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COMPOSITION OPERATORS ON THE SPACE OF HARMONIC MAPPING $\mathcal{V}^{\mathcal{H}}$

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Abstract. This paper sought to introduce a new space $\mathcal{V}^{\mathcal{H}}$ of harmonic mappings that is an extension of the space of analytic functions $\mathcal{V}$ whose first derivatives are in the classical Zygmund space. We obtain several characterizations of the space $\mathcal{V}^{\mathcal{H}}$. Lastly, the boundedness and compactness of the operator C_φ acting on $\mathcal{V}^{\mathcal{H}}$ are investigated.

1. Introduction

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disk. Assume $H(\mathbb{D})$ be the class of analytic functions on the unit disk $\mathbb{D}$, and let $\mathcal{H}ar(\mathbb{D})$ be the class of harmonic mappings on the unit disk $\mathbb{D}$. A harmonic mapping f precisely on $\mathbb{D}$ is a complex-valued function such that

$$\Delta f := 4 f_{\zeta\bar{\zeta}} \equiv 0 \quad \text{on } \mathbb{D},$$

where $f_{\zeta\bar{\zeta}}$ represents the second-order complex partial derivatives of f.

It is well known that $f \in \mathcal{H}ar(\mathbb{D})$ has a representation of the form $h + \overline{k}$, where $h, k \in H(\mathbb{D})$. The uniqueness of the representation holds if we impose the condition $k(0) = 0$.

The operator theory of spaces of analytic functions on a various settings on the unit disk has been completely analyzed and a enormous amount of research papers on this matter have appeared in the literature, but the study of a similarly coverage in the harmonic setting is still limited.

In recent years, some papers have concentrated on the study of harmonic mappings. Besides [3], for characterization of Bloch type spaces of harmonic mapping, see [5], for harmonic zygmund spaces. In [6], the authors investigate the compactness and boundedness of C_φ mapping into weighted Banach spaces of harmonic mappings. We also encourage the reader to see the additional references related to the harmonic mappings such as [[7],[10], [8], [2],[4], [19], [18], [22], [20], [17],[11], [12], [14], [15], [16], [20], [13]].

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The results carried out in [21] bring the interesting question for whether we can extend the space $\mathcal{V}$ to the harmonic setting and study the operator theoretic properties of C_φ.

2. BACKGROUND

In complex function theory, the space of analytic functions called the iterated weighted-type space denoted by $\{\mathcal{V}_n : n \in \mathbb{N}\}$ for $n \in \{\mathbb{N} \cup \{0\}\}$ has been introduced and investigated recently. The space $\{\mathcal{V}_n : n \in \mathbb{N}\}$ defined as

$$\mathcal{V}_n := \left\{ h \in H(\mathbb{D}) : \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|h^{(n)}(\zeta)| < \infty \right\}.$$

For an excellent reference see [9].

The space $\mathcal{V}_n$, for $n = 0, 1, 2$ and 3, reduced to the well-known growth space $\mathcal{A}^{-1}$, the Bloch space $\mathcal{B}$, the Zygmund space $\mathcal{Z}$, and the space $\mathcal{V}$, respectively. Moreover, the norms defined on them is given by

$$\begin{aligned}
\|h\|_{\mathcal{A}^{-1}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|h(\zeta)|, \\
\|h\|_{\mathcal{B}} &:= |h(0)| + \|h\|_{s\mathcal{B}}, \\
\|h\|_{\mathcal{Z}} &:= |h(0)| + |h'(0)| + \|h\|_{s\mathcal{Z}}, \\
\|h\|_{\mathcal{V}} &:= |h(0)| + |h'(0)| + |h''(0)| + \gamma(h).
\end{aligned}$$

where the semi-norms defined as the following:

$$\begin{aligned}
\|h\|_{s\mathcal{B}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|h'(\zeta)|, \\
\|h\|_{s\mathcal{Z}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|h''(\zeta)|, \\
\gamma(h) &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|h^{(3)}(\zeta)|.
\end{aligned}$$

In [3],[5], the authors extend $\{V_n : n \in \mathbb{N}\}$ for $n = 0, 1, 2$ for harmonic mappings f as follows:

$$\begin{aligned}
\|f\|_{\mathcal{A}^{-1}_{\mathcal{H}}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)|f(\zeta)|, \\
\|f\|_{\mathcal{B}^{\mathcal{H}}} &:= |f(0)| + \|f\|_{s\mathcal{B}^{\mathcal{H}}}, \\
\|f\|_{\mathcal{Z}^{\mathcal{H}}} &:= |f(0)| + |h_\zeta(0)| + |f_{\bar{\zeta}}(0)| + \|f\|_{s\mathcal{Z}^{\mathcal{H}}}.
\end{aligned}$$

such that

$$\begin{aligned}
\|f\|_{s\mathcal{B}^{\mathcal{H}}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)\big(|f_\zeta(\zeta)| + |f_{\bar{\zeta}}(\zeta)|\big). \\
\|f\|_{s\mathcal{Z}^{\mathcal{H}}} &:= \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2)\big(|f_{\zeta\zeta}(\zeta)| + |f_{\overline{\zeta\zeta}}(\zeta)|\big).
\end{aligned}$$

In the next section, we introduce the extension of the space $\mathcal{V}$ of analytic functions to the harmonic settings.

3. The space of harmonic mappings $\mathcal{V}^{\mathcal{H}}$

We start this section by introducing the space $\mathcal{V}^{\mathcal{H}}$.

Definition 3.1. *The space $\mathcal{V}^{\mathcal{H}}$ is the set of all $f \in \mathcal{H}ar(\mathbb{D})$ such that*

$$\gamma^{\mathcal{H}}(f) := \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big[|\frac{\partial^3}{\partial\zeta^3}f(\zeta)| + |\frac{\partial^3}{\partial\overline{\zeta}^3}f(\zeta)|\big] < \infty. \tag{3.1}$$

Remark 3.1. *When $f \in H(\mathbb{D})$, we have $\frac{\partial^3}{\partial\overline{\zeta}^3}f(\zeta) = 0$, and this definition agrees with (2.1). Observe also that when $f = h + \overline{k}$, we have $\frac{\partial^3}{\partial\zeta^3}f(\zeta) = h^{(3)}(\zeta)$ and $\frac{\partial^3}{\partial\overline{\zeta}^3}f(\zeta) = \overline{k^{(3)}}$, then (3.1) becomes*

$$\gamma^{\mathcal{H}}(f) := \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big[|h^{(3)}(\zeta)| + |k^{(3)}(\zeta)|\big] < \infty. \tag{3.2}$$

This implies that

$$\frac{1}{2}\Big(\gamma(h)+\gamma(k)\Big) \leq \max\{\gamma(h)+\gamma(k) \leq \gamma^{\mathcal{H}}(f) \leq \gamma(h)+\gamma(k). \tag{3.3}$$

Theorem 3.1. *$\mathcal{V}^{\mathcal{H}}$ is a Banach space under the norm*

$$\begin{aligned}\|f\|_{\mathcal{V}^{\mathcal{H}}} &:= |f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta\zeta}}(0)| + \gamma^{\mathcal{H}}(f)\\ &= |f(0)| + \|f_\zeta + f_{\overline{\zeta}}\|_{\mathcal{Z}^{\mathcal{H}}}.\end{aligned}$$

Proof. For each $n \in \mathbb{N}$, suppose that (f_n) is a Cauchy sequence in $\mathcal{V}^{\mathcal{H}}$. Clearly, $(f_n(0))$ is Cauchy in $\mathbb{C}$ which is complete, thus

$$f(0) := \lim_{n\to\infty} f_n(0).$$

Furthermore, the sequence $\big((f_n)_\zeta + (f_n)_{\overline{\zeta}}\big)$ is Cauchy in $\mathcal{Z}^{\mathcal{H}}$, which is complete see [1]. Thus,

$$\lim_{n\to\infty}\big((f_n)_\zeta + (f_n)_{\overline{\zeta}}\big) \in \mathcal{Z}^{\mathcal{H}}.$$

Let $F \in \mathcal{Z}^{\mathcal{H}}$ such that

$$\|(f_n)_\zeta + (f_n)_{\overline{\zeta}} - F\|_{\mathcal{Z}^{\mathcal{H}}} \to 0, \text{ as } n \to \infty.$$

Let $F = H + \overline{K}$ with $K(0) = 0$. Then $H, K \in \mathcal{Z}$,

$$\|(f_n)_\zeta - H\|_{\mathcal{Z}} \to 0 \qquad \text{and} \qquad \|\overline{(f_n)_{\overline{\zeta}}} - K\|_{\mathcal{Z}} \to 0.$$

Define

$$h(\zeta) = f(0) + \int_0^\zeta H(z)\,dz \qquad \text{and} \qquad k(\zeta) = \int_0^\zeta G(z)\,dz.$$

Then $h, k \in H(\mathbb{D})$, $h' = H$ and $k' = K$. Thus $f, g \in \mathcal{V}$, such that $h(0) = f(0)$ and $k(0) = 0$.

Define $f(\zeta) := h(\zeta) + \overline{k(\zeta)}$,

$$\begin{aligned}\|f_n - f\|_{\mathcal{V}^{\mathcal{H}}} &= |f_n(0) - f(0)| + |(f_n)_\zeta(0) - H(0)|\\ &\quad + |\overline{(f_n)_{\overline{\zeta}}(0)} - K(0)| + |(f_n)_{\zeta\zeta}(0) - H(0)|\\ &\quad + |\overline{(f_n)_{\overline{\zeta\zeta}}(0)} - K(0)| + \gamma^{\mathcal{H}}((f_n)_\zeta + (f_n)_{\overline{\zeta}} - H - \overline{K}),\end{aligned} \tag{3.4}$$

which converges to 0 as $n \to \infty$. Therefore, $f_n \to f$ in $\mathcal{V}^{\mathcal{H}}$ and hence the space is complete.

The next theorem obtains the relationship between the semi-norm of $h \in \mathcal{V}$ and the semi-norm of $f \in \mathcal{V}^{\mathcal{H}}$.

Proposition 3.1. *For $h \in H(\mathbb{D})$, and $f \in \mathcal{H}ar(\mathbb{D})$. Consider that f is either the real or imaginary part of h. Then*

$$\gamma^{\mathcal{H}}(f) = \gamma(h)$$

Proof. First assume $f = Re(h)$. Then $f = \frac{1}{2}h + \frac{1}{2}\bar{h}$.
For $f \in \mathcal{V}^{\mathcal{H}}$, we have

$$\gamma^{\mathcal{H}}(f) = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big[\frac{1}{2}|h^{(3)}(\zeta)| + \frac{1}{2}|h^{(3)}(\zeta)|\big] = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|h^{(3)}(\zeta)| = \gamma(h). \tag{3.5}$$

Next assume $f = Im(h)$, then $f = \frac{1}{2i}h - \frac{1}{2i}\bar{h}$. So

$$\gamma^{\mathcal{H}}(f) := \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big(|\frac{1}{2i}h^{(3)}(\zeta)| + |\frac{1}{2i}h^{(3)}(\zeta)|\big) = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|h^{(3)}(\zeta)| = \gamma(h). \tag{3.6}$$

The next theorem shows the relationship between the harmonic mapping f and the analytic functions in its representation form.

Proposition 3.2. *Let $f = h + \overline{k}$ be a harmonic mapping. Then*

$$f \in \mathcal{V}^{\mathcal{H}} \iff h, k \in \mathcal{V}.$$

Proof. For $f = h + \overline{k}$ with h and k as in the statement. Then

$$\begin{aligned}\gamma^{\mathcal{H}}(f) &:= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big[|h^{(3)}(\zeta)| + |k^{(3)}(\zeta)|\big] \\ &= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|h^{(3)}(\zeta)| + \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|k^{(3)}(\zeta)| \\ &= \gamma(h) + \gamma(k).\end{aligned}$$

which is finite. Therefore, $f \in \mathcal{V}^{\mathcal{H}}$.

Conversely, assume $f \in \mathcal{V}^{\mathcal{H}}$, then by (3.2), we see that

$$\gamma(h) \le \gamma^{\mathcal{H}}(f) < \infty.$$

$$\gamma(k) \le \gamma^{\mathcal{H}}(f) < \infty.$$

Therefore, h and k are in $\mathcal{V}$.

Now, we wish to show that $\mathcal{V}^{\mathcal{H}}$ is properly contained in $\mathcal{Z}^{\mathcal{H}}$, so we shall make use of the following lemma that introduced in [9], for analytic settings.

Lemma 3.1. *The space $\mathcal{V}$ is properly contained in $\mathcal{Z}$. Furthermore, for $h \in H(\mathbb{D})$ we have*

$$\|h\|_{\mathcal{Z}} \le \|h\|_{\mathcal{V}}. \tag{3.7}$$

Theorem 3.2. *$\mathcal{V}^{\mathcal{H}}$ is contained in $\mathcal{Z}^{\mathcal{H}}$. Moreover,*

$$\|f\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2\|f\|_{\mathcal{V}^{\mathcal{H}}}. \tag{3.8}$$

Proof. For $h \in H(\mathbb{D})$ by lemma (3.1), we have

$$|h(0)| + |h'(0)| + \|h\|_{s\mathcal{Z}} \leq |h(0)| + |h'(0)| + |h''(0)| + \gamma(h). \tag{3.9}$$

That implies

$$\|h\|_{s\mathcal{Z}} \leq |h''(0)| + \gamma(h). \tag{3.10}$$

(3.11)

Likewise,

$$\|k\|_{s\mathcal{Z}} \leq |k''(0)| + \gamma(k). \tag{3.12}$$

For a harmonic mapping $f = h + \overline{k}$, we know that

$$\begin{aligned} \|f\|_{s\mathcal{Z}^{\mathcal{H}}} &\leq \|h\|_{s\mathcal{Z}} + \|k\|_{s\mathcal{Z}} \\ &\leq |h''(0)| + \gamma(h) + |k''(0)| + \gamma(k) \\ &\leq |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta\zeta}}(0)| + 2\gamma^{\mathcal{H}}(f). \end{aligned}$$

Therefore, add $|f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)|$ to both sides of the above inequality to get

$$\begin{aligned} |f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)| + \|f\|_{s\mathcal{Z}^{\mathcal{H}}} &\leq |f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta\zeta}}(0)| + 2\gamma^{\mathcal{H}}(f) \\ &\leq 2\Big[|f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta\zeta}}(0)| + \gamma^{\mathcal{H}}(f)\Big]. \end{aligned}$$

Thus we obtain

$$\|f\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2\|f\|_{\mathcal{V}^{\mathcal{H}}}. \tag{3.13}$$

The following result shows that the space $\mathcal{V}^{\mathcal{H}}$ is continuously embbeding in the space of continuous functions on the closed unit disk.

Theorem 3.3. *The space $\mathcal{V}^{\mathcal{H}}$ is contained in $C(\overline{\mathbb{D}})$. Moreover, for $f \in \mathcal{V}^{\mathcal{H}}$, we have*

$$\|f\|_\infty \leq 2\log 2\|f\|_{\mathcal{V}^{\mathcal{H}}}.$$

Proof. Let $f \in \mathcal{V}^{\mathcal{H}}$ such that $f = h + \overline{k}$, and $k(0) = 0$. Then we have

$$\begin{aligned} |f(\zeta)| &\leq |h(\zeta)| + |k(\zeta)| \\ &\leq |h(0)| + |h'(0)| + |h''(0)| + \log 2\|h'\|_{s\mathcal{Z}} + |k'(0)| + |k''(0)| + \log 2\|k'\|_{s\mathcal{Z}} \\ &= |f(0)| + |f_\zeta(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta}}(0)| + |f_{\overline{\zeta\zeta}}(0)| + \log 2\Big[\|f_\zeta\|_{s\mathcal{Z}} + \|f_{\overline{\zeta}}\|_{s\mathcal{Z}}\Big] \\ &\leq |f(0)| + |f_\zeta(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta}}(0)| + |f_{\overline{\zeta\zeta}}(0)| + \log 2\Big[\gamma(h) + \gamma(k)\Big] \\ &\leq |f(0)| + |f_\zeta(0)| + |f_{\overline{\zeta}}(0)| + |f_{\zeta\zeta}(0)| + |f_{\overline{\zeta\zeta}}(0)| + 2\log 2\gamma^{\mathcal{H}}(f) \\ &\leq 2(\log 2)\|f\|_{\mathcal{V}^{\mathcal{H}}}. \end{aligned}$$

Therefore, $\|f\|_\infty \leq 2\log 2\|f\|_{\mathcal{V}^{\mathcal{H}}}$ as desired. □

The following result shows that, as in the analytic setting, the space $\mathcal{V}^{\mathcal{H}}$ is an automorphism invariant.

Theorem 3.4. *The space $\mathcal{V}^{\mathcal{H}}$ is an automorphism invariant.*

Proof. Since every conformal automorphism of $\mathbb{D}$ is the composite of a mapping of the form φ_b and a rotation, and rotations clearly do not affect the seminorm. it is enough to obtain that for any $f \in \mathcal{V}^{\mathcal{H}}$ and $b \in \mathbb{D}$, $f \circ \varphi_b \in \mathcal{V}^{\mathcal{H}}$.

Fix $b \in \mathbb{D}$, and set

$$\varphi_b(\zeta) = \frac{b - \zeta}{1 - \bar{b}\zeta}.$$

Moreover, for all $\zeta \in \mathbb{D}$

$$|\varphi_b'(\zeta)| = \frac{1 - |b|^2}{|1 - \bar{b}\zeta|^2}.$$

$$|\varphi_b''(\zeta)| = \frac{2|b|(1 - |b|^2)}{|1 - \bar{b}\zeta|^3}.$$

$$|\varphi_b^{(3)}(\zeta)| = \frac{6|b|^2(1 - |b|^2)}{|1 - \bar{b}\zeta|^4}.$$

Then, we have

$$\begin{aligned} \left|\frac{\partial^3}{\partial\zeta^3}(f \circ \varphi_b)(\zeta)\right| &= \left|\varphi_b^{(3)}(\zeta)(f_\zeta \circ \varphi_b)(\zeta) + 3\varphi_b''(\zeta)(f_\zeta \circ \varphi_b)_\zeta(\zeta) + \varphi_b'(\zeta)^2(f_{\zeta\zeta} \circ \varphi_b)_\zeta(\zeta)\right| \\ \left|\frac{\partial^3}{\partial\bar{\zeta}^3}(f \circ \varphi_b)(\zeta)\right| &= \left|\varphi_b^{(3)}(\zeta)(f_{\bar{\zeta}} \circ \varphi_b)(\zeta) + 3\varphi_b''(\zeta)(f_{\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta) + \varphi_b'(\zeta)^2(f_{\bar{\zeta}\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta)\right| \end{aligned}$$

Thus,

$$\begin{aligned} \gamma^{\mathcal{H}}(f \circ \varphi_b)(\zeta) &= \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)\Big[\left|\frac{\partial^3}{\partial\zeta^3}(f \circ \varphi_b)(\zeta)\right| + \left|\frac{\partial^3}{\partial\bar{\zeta}^3}(f \circ \varphi_b)(\zeta)\right|\Big] \\ &= \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)\Big[\left|\varphi_b^{(3)}(\zeta)(f_\zeta \circ \varphi_b)(\zeta) + 3\varphi_b''(\zeta)(f_\zeta \circ \varphi_b)_\zeta(\zeta) + \varphi_b'(\zeta)^2(f_{\zeta\zeta} \circ \varphi_b)_\zeta(\zeta)\right| \\ &+ \left|\varphi_b^{(3)}(\zeta)(f_{\bar{\zeta}} \circ \varphi_b)(\zeta) + 3\varphi_b''(\zeta)(f_{\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta) + \varphi_b'(\zeta)^2(f_{\bar{\zeta}\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta)\right|\Big] \\ &\le \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b^{(3)}(\zeta)||(f_\zeta \circ \varphi_b)(\zeta)| + 3\sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b''(\zeta)||(f_\zeta \circ \varphi_b)_\zeta(\zeta)| \\ &+ \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b'(\zeta)|^2|(f_{\zeta\zeta} \circ \varphi_b)_\zeta(\zeta)| \\ &+ \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b^{(3)}(\zeta)||(f_{\bar{\zeta}} \circ \varphi_b)(\zeta)| + 3\sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b''(\zeta)||(f_{\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta)| \\ &+ \sup_{\zeta \in \mathbb{D}}(1 - |\zeta|^2)|\varphi_b'(\zeta)|^2|(f_{\bar{\zeta}\bar{\zeta}} \circ \varphi_b)_{\bar{\zeta}}(\zeta)|. \end{aligned}$$

Thus

$$\begin{aligned}
\gamma^{\mathcal{H}}(f\circ\varphi_b)(\zeta) &= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b^{(3)}(\zeta)|\Big[|(f_\zeta\circ\varphi_b)(\zeta)|+|(f_{\overline{\zeta}}\circ\varphi_b)(\zeta)|\Big]\\
&+ 3\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b''(\zeta)|\Big[|(f_\zeta\circ\varphi_b)_\zeta(\zeta)|+|(f_{\overline{\zeta}}\circ\varphi_b)_{\overline{\zeta}}(\zeta)|\Big]\\
&+ \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b'(\zeta)|^2\Big[|(f_{\zeta\zeta}\circ\varphi_b)_\zeta(\zeta)|+|(f_{\overline{\zeta\zeta}}\circ\varphi_b)_{\overline{\zeta}}(\zeta)|\Big]\\
&\le \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b^{(3)}(\zeta)|\|f\|_{\mathcal{V}^{\mathcal{H}}}+3\sup_{\zeta\in\mathbb{D}}|\varphi_b''(\zeta)|\|((f_\zeta+f_{\overline{\zeta}})\circ\varphi_b)_\zeta(\zeta)\|_{s\mathcal{B}^H}\\
&+ \sup_{\zeta\in\mathbb{D}}|\varphi_b'(\zeta)|^2\|((f_{\zeta\zeta}+f_{\overline{\zeta\zeta}})\circ\varphi_b)_\zeta(\zeta)\|_{s\mathcal{B}^H}.
\end{aligned}$$

The space $\mathcal{B}^H$ is a Möbius invariant, and recall that

$$\|((f_\zeta+f_{\overline{\zeta}})\circ\varphi_b)(\zeta)\|_{s\mathcal{B}^H}=\|(f_\zeta+f_{\overline{\zeta}})(\zeta)\|_{s\mathcal{B}^H}=\|f\|_{s\mathcal{Z}^{\mathcal{H}}}\le\|f\|_{\mathcal{Z}^{\mathcal{H}}}\le 2\|f\|_{\mathcal{V}^{\mathcal{H}}}.$$

$$\|(f_{\zeta\zeta}+f_{\overline{\zeta\zeta}})\circ\varphi_b(\zeta)\|_{s\mathcal{B}^H}=\|f_{\zeta\zeta}+f_{\overline{\zeta\zeta}}(\zeta)\|_{s\mathcal{B}^H}=\|(f_\zeta+f_{\overline{\zeta}})(\zeta)\|_{s\mathcal{Z}^{\mathcal{H}}}=\gamma^{\mathcal{H}}(f)\le\|f\|_{\mathcal{V}^{\mathcal{H}}}.$$

Therefore, we have

$$\begin{aligned}
\gamma^{\mathcal{H}}(f\circ\varphi_b)(\zeta) &\le \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b^{(3)}(\zeta)|\|f\|_{\mathcal{V}^{\mathcal{H}}}+6\sup_{\zeta\in\mathbb{D}}|\varphi_b''(\zeta)|\|f\|_{\mathcal{V}^{\mathcal{H}}}+\sup_{\zeta\in\mathbb{D}}|\varphi_b'(\zeta)|^2\|f\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \Big[\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)|\varphi_b^{(3)}(\zeta)|+6\sup_{\zeta\in\mathbb{D}}|\varphi_b''(\zeta)|+\sup_{\zeta\in\mathbb{D}}|\varphi_b'(\zeta)|^2\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma^{\mathcal{H}}(f\circ\varphi_b)(\zeta) &\le \Big[\sup_{\zeta\in\mathbb{D}}\frac{6|b|^2(1-|\zeta|^2)(1-|b|^2)}{|1-\bar{b}\zeta|^4}+\sup_{\zeta\in\mathbb{D}}\frac{12|b|(1-|b|^2)}{|1-\bar{b}\zeta|^3}+\sup_{\zeta\in\mathbb{D}}\frac{(1-|b|^2)^2}{|1-\bar{b}\zeta)|^4}\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}\\
&\le \Big[\sup_{\zeta\in\mathbb{D}}\frac{6(1-|\zeta|^2)(1-|b|^2)}{(1-|\zeta|)(1-|b|)^3}+\sup_{\zeta\in\mathbb{D}}\frac{12(1-|b|^2)}{(1-|b|)^3}+\sup_{\zeta\in\mathbb{D}}\frac{(1-|b|^2)^2}{(1-|b|)^4}\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \Big[\frac{12(1+|b|)}{(1-|b|)^2}+\frac{12(1+|b|)}{(1-|b|)^2}+\frac{(1+|b|)^2}{(1-|b|)^2}\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \Big[\frac{(1+|b|)(25+|b|)}{(1-|b|)^2}\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \Big[\frac{(1+|\varphi_b(0)|)(25+|\varphi_b(0)|)}{(1-|\varphi_b(0)|)^2}\Big]\|f\|_{\mathcal{V}^{\mathcal{H}}}.
\end{aligned}$$

Therefore, $\gamma^{\mathcal{H}}(f\circ\varphi_b)(\zeta)\in\mathcal{V}^{\mathcal{H}}$. □

Definition 3.2. *The little subspace $\mathcal{V}_0^{\mathcal{H}}$ is the set of all elements in $\mathcal{V}^{\mathcal{H}}$ such that:*

$$\mathcal{V}_0^{\mathcal{H}}:=\{f\in\mathcal{V}^{\mathcal{H}}:\lim_{|\zeta|\to1}(1-|\zeta|^2)\big[|\frac{\partial^3}{\partial\zeta^3}f(\zeta)|+|\frac{\partial^3}{\partial\overline{\zeta}^3}f(\zeta)|\big]=0\}. \tag{3.14}$$

Theorem 3.5. *The little subspace $\mathcal{V}_0^{\mathcal{H}}$ is a closed separable.*

Proof. Suppose the sequence $f_n\in\mathcal{V}_0^{\mathcal{H}}$ converges in norm to $f\in\mathcal{V}^{\mathcal{H}}$, thus the sequence $\{(f_n)_\zeta+(f_n)_{\overline{\zeta}}\}\in\mathcal{Z}_0^{\mathcal{H}}$, and

$$\|\{(f_n)_\zeta + (f_n)_{\bar{\zeta}}\} - \{f_\zeta + f_{\bar{\zeta}}\}\|_{\mathcal{Z}^{\mathcal{H}}} \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\mathcal{Z}_0^{\mathcal{H}}$ is closed subspace of $\mathcal{Z}^{\mathcal{H}}$, we get $f_\zeta + f_{\bar{\zeta}} \in \mathcal{Z}_0^{\mathcal{H}}$ which implies $f \in \mathcal{V}_0^{\mathcal{H}}$. Therefore, $\mathcal{V}_0^{\mathcal{H}}$ is closed.

For separability, it is sufficient to approximate any $f \in \mathcal{V}_0^{\mathcal{H}}$ by a sequence in the form $A_n + \overline{B_n}$, where A_n and B_n are polynomials in $\mathcal{V}$

Let $f = h + \overline{k} \in \mathcal{V}_0^{\mathcal{H}}$, therefore, $h', k' \in \mathcal{Z}_0$. The functions h' and k' approximated in norm by sequences of polynomials a_n and b_n, respectively.

For $n \in \mathbb{N}$, let

$$A_n(\zeta) = f(0) + \int_0^\zeta a_n(z)\,dz$$

and

$$B_n(\zeta) = \int_0^\zeta b_n(z)\,dz.$$

Thus, A_n and B_n are polynomials in $\mathcal{V}$ such that $A_n(0) = f(0)$, $B_n(0) = 0$, and as $n \to \infty$, we have

$$\|f - (A_n + \overline{B_n})\|_{\mathcal{V}^{\mathcal{H}}} \le \|h'(0) - a_n\|_{\mathcal{Z}} + \|k' - b_n\|_{\mathcal{Z}} \to 0.$$

The polynomials in $\mathcal{V}$ with coefficients in $\mathbb{Q}[i]$ are a countable dense subset of $\mathbb{C}[\zeta]$. Therefore, $\mathcal{V}_0^{\mathcal{H}}$ is separable. □

4. Composition operators on the space $\mathcal{V}^{\mathcal{H}}$

The composition operators C_φ induced by an analytic or conjugate analytic of self map on the region $\mathbb{D}$, defined as

$$C_\varphi f = f \circ \varphi.$$

Now, we move our attention to investigate the boundedness and the compactness of the above operator on the space $\mathcal{V}^{\mathcal{H}}$.

Recall the following lemmas that were introduced in [21]. Both of them will be used in the proof of the main theorems in this section.

Lemma 4.1. *For φ be analytic self-map of the unit disk. Then*

(1) $C_\varphi : \mathcal{V} \to \mathcal{V}$ *is bounded..*
(2) $\varphi \in \mathcal{V}$ *and*

$$\sup_{\zeta\in\mathbb{D}}(1 - |\zeta|^2)\big(|\varphi'(\zeta)\varphi''(\zeta)\log\frac{e}{1 - |\varphi(\zeta)|^2}\big) < \infty.$$

Lemma 4.2. *For φ be analytic self-map of the unit disk. Then*

(1) $C_\varphi : \mathcal{V} \to \mathcal{V}$ *is compact..*
(2) $\frac{(1-|\zeta|^2)|\varphi'(\zeta)^3|}{1-|\varphi(\zeta)|^2} = 0$ *and*

$$\lim_{|\varphi(\zeta)\to 1}(1 - |\zeta|^2)\big(|\varphi'(\zeta)\varphi''(\zeta)\log\frac{e}{1 - |\varphi(\zeta)|^2}\big) = 0.$$

Theorem 4.1. *For φ be analytic self-map of the unit disk. Then*

(1) $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ *is bounded.*
(2) $C_\varphi : \mathcal{V} \to \mathcal{V}$ *is bounded..*
(3) $\varphi \in \mathcal{V}$ *and*

$$\sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\big(|\varphi'(\zeta)\varphi''(\zeta)\log\frac{e}{1-|\varphi(\zeta)|^2}\big)<\infty.$$

Proof. The equivalent of (2) and (3) is due to Lemma (4.1). It is enough to show that (1) $\iff$ (2).

Assume that $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ is bounded. Since $\mathcal{V}$ is a subspace of $\mathcal{V}^{\mathcal{H}}$, we have $\|f\|_{\mathcal{V}} = \|f\|_{\mathcal{V}^{\mathcal{H}}}$ for each $f \in \mathcal{V}$. Let $h \in \mathcal{V}$, $h \circ \varphi$ is an analytic function in $\mathcal{V}^{\mathcal{H}}$, and also in $\mathcal{V}$. Therefore, $C_\varphi : \mathcal{V} \to \mathcal{V}$ is bounded.

Conversely, now consider $C_\varphi : \mathcal{V} \to \mathcal{V}$ is bounded. Let $f \in \mathcal{V}^{\mathcal{H}}$ such that $f = h + \overline{k}$ and $k(0) = 0$. Recall that when $h, k \in \mathcal{V}$, then $h \circ \varphi, k \circ \varphi \in \mathcal{V}$, and

$$\|h\circ\varphi\|_{\mathcal{V}} \le \|C_\varphi\|_{\mathcal{V}\to\mathcal{V}}\|h\|_{\mathcal{V}}.$$

Likewise,

$$\|k\circ\varphi\|_{\mathcal{V}} \le \|C_\varphi\|_{\mathcal{V}\to\mathcal{V}}\|k\|_{\mathcal{V}}.$$

Therefore, $f \circ \varphi = h \circ \varphi + \overline{k \circ \varphi} \in \mathcal{V}^{\mathcal{H}}$, and

$$\|h\circ\varphi\|_{\mathcal{V}^{\mathcal{H}}} \le \|f\circ\varphi\|_{\mathcal{V}} + \|k\circ\varphi\|_{\mathcal{V}}.$$

Since C_φ is bounded as an operator on $\mathcal{V}$, we obtain that $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ is bounded. □

Proposition 4.1. *Let $\{h_n\}$ be a sequence in $H(\mathbb{D})$. If $\{h_n\}$ is in the unit ball of $\mathcal{V}$, then $\{h_n\}$ is in the unit ball of $\mathcal{V}^{\mathcal{H}}$ as well.*

Proof. Assume that $\{h_n\}$ be in the unit ball of $\mathcal{V}$, then $\{h_n\} \in \mathcal{V}^{\mathcal{H}}$ since $\mathcal{V} \subset \mathcal{V}^{\mathcal{H}}$, and

$$\|h_n\|_{\mathcal{V}^{\mathcal{H}}} = \|h_n\|_{\mathcal{V}}.$$

Thus, $\{h_n\}$ be in the unit ball of $\mathcal{V}^{\mathcal{H}}$. □

Theorem 4.2. *For φ be analytic self-map of the unit disk. Then*

(1) $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ *is compact.*
(2) $C_\varphi : \mathcal{V} \to \mathcal{V}$ *is compact..*
(3) $\frac{(1-|\zeta|^2)|\varphi'(\zeta)^3|}{1-|\varphi(\zeta)|^2} = 0$ *and*

$$\lim_{|\varphi(\zeta)|\to 1}(1-|\zeta|^2)\big(|\varphi'(\zeta)\varphi''(\zeta)\log\frac{e}{1-|\varphi(\zeta)|^2}\big)=0.$$

Proof. The equivalence of (2) and (3) follows directly from Lemma 4.2. It is enough to show the equivalent of (1) and (2).

Assume $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ is compact. Then $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ is bounded, so by Theorem 4.1, we have $C_\varphi : \mathcal{V} \to \mathcal{V}$ is bounded.

Let (h_n) be a sequence of analytic functions in the unit ball of $\mathcal{V}$, by proposition 4.1, we have (h_n) is in the unit ball of $\mathcal{V}^{\mathcal{H}}$ as well. Thus, there is a subsequence (h_{n_j}) and $h \in \mathcal{V}^{\mathcal{H}}$ such that:

$$\|h_{n_j}\circ\varphi - h\|_{\mathcal{V}^{\mathcal{H}}} \to 0, \quad \text{as} \quad n\to\infty.$$

By Theorem 3.3, we have converges in $\|\cdot\|_{\mathcal{V}^{\mathcal{H}}}$ implies uniform convergence on $\overline{\mathbb{D}}$, and the function h is the uniform limit on compact sets of (h_n). Thus, $h \in H((\mathbb{D})$. Moreover, $h \in \mathcal{V}$.

For any functions on $\mathcal{V}$, we have

$$\|\cdot\|_{\mathcal{V}^{\mathcal{H}}} = \|\cdot\|_{\mathcal{V}}.$$

Therefore, $C_\varphi : \mathcal{V} \to \mathcal{V}$ is compact.

Conversely, consider $C_\varphi : \mathcal{V} \to \mathcal{V}$ is compact. Let (f_n) be a sequence of harmonic mappings in the unit ball of $\mathcal{V}^{\mathcal{H}}$, such that $\|f_n\|_{\mathcal{V}^{\mathcal{H}}} = 1$.

For $n \in \mathbb{N}$, there exist $h_n, k_n \in H(\mathbb{D})$ such that $f_n = h_n + \overline{k_n}$ and $g_n(0) = 0$. Moreover, $h_n, k_n \in \mathcal{V}$. Due to the compactness of C_φ on $\mathcal{V}$, there exist a subsequence (h_{n_j}) and $h \in \mathcal{V}$ such that

$$\|h_{n_j} \circ \varphi - h\|_{\mathcal{V}} \to 0, \quad \text{as } j \to \infty.$$

For the bounded sequence (k_{n_j}) in $\mathcal{V}$ there exist a subsequence $(k_{n_{j_i}})$ and $k \in \mathcal{V}$ such that

$$\|k_{n_{j_i}} \circ \varphi - k\|_{\mathcal{V}^{\mathcal{H}}} \to 0, \quad \text{as } i \to \infty.$$

Let $f = h + \overline{k}$, then by Theorem 3.2, we have $f \in \mathcal{V}^{\mathcal{H}}$ and

$$\|f_{n_{j_i}} \circ \varphi - f\|_{\mathcal{V}^{\mathcal{H}}} \le \|h_{n_{j_i}} \circ \varphi - h\|_{\mathcal{V}} + \|k_{n_{j_i}} \circ \varphi - k\|_{\mathcal{V}} \to 0,$$

as $i \to \infty$. Therefore $C_\varphi : \mathcal{V}^{\mathcal{H}} \to \mathcal{V}^{\mathcal{H}}$ is compact.

□

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