^0(A, \varphi, \sigma, f)$ and ε is an arbitrary positive number, then there exists an index p_0 such that for every $p \geq p_0$ and all m,

$$R_p = \frac{1}{h_p}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) < \varepsilon$$

Thus, we can also find $K > 0$ such that $R_p \leq K$ for all $p = 1, 2, \ldots$. Now let s be any integer with $k_{r-1} \leq s \leq k_r$, then we obtain, for all m

$$I = \frac{1}{s}\sum_{j=1}^{s} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) \leq \frac{1}{k_{r-1}}\sum_{j=1}^{k_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{r-1}}\sum_{p=1}^{p_0}\sum_{j\in I_p} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)$$

$$I_2 = \frac{1}{k_{r-1}}\sum_{p=p_{0+1}}^{s}\sum_{j\in I_p} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)$$

It is easy to see that,

$$\begin{aligned}
I_1 &= \frac{1}{k_{r-1}}\sum_{p=1}^{p_0}\sum_{j\in I_p} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)\\
&= \frac{1}{k_{r-1}}\Big(\sum_{j\in I_1} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)+...+\sum_{j\in I_{p_0}} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)\Big)\\
&\le \frac{1}{k_{r-1}}(h_1R_1+...+h_{p_0}R_{p_0}),\\
&\le \frac{1}{k_{r-1}} j_0 k_{p_0} sup_{1\le i\le p_0} R_i,\\
&\le \frac{p_0 k_{p_0}}{k_{r-1}} K.
\end{aligned}$$

Further, we have for all m

$$\begin{aligned}
I_2 &= \frac{1}{k_{r-1}}\sum_{p=p_0+1}^{s}\sum_{j\in I_p} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)\\
&= \frac{1}{k_{r-1}}\sum_{p=p_o+1}^{s}\Big(\frac{1}{h_p}\sum_{j\in I_p} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)h_p\\
&\le \varepsilon\frac{1}{k_{r-1}}\sum_{p=p_0+1}^{s} h_p,\\
&\le \varepsilon\frac{k_r}{k_{r-1}},\\
&= \varepsilon q_r < \varepsilon.M.
\end{aligned}$$

Thus $I \le \frac{p_o k_{p_o}}{k_{r-1}} K + \varepsilon.M$. Hence, $x \in w(A,\varphi,\sigma,f)$.
The proof of (c) follows from (a) and (b). This completes the proof. □

Theorem 2.2. $N_\theta^0(A,\varphi,\sigma) \subset N_\theta^0(A,\varphi,\sigma,f)$.

Proof. Let $x \in N_\theta^0(A,\varphi,\sigma)$. For a given $\varepsilon > 0$ we consider $0 < \delta < 1$ such that $f(x) < \varepsilon$ for every $x \in [0,\delta]$. We can write

$$\frac{1}{h_r}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big) = S_1 + S_2,$$

where $S_1 = \frac{1}{h_r}\sum_{j\in I_r} f\Big(\big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\big|\Big)$ and this sum is taken over

$$\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big| \le \delta$$

and

$$S_2 = \frac{1}{h_r}\sum_{j\in I_r} f\Big(\Big|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\Big|\Big)$$

and this sum is taken over

$$\left|\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|)\right| > \delta.$$

We write $S_1 = \frac{1}{h_r}\sum_{j\in I_r} f\Big(\delta\Big) = f(\delta) < \varepsilon$ and furthermore

$$S_2 = f(1)\frac{1}{\delta}\frac{1}{h_r}\sum_{j\in I_r}\sum_{k=1}^{\infty} a_{jk}\varphi(|x_{\sigma^k(m)}|).$$

Hence we have $x \in N_\theta^0(\mathbf{A}, \varphi, \sigma, f)$.
This completes the proof. □

3. Conclusion

The definition and conclusion given here can be used as theoretical tool to study sequence space under infinite real matrix. The contribution of the paper is to study some properties of the new sequence space by using φ and modulus functions. Also some inclusion relations are presented.

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