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ON EXTENDED b-METRIC SPACES: RELAXING COMPLETENESS CRITERION AND EXPLORING BEHAVIOR

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Abstract. In this study, we demonstrate that fixed points can occur in extended b-metric spaces that are not necessarily complete. We must employ a novel form of contraction conditions in order to relax the completeness criterion. It has been investigated how fixed points behave in both partially complete and orbitally complete spaces. We provide some instances to highlight our findings.

1. Introduction

The Contraction Principle by Banach (BCP) has had a substantial impact on the advancement of fixed point theory. Banach [4] offered a clarification in relation to the fixed point's distinctiveness for a contraction map in metric spaces that are complete. Researchers have broadly generalized the BCP, either by expanding the idea of space or the contraction. It is challenging to name every generalization.

By establishing the notion of a b-metric space in 1989, Bakhtin [5] expanded an explanation of a metric space. With regard to b-metric spaces, BCP has many different generalizations, see [13, 20, 21, 23, 24, 27].
On the other hand, some mathematicians broaden the BCP by expanding the definition of the contraction condition. Among these generalizations are those of Edelstein [8], Boyd-Wong [6], Meir-Keeler [9], ćirić [7], Khan et al. [11], Kirk [10] and kannan [12].

With regard to ordered metric spaces, one of the earliest generalisations was established by Ran-Reurings [19] in 2004. The nonlinear contractive condition is only believed to apply to comparable elements, which is the primary distinction between the Ran-Reurings theorem and the Banach Contraction Principle. See [2, 14, 15, 16, 17, 18, 22] for a list of publications that have been reported in ordered metric spaces.

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Rashid et al. [15] established the concept of t-property and t-contractions in an ordered metric space in 2019. It is now possible to explore fixed points in incomplete spaces. See [3, 24, 25, 30] for a list of fixed point findings that were produced with this configuration.

By introducing the concept of an extended b-metric space, which expands on the idea of a b-metric space, Kamran et al. [26] completed another novel work in 2017. In this new environment, certain fixed point findings are demonstrated, see [1, 28, 29].

First, let's review some definitions.

Definition 1.1. ***(b-metric space)***[5]. *Suppose $\Omega : X^2 \to \mathbb{R}^+$ is a map on a nonempty set X, satisfying:*
(i_1) *$\Omega(\jmath, \eta) = 0$ if and only if $\jmath = \eta$,*
(i_2) *$\Omega(\jmath, \eta) = \Omega(\eta, \jmath)$,*
(i_3) *$\Omega(\jmath, \eta) \leq s[\Omega(\jmath, z) + \Omega(z, \eta)]$, $s \geq 1$*
for all $\jmath, \eta, z \in X$. If so, Ω is referred to a b-metric on X and (X, Ω) is referred to a b-metric space. It should be noted that for $s = 1$, a b-metric space reduces to a metric space. Thus, every metric space is a b-metric space with $s = 1$, but, generally speaking, not all b-metric spaces are metric spaces.

Definition 1.2. ***(Extended b-metric space)***[26].
Suppose that X is a nonvoid set and $b_\phi : X^2 \to \mathbb{R}^+$, where $\phi : X \times X \to [1, \infty)$, is a function satisfying:
$(*_1)$ *$b_\phi(\jmath, \eta) = 0$ iff $\jmath = \eta$,*
$(*_2)$ *$b_\phi(\jmath, \eta) = b_\phi(\eta, \jmath)$,*
$(*_3)$ *$b_\phi(\jmath, \eta) \leq \phi(\jmath, \eta)[b_\phi(\jmath, z) + b_\phi(z, \eta)]$,*
for all $\jmath, \eta, z \in X$. If so, b_ϕ is referred to an extended b-metric on X and (X, b_ϕ) is referred to an extended b-metric space. It should be noted that for $\phi(\jmath, \eta) = 1$, an extended b-metric space reduces to a b-metric space. Each b-metric space is therefore an extended b-metric space, but not the other way around.

In this paper, we establish that fixed points exist in an extended b-metric space where the completeness is missed. We incorporate new types of contraction conditions in order to relax the completeness condition.

2. Main Results

Our obtained results are based on weakness of completness.

2.1. On $\bar{\mathfrak{f}}$-orbitally completeness.

The idea of $\bar{\mathfrak{f}}$-orbitally complete extended b-metric spaces has been used in this section. A novel method has been used to derive the presence of a fixed point using this idea. Let's start by going through a definition.

Definition 2.1. *Given an extended b-metric space (X, b_ϕ), let h be a self-map. State that for $S \subseteq X$ with $\varsigma \in S$*

$$O_S(\varsigma, n) = \{\varsigma, h\varsigma, h^2\varsigma, ..., h^n\varsigma\}$$

and

$$O_S(\varsigma,\infty)=\{\varsigma,h\varsigma,h^2\varsigma,...\}.$$

(X,b_ϕ) *is said to be* $\bar{h}$*-orbitally complete with regard to* S*, if each Cauchy sequence discovered in* $O_S(\varsigma,\infty)$ *must converge in* S*.*

We now provide an illustration of Definition 2.1 using examples.

Example 2.2. *Suppose* $X=[0,2)\cap\mathbb{Q}$*. We define an extended b-metric* $b_\phi : X^2\to[0,\infty)$ *by*

$$\begin{aligned} b_\phi(\jmath,\eta) &= 0, \text{ iff } \jmath=\eta \\ &= \jmath+\eta, \text{ iff } \jmath\neq\eta. \end{aligned}$$

We take $S=[0,1)\cap\mathbb{Q}$ *and define a self map* h *on* X *by* $h(\jmath)=\frac{\jmath}{3}$*. Let us define* $\phi : X^2\to[1,\infty)$ *by* $\phi(\jmath,\eta)=8+\eta-\jmath$*.*
We first prove $b_\phi(\jmath,\eta)\le\phi(\jmath,\eta)[b_\phi(\jmath,z)+b_\phi(z,\eta)]$ *in two cases:*
Case I : when $\jmath=\eta$*, then*
$\phi(\jmath,\eta)[b_\phi(\jmath,z)+b_\phi(z,\eta)]\ge 0=b_\phi(\jmath,\eta)$*.*

Case II : Now, for $\jmath\neq\eta$*, in order to prove* $b_\phi(\jmath,\eta)\le\phi(\jmath,\eta)[b_\phi(\jmath,z)+b_\phi(z,\eta)]$*, we make it clear that* $b_\phi(\jmath,\eta)\le 4,\ \forall \jmath,\eta\in X$ *and for these values* $\phi(\jmath,\eta)[b_\phi(\jmath,z)+b_\phi(z,\eta)]>8$*.*

Now, it remains to prove that (X,b_ϕ) *is* $\bar{h}$*-orbitally complete extended b-metric space with regard to* S*.*
Let $\varsigma\in S$ *and* $\{a_n\}$ *be any Cauchy sequence in* $O_S(\varsigma,\infty)$*, then*
$b_\phi(a_n,0)=a_n+0=\frac{\varsigma}{3^n}\to 0\in S$ *as* $n\to\infty$*. It proves that* (X,b_ϕ) *is* $\bar{h}$*-orbitally complete with regard to* S*.*

Example 2.3. *Let* $X=\{1,2,3\}$ *equipped with an extended b-metric* $b_\phi : X^2\to[0,\infty)$ *given as*

$$b_\phi(1,1)=b_\phi(2,2)=b_\phi(3,3)=0$$
$$b_\phi(1,2)=b_\phi(2,1)=60, b_\phi(1,3)=b_\phi(3,1)=900, b_\phi(2,3)=b_\phi(3,2)=500.$$

Here, $\phi : X\times X\to[1,\infty)$ *is defined by* $\phi(\jmath,\eta)=\jmath^2+\eta^2+1$*. Dene* $h : X\to X$ *by*

$$h(1)=1, h(2)=3 \text{ and } h(3)=2.$$

We take $S=\{1\}$*. Then, for any* $\varsigma\in S$*, we have*

$$O_S(\varsigma,\infty)=\{1,1,1,...,1,...\}.$$

It is clear that (X,b_ϕ) *is* $\bar{h}$*-orbitally complete extended b-metric space with regard to* S*.*

It is crucial to prove the subsequent lemma before proving our first main result.

Lemma 2.4. *Let "*(X,b_ϕ) *be an extended b-metric space. Then for any sequence* $\{\jmath_n\}$ *in* X*, we have*

$$\begin{aligned} b_\phi(\jmath_n,\jmath_m) \le\ & \phi(\jmath_n,\jmath_m).b_\phi(\jmath_n,\jmath_{n+1})+\phi(\jmath_n,\jmath_m).\phi(\jmath_{n+1},\jmath_m).b_\phi(\jmath_{n+1},\jmath_{n+2}) \\ & +...+\phi(\jmath_n,\jmath_m).\phi(\jmath_{n+1},\jmath_m).\phi(\jmath_{n+2},\jmath_m)...\phi(\jmath_{m-1},\jmath_m).b_\phi(\jmath_{m-1},\jmath_m), \end{aligned}$$

for all $n, m \in \mathbb{N}$ *with* $n < m$.

Proof. By using triangular inequality, we have

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) &\leq \phi(\jmath_n, \jmath_m)[b_\phi(\jmath_n, \jmath_{n+1}) + b_\phi(\jmath_{n+1}, \jmath_m)] \\ &= \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \phi(\jmath_n, \jmath_m).b_\phi(\jmath_{n+1}, \jmath_m). \end{aligned}$$

Again using triangular inequality, we have

$$b_\phi(\jmath_n, \jmath_m) \leq \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \phi(\jmath_n, \jmath_m)\Big[\phi(\jmath_{n+1}, \jmath_m)[b_\phi(\jmath_{n+1}, \jmath_{n+2}) + b_\phi(\jmath_{n+2}, \jmath_m)]\Big].$$

Continuing in this way, we get

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ & +... + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-2}, \jmath_m).b_\phi(\jmath_{m-2}, \jmath_m) \\ & +\phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-2}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m). \end{aligned}$$

Since $\phi \geq 1$, we get that

$$\begin{aligned} & \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-2}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m) \\ \leq\ & \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-2}, \jmath_m).\phi(\jmath_{m-2}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m). \end{aligned}$$

Hence,

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ & +... + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-1}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m)." \end{aligned}$$

□

Theorem 2.5. *Let* $(X, b_\phi, \preceq)$ *be an ordered extended b-metric space and* f *be a self map on* X. *Suppose that the set "* $S = \{\jmath \in X : \jmath \preceq f\jmath\}$ *is nonempty with* $\jmath_0 \in S$ *and that*

$$b_\phi(\eta, f\eta) \leq \frac{1}{[\phi(\eta, f\eta) + 1]} b_\phi(\jmath, f\jmath) \tag{2.1}$$

for all $\jmath, \eta \in S$ *with* $\jmath \prec \eta$ *and a function* $\phi : X \times X \to [1, \infty]$. *If further,* $f(S) \subseteq S$ *and* $(X, b_\phi, \preceq)$ *is* $\bar{f}$*-orbitally complete with respect to* S *and the mapping* $\phi : X \times X \to [1, \infty)$ *is such that for all* $\jmath, \eta \in X$ *with* $\jmath \prec \eta$ *implies* $\phi(\jmath, z) \geq \phi(\eta, z)$ *for any* $z \in X$ *and* $\lim\limits_{m,n\to\infty} \phi(f^m\jmath_0, f^n\jmath_0)$ *should be finite, then* f *has a fixed point in* S.

Proof. According to the presumption that $S \neq \emptyset$, let $\jmath_0 \in S$. After that, $f(\jmath_0) \in S$. If $f(\jmath_0) = \jmath_0$, the proof is done. If not, we pick $\jmath_1 = f(\jmath_0) \neq \jmath_0$. Now, if $f(\jmath_1) = \jmath_1$ in $f(\jmath_1) \in S$, we are done. Alternatively, select $\jmath_2 = f(\jmath_1)$. $f(\jmath_2)$ is now in S. As a result of continuing the procedure, we have a sequence $\{\jmath_n\}$ in S where $\jmath_n \prec \jmath_{n+1}$ for all n, and

$$\jmath_{n+1} = f(\jmath_n). \tag{2.2}$$

With $\jmath_0, \jmath_1 \in S$ and $\jmath_0 \prec \jmath_1$, followed by (2.1), we have

$$b_\phi(\jmath_1, f(\jmath_1)) \leq \frac{1}{[\phi(\jmath_1, f\jmath_1) + 1]} b_\phi(\jmath_0, f(\jmath_0)). \tag{2.3}$$

Again, as $\jmath_1, \jmath_2 \in S$ with $\jmath_1 \prec \jmath_2$, then by (2.1), we have

$$b_\phi(\jmath_2, f(\jmath_2)) \leq \frac{1}{[\phi(\jmath_2, f\jmath_2) + 1]} b_\phi(\jmath_1, f(\jmath_1)). \tag{2.4}$$

Using (2.3) in (2.4), we get

$$b_\phi(\jmath_2, f(\jmath_2)) \leq \frac{1}{[\phi(\jmath_2, f\jmath_2)+1]}.\frac{1}{[\phi(\jmath_1, f\jmath_1)+1]}b_\phi(\jmath_0, f(\jmath_0)).$$

Using the fact that for $\jmath_1 \prec \jmath_2 \implies \phi(\jmath_1, f\jmath_1) \geq \phi(\jmath_2, f\jmath_2)$, we have

$$b_\phi(\jmath_2, f(\jmath_2)) \leq \frac{1}{[\phi(\jmath_2, f\jmath_2)+1]^2}.b_\phi(\jmath_0, f(\jmath_0)).$$

Continuing in this way, we get

$$b_\phi(\jmath_n, \jmath_{n+1}) = b_\phi(\jmath_n, f(\jmath_n)) \leq \frac{1}{[\phi(\jmath_n, f\jmath_n)+1]^n}b_\phi(\jmath_0, f(\jmath_0)). \tag{2.5}$$

We now demonstrate that in $O_S(\jmath_0, \infty)$, $\{\jmath_n\}$ is a Cauchy sequence.
For $n, m \in \mathbb{N}$ with $n < m$ and by Lemma 2.4, we obtain

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \theta(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ & +... + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-1}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m) \end{aligned} \tag{2.6}$$

Since $\{\jmath_n\}$ is a strictly increasing sequence, by using the definition of ϕ, we have

$$\phi(\jmath_{m-1}, \jmath_m) \leq \phi(\jmath_{m-2}, \jmath_m) \leq ... \leq \phi(\jmath_{n+1}, \jmath_m) \leq \phi(\jmath_n, \jmath_m). \tag{2.7}$$

Now, by using (2.7) in (2.6), we get

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + [\phi(\jmath_n, \jmath_m)]^2.b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ & + [\phi(\jmath_n, \jmath_m)]^3.b_\phi(\jmath_{n+2}, \jmath_{n+3}) + ... + [\phi(\jmath_n, \jmath_m)]^{m-n-1}.b_\phi(\jmath_{m-1}, \jmath_m) \end{aligned} \tag{2.8}$$

By using (2.5) in (2.8), we get

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \frac{\phi(\jmath_n, \jmath_m)}{[\phi(\jmath_n, f\jmath_n)+1]^n}.b_\phi(\jmath_0, f(\jmath_0)) + \frac{[\phi(\jmath_n, \jmath_m)]^2}{[\phi(\jmath_{n+1}, f\jmath_{n+1})+1]^{n+1}}.b_\phi(\jmath_0, f(\jmath_0)) \\ & +... + \frac{[\phi(\jmath_n, \jmath_m)]^{m-n-1}}{[\phi(\jmath_{m-1}, f\jmath_{m-1})+1]^{m-1}}.b_\phi(\jmath_0, f(\jmath_0)) \\ \leq\ & \left[\left[\frac{\phi(\jmath_n, \jmath_m)}{\phi(\jmath_n, f\jmath_n)+1}\right]^n + \left[\frac{\phi(\jmath_n, \jmath_m)}{\phi(\jmath_{n+1}, f\jmath_{n+1})+1}\right]^{n+1}\right. \\ & \left. +... + \left[\frac{\phi(\jmath_n, \jmath_m)}{\phi(\jmath_{m-1}, f\jmath_{m-1})+1}\right]^{m-1}\right].b_\phi(\jmath_0, f(\jmath_0)).\text{''} \end{aligned}$$

Using the assumption that $\lim_{m,n\to\infty} \phi(\jmath_m, \jmath_n)$ is finite, we have $\lim_{m,n\to\infty} \frac{\phi(\jmath_m, \jmath_n)}{\phi(\jmath_n, f\jmath_n)+1} = \lambda < 1$. We have from above equation,

$$\begin{aligned} \lim_{m,n\to\infty} b_\phi(\jmath_n, \jmath_m) \leq\ & \lim_{m,n\to\infty}[\lambda^n + \lambda^{n+1} + \lambda^{n+2} + ... + \lambda^{m-1}]b_\phi(\jmath_0, f(\jmath_0)), \\ \leq\ & \lim_{m,n\to\infty}[\lambda^n + \lambda^{n+1} + ...]b_\phi(\jmath_0, f(\jmath_0)), \\ \to\ & 0, \ as\ n, m \to \infty. \end{aligned} \tag{2.9}$$

That is,

$$\lim_{n,m\to\infty} b_\phi(\jmath_n, \jmath_m) = 0.$$

This demonstrates that in $O_S(\jmath_0, \infty)$, $\{\jmath_n\}$ is a strictly increasing Cauchy sequence. However, because $(X, b_\phi, \preceq)$ is $\bar{\text{f}}$-orbitally complete with regard to S, there is $\varsigma \in S$, such that $\jmath_n \to \varsigma$ and $\jmath_n \prec \varsigma$, $\forall$ n.

We now demonstrate that ς is a fixed point of f in S.
Since $\jmath_n, \varsigma \in S$ with $\jmath_n \prec \varsigma$ for all n, by (2.1) and (2.5), we have

$$\begin{aligned} b_\phi(\varsigma, f(\varsigma)) &\leq \frac{1}{\phi(\varsigma, f\varsigma)+1} b_\phi(\jmath_n, f(\jmath_n)) \\ &\leq \frac{1}{\phi(\varsigma, f\varsigma)+1} \cdot \frac{1}{[\phi(\jmath_n, f\jmath_n)+1]^n} b_\phi(\jmath_0, f(\jmath_0)) \\ &\rightarrow 0, \ as\ n \to \infty. \end{aligned}$$

Thus, we have $b_\phi(\varsigma, f(\varsigma)) = 0$, which implies $f(\varsigma) = \varsigma$. Thus, ς is a fixed point of f in S. □

We give an example to demonstrate Theorem 2.5.

Example 2.6. *Let $X = \{1, 2, 3\}$ and " $\preceq$ " is defined as natural ordering " $\leq$ ". We define $b_\phi : X^2 \to [0, \infty)$ by*

$$b_\phi(1,1) = b_\phi(2,2) = b_\phi(3,3) = 0$$

$$b_\phi(1,2) = b_\phi(2,1) = 800, b_\phi(1,3) = b_\phi(3,1) = 100, b_\phi(2,3) = b_\phi(3,2) = 300.$$

$\phi : X \times X \to [1, \infty)$ is defined by $\phi(\jmath, \eta) = 7 + \eta - \jmath$. Then one can quickly confirm that $(X, b_\phi, \preceq)$ is an ordered extended b-metric space.

Now, specify $f : X \to X$ by

$$f(1) = 3, f(2) = 1 \text{ and } f(3) = 3.$$

Then clearly $S = \{1, 3\}$ and $f(S) \subseteq S$. It can be easily verified that for all $\jmath, \eta \in X$ with $\jmath < \eta$ implies $\phi(\jmath, z) \geq \phi(\eta, z)$ for any $z \in X$.
By the observation that for any $\jmath \in S$, $f^n(\jmath) = 3$ for all $n \geq 2$, we can deduce that $(X, b_\phi, \preceq)$ is $\bar{f}$-orbitally complete with regard to S. Finally, (2.1) can be easily verified by using the fact that $b_\phi(3, f(3)) = 0$.
Thus, all of the requirements of Theorem 2.5 have been met and hence f has a fixed point in S, which is clearly $3 \in S$.

Corollary 2.7. *From (2.5) and (2.9), it is evident that Theorem 2.5 still applies when Equation (2.1) is changed to*

$$b_\phi(\eta, f\eta) \leq \frac{1}{\phi(\eta, f\eta) + t} b_\phi(\jmath, f\jmath)$$

for all $\jmath, \eta \in S$ with $\jmath \prec \eta$ and $t > 0$.

2.2. **On partial completeness.** Before proceeding to main theorem, we first define a partially complete extended b-metric space.

Definition 2.8. *$(X, b_\phi, \preceq)$ is a representation of an ordered extended b-metric space. Each strictly growing Cauchy sequence in S must have a strict upper bound in S in order for the triplet $(X, b_\phi, \preceq)$ to be considered partially complete with regard to S, i.e. $\exists\ \jmath \in S$ such that $\jmath_n \prec \jmath$, for each $n \in \mathbb{N}$,*

Theorem 2.9. *Given f as a self map on an extended ordered b-metric space $(X, b_\phi, \preceq)$ and define the mapping " $\phi : X^2 \to [1, \infty)$ in such a way that for all $\jmath, \eta \in X$ with $\jmath \prec \eta \implies \phi(\jmath, z) \geq \phi(\eta, z)$ for any $z \in X$ and $\lim_{m,n\to\infty} \phi(\jmath_m, \jmath_n) < \frac{1}{\lambda}$,* *here* $\jmath_n = f^n \jmath_0,\ n = 1, 2, ...$. *Also, we assume that,* $S = \{\jmath \in X : \jmath \preceq f\jmath\}$ *is non empty and*

$$b_\phi(\eta, f\eta) \leq \lambda b_\phi(\jmath, f\jmath),\ \forall\ \jmath, \eta \in S \textit{ with } \jmath \prec \eta, \tag{2.10}$$

with $\lambda \in [0, 1)$. *Further, if* $f(S) \subseteq S$ *and* $(X, b_\phi, \preceq)$ *is partially complete with regard to* S, *then* f *has a fixed point in* S.

Proof. Following the proof of Theorem 2.5, we get a strictly growing Cauchy sequence $\{\jmath_n\}$ in S such that

$$\jmath_{n+1} = f(\jmath_n)$$

and

$$b_\phi(\jmath_n, f\jmath_n) \leq \lambda^n.b_\phi(\jmath_0, f\jmath_0). \tag{2.11}$$

We now prove that $\{\jmath_n\}$ is a Cauchy sequence in S.
By using Lemma 2.4, we have for $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) \leq\ & \phi(\jmath_n, \jmath_m).b_\phi(\jmath_n, \jmath_{n+1}) + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ & +... + \phi(\jmath_n, \jmath_m).\phi(\jmath_{n+1}, \jmath_m).\phi(\jmath_{n+2}, \jmath_m)...\phi(\jmath_{m-1}, \jmath_m).b_\phi(\jmath_{m-1}, \jmath_m) \end{aligned} \tag{2.12}$$

Since $\{\jmath_n\}$ is strictly increasing sequence, then by using the definition of ϕ, we have

$$\phi(\jmath_{m-1}, \jmath_m) \leq \phi(\jmath_{m-2}, \jmath_m) \leq ... \leq \phi(\jmath_{n+1}, \jmath_m) \leq \phi(\jmath_n, \jmath_m). \tag{2.13}$$

Using (2.11) and (2.13) in (2.12), we get

$$\begin{aligned} b_\phi(\jmath_n, \jmath_m) &\leq [\phi(\jmath_n, \jmath_m)].b_\phi(\jmath_n, \jmath_{n+1}) + [\phi(\jmath_n, \jmath_m)]^2 b_\phi(\jmath_{n+1}, \jmath_{n+2}) \\ &\quad +... + [\phi(\jmath_n, \jmath_m)]^{m-n-1} b_\phi(\jmath_{m-1}, \jmath_m), \\ &\leq [\phi(\jmath_n, \jmath_m)]\lambda^n b_\phi(\jmath_0, f\jmath_0) + [\phi(\jmath_n, \jmath_m)]^2 \lambda^{n+1} b_\phi(\jmath_0, f\jmath_0) \\ &\quad +... + [\phi(\jmath_n, \jmath_m)]^{m-n-1} \lambda^{m-1} b_\phi(\jmath_0, f\jmath_0), \\ &\leq \Big[[\phi(\jmath_n, \jmath_m)\lambda]^n + [\phi(\jmath_n, \jmath_m)\lambda]^{n+1} + ... + [\phi(\jmath_n, \jmath_m)\lambda]^{m-1}\Big] b_\phi(\jmath_0, f\jmath_0), \\ &= [t^n + t^{n+1} + ...] b_\phi(\jmath_0, f\jmath_0);\ \text{where } t = \phi(\jmath_n, \jmath_m)\lambda. \end{aligned}$$

Note that $\lim\limits_{m,n\to\infty} t < \frac{1}{\lambda}.\lambda = 1$, say $\lim\limits_{m,n\to\infty} t = s < 1$.
So, $\lim\limits_{m,n\to\infty} b_\phi(\jmath_n, \jmath_m) = 0$.

This proves that $\{\jmath_n\}$ is a Cauchy sequence. Now, $(X, b_\phi, \preceq)$ being a partially complete with respect to S, therefore there is $w \in S$, such that $\jmath_n \prec w,\ \forall n \in \mathbb{N}$. Thus, by using (2.10) and (2.11)," we have

$$\begin{aligned} b_\phi(w, fw) &\leq \lambda b_\phi(\jmath_n, f\jmath_n), \\ &\leq \lambda^{n+1} b_\phi(\jmath_0, f\jmath_0), \\ &\to 0,\ \text{as } n \to 0. \end{aligned}$$

It implies that $b_\phi(w, fw) = 0$ and hence f has a fixed point in S. □

We provide some illustrations to back up our conclusions.

Example 2.10. *Let* $X = (-1, 0]$. " $\preceq$ " *is defined as natural ordering* " $\leq$ ". *We define the metric* $b_\phi : X^2 \to [0, \infty)$ *by*

$$\begin{aligned} b_\phi(\jmath, \eta) &= 0,\ \textit{iff } \jmath = \eta \\ &= 3 + \jmath + \eta,\ \textit{iff } \jmath \neq \eta. \end{aligned}$$

Further, if we specify $\phi : X^2 \to [1, \infty)$ *by* $\phi(\jmath, \eta) = 5 + \eta - \jmath$*, then we can easily verify that* $(X, b_\phi, \preceq)$ *is an ordered extended b-metric space.*
Now, we consider $f : X \to X$ *by* $f(\jmath) = \frac{\jmath}{2}$*. Take* $S = \{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, ...\} \cup \{0\}$ *and* $\lambda = \frac{1}{7}$*.*
Here, $f(S) \subseteq S$ *and for any* $\jmath_0 \in S$*, we can show that* $\jmath_k = f^k(\jmath_0) = -\frac{1}{2^k}$ *for some* $k \in \mathbb{N} \cup \{0\}$*. Thus,*

$$\phi(\jmath_m, \jmath_n) = 5 - \frac{1}{2^n} + \frac{1}{2^m}$$

and hence $\lim_{m,n\to\infty} \phi(\jmath_m, \jmath_n) = 5 < 7 = \frac{1}{\lambda}$*.*
Now, we will prove (2.10). Let $\jmath, \eta \in S$ *with* $\jmath < \eta$*, we have*

$$\begin{aligned} \lambda b_\phi(\jmath, f\jmath) - b_\phi(\eta, f\eta) &= \frac{1}{7}[b_\phi(\jmath, \frac{\jmath}{2})] - b_\phi(\eta, \frac{\eta}{2}) \\ &= \frac{1}{7}[3 + \jmath + \frac{\jmath}{2}] - [3 + \eta + \frac{\eta}{2}] \\ &\geq 0. \end{aligned}$$

Finally, it remains to show that $(X, b_\phi, \preceq)$ *is partially complete with regard to* S*. Since* $0 \in S$ *is the strict upper bound of each element of* S *and hence to every increasing Cauchy sequence in* S*.*
Thus, all of the requirements of Theorem 2.9 have been met, and so f *has a fixed point in* S*.*

3. Conclusion

The results of this study have a wide range of ramifications and prospective uses. By providing knowledge about their attributes and qualities, they aid in the analysis of incomplete spaces. Additionally, optimisation and algorithm design, mathematical modelling, network analysis, dynamical systems, and numerical analysis can all benefit from the study's findings. The investigation presented in this study broadens our comprehension of fixed points in extended b-metric spaces, providing new directions for further investigations. A useful tool for analysing and dealing with spaces that do not satisfy the conventional completeness requirement is the innovative contraction condition that is here introduced.

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