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FIXED POINT RESULTS VIA Ψ-CONTRACTION IN C^*-ALGEBRA VALUED GENERALIZED METRIC SPACES WITH APPLICATIONS

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Abstract. In this manuscript, using the concept of C^*-algebra valued generalized metric spaces, fixed point results for self mappings with contractive conditions related to a nondecreasing function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\lim_{m\to\infty} \psi^m(s) = 0,\ \forall\ s \in (0, +\infty)$, are established. As an application, uniqueness and existence result for integral equation are also provided.

1. Introduction and Preliminaries

In 1947, Segal [2] initiate the word "C^*-algebra" to explain a "uniformly closed, self adjoint algebra of bounded operators on a Hilbert space". A C^*-algebra[3] is regularly use to describe a physical structure in statistical mechanics and quantum field theory and it has subsequently emerged as a significant area of research. As we have known that, the most important result in the theory of fixed point is the Banach contractive principle. The principle has been extensively used in various branches of physics and mathematics. There are numerous generalizations for such a principle. Generally, the principle has been divided in two ways. On one side, usual contraction conditions was changed by a weakly contraction. While on other side, different types of metric spaces replaced the action spaces[3, 4].
The first generalization in this direction is Edelstein's contractive condition [5] in which the Banach condition is changed by picking different points from the space Z and allowing constant $\beta = 1$. Later, Rakotch[6] developed a contractive condition in which a monotonic decreasing function $\beta : [0, 1) \to [0, 1)$ replaces the constant β of the contraction condition. That is,

$$d(S\eta, S\zeta) \leq \beta(t) d(\eta, \zeta), \quad \forall \quad \eta, \zeta \in Z.$$

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Later on, it was generalized by Kannan[7], in which he has changed the contraction condition by

$$d(S\eta, S\zeta) \leq \beta\{d(\eta, S\eta) + d(\zeta, S\zeta)\}, \quad \text{where} \quad 0 < \beta < \frac{1}{2}.$$

Following Kannan, Chatterjea[8] established a fixed point theorem for operators that satisfy the given criteria $\forall\ \eta, \zeta \in Z$.

$$d(S\eta, S\zeta) \leq \beta\{d(\eta, S\zeta) + d(\zeta, S\eta)\}, \quad \text{where} \quad 0 < \beta < 1.$$

Especially Mustafa in collaboration with Sims [9] established the idea of generalized metric space and proved various results of fixed point. Since then, many researchers studied the theory of fixed point in generalized metric spaces and numerous significant results are obtained [10, 11, 12, 13]. In 2014, Ma et al. [4] developed the idea of C^*-algebra valued metric spaces and presented several fixed point results by giving the structure of contraction mapping with reference to C^*-algebra, corresponding to the Banach contractive principle. Also they used expansive mappings and proved various fixed point results.
After that, Kamran et al. [21] extended the result of Ma et al.[4] to C^*-algebra valued b-metric space and they generalized the Banach contractive principle. For some interesting fixed point results, see, also, the papers of Humaira et al. [38] , Hussain et al. [39] , Shoaib et al. ([40, 41]) and Zada et al. [42] .

In this paper we used the idea of Shatanawi [1] and presented various fixed point theorems in C^*-algebra valued generalized metric spaces. Our result generalizes the result of Mustafa and Sims[9], Ma et al.[4] and Kamran et al.[21]. Now we summarize the necessary definitions and propositions that should be needed throughout the paper. Throughout the paper, $\mathbb{M}$ will denote unital C^*-algebra and $\mathbb{N}$ will denote natural number.

Definition 1.1. [4] An algebra $\mathbb{M}$ over $\mathbb{C}$ (complex number) is known as $*$-algebra if there is an involution $* : \mathbb{M} \to \mathbb{M}$ satisfy the following axioms

$$\begin{aligned}
(\eta + \zeta)^* &= \eta^* + \zeta^*, \\
(\alpha\eta)^* &= \overline{\alpha}\eta^*, \\
(\eta\zeta)^* &= \zeta^*\eta^*, \\
(\eta^*)^* &= \eta, \quad \text{for all} \quad \eta, \zeta \in \mathbb{M}, \quad \text{and} \quad \alpha \in \mathbb{C}.
\end{aligned}$$

Definition 1.2. [4] A Banach $*$-algebra $(\mathbb{M}\ , *)$ is said to be a C^*-algebra if

$$\|\eta\eta^*\| = \|\eta\|^2,$$

$\forall\ \eta \in \mathbb{M}$.

Example 1.1. [4] *Suppose Z be a compact Hausdroff space and $C_\circ(Z)$ denote all continuous function. The involution map $* : C_\circ(Z) \to C_\circ(Z)$ is defined by*

$$g^*(a) = \overline{g}(a), \qquad \forall \quad a \in Z.$$

Then $C_\circ(Z)$ is a C^*-algebra. Define norm on $C_\circ(Z)$ as

$$\|g(a)\| = \sup_{a \in Z} |g(a)|.$$

This norm is submultiplicative. In fact

$$\begin{aligned}\|gf\| &= \sup_{a\in Z}|g(a)f(a)| \\ &\le \sup_{a\in Z}|g(a)|\sup_{a\in Z}|f(a)| \\ &= \|g\|\|f\|.\end{aligned}$$

Also,

$$\begin{aligned}\|gg^*\| &= \sup_{a\in Z}|g(a)g^*(a)| \\ &= \sup_{a\in Z}|g(a)\overline{g}(a)|, \\ &= \sup_{a\in Z}|g(a)|^2.\end{aligned}$$

Thus $C_\circ(X)$ is a C^*-algebra.

Definition 1.3. [1] Suppose $Z \neq \emptyset$ and $G : Z^3 \to [0, +\infty)$ is a map, which satisfies the following the conditions:

$$\begin{aligned}G(\eta,\zeta,\theta) &= 0, \text{ iff } \eta=\zeta=\theta, \\ G(\eta,\eta,\zeta) &> 0, \text{ with } \eta\neq\zeta, \\ G(\eta,\eta,\zeta) &\le G(\eta,\zeta,\theta), \text{ with } \theta\neq\zeta, \\ G(\eta,\zeta,\theta) &= G(\eta,\theta,\zeta)=G(\zeta,\theta,\eta)=\cdots, \\ G(\eta,\zeta,\theta) &\le G(\eta,a,a)+G(a,\zeta,\theta), \quad \forall\ \eta,\zeta,\theta,a\in Z.\end{aligned}$$

Then Z together with a map G is known as generalized metric space.

Example 1.2. [1] *Assume that $(\mathbb{R}, d)$ be the usual metric space. Now a map G is defined by*

$$G(\eta,\zeta,\theta) = d(\eta,\zeta) + d(\zeta,\theta) + d(\eta,\theta), \qquad \forall\ \eta,\zeta,\theta\in\mathbb{R}.$$

Then clearly $\mathbb{R}$ with a map G is a generalized metric space.

The next definition is taken from [36].

Definition 1.4. Suppose $Z \neq \emptyset$ and $G : Z^3 \to \mathbb{M}$(C^*-algebra) is a map satisfying

$$\begin{aligned}G(\eta,\zeta,\theta) &= 0 \quad \text{iff} \quad \eta=\zeta=\theta, \\ G(\eta,\eta,\zeta) &> 0 \quad \text{with} \quad \eta\neq\zeta, \\ G(\eta,\eta,\zeta) &\le G(\eta,\zeta,\theta) \quad \text{with} \quad \zeta\neq\theta, \\ G(\eta,\zeta,\theta) &= G(\zeta,\theta,\eta)=G(\eta,\theta,\zeta)=\cdots, \\ G(\eta,\zeta,\theta) &\le G(\eta,a,a)+G(a,\zeta,\theta), \quad \forall\ \eta,\zeta,\theta,a\in Z \qquad \text{(rectangle inequality)}.\end{aligned}$$

Then $(Z, \mathbb{M}, G)$ is known as C^*-algebra valued generalized metric space.

A "C^*-algebra valued generalized metric space" is now referred to as a 'C^*-valued generalized metric space'.

Example 1.3. [36] *Suppose $X = \{\eta, \zeta\}$ and $M_2(\mathbb{C})$ of 2×2 matrices is identified by $B(\mathbb{C}^2)$ be a C^*-algebra. Suppose*

$$\begin{aligned}G(\eta,\eta,\eta) &= G(\zeta,\zeta,\zeta)=0, \\ G(\eta,\eta,\zeta) &= I, \\ G(\eta,\zeta,\zeta) &= 2I,\end{aligned}$$

enlarge G to $Z \times Z \times Z$ by using symmetry. Then the triplet (Z, M_2, G) is a C^*-valued generalized metric space.

Example 1.4. [36] *Suppose $Z = \mathbb{C}$, $\mathbb{M} = M_3(\mathbb{C})$ and $\alpha, \beta > 0$, set*

$$G(\eta,\zeta,\theta) = \begin{bmatrix} g(\eta,\zeta,\theta) & 0 & 0 \\ 0 & \alpha g(\eta,\zeta,\theta) & 0 \\ 0 & 0 & \beta g(\eta,\zeta,\theta) \end{bmatrix},$$

where $g(\eta,\zeta,\theta) = |\eta-\zeta|+|\zeta-\theta|+|\theta-\eta|$. Then $(Z,\mathbb{M},G)$ is a C^*-valued generalized metric space.

Definition 1.5. [36] A sequence $\{\eta_c\}$ in Z is known as G-convergent to $\eta \in Z$ with respect to $(Z,\mathbb{M},G)$ if

$$\lim_{c,d\to\infty} G(\eta,\eta_c,\eta_d) = 0.$$

That is, $\forall\ \epsilon > 0$, $\exists\ k \in \mathbb{N}$ so that

$$\|G(\eta,\eta_c,\eta_d)\| < \epsilon, \qquad \forall \quad c,d > k.$$

The next proposition is essentially contains in [36], but we will prove it for convenience.

Proposition 1.1. *Suppose $(Z,\mathbb{M},G)$ be a C^*-valued generalized metric space, $\{\eta_c\} \subseteq Z$, $\eta \in Z$. Then all are equivalent*

$$\eta_c \to \eta \quad as \quad c \to \infty \tag{1}$$

$$G(\eta_c,\eta_c,\eta) \to 0 \quad as \quad c \to \infty \tag{2}$$

$$G(\eta_c,\eta,\eta) \to 0 \quad as \quad c \to \infty. \tag{3}$$

Proof. **(1)** $\Rightarrow$ **(2)** if $\eta_c \xrightarrow{G} \eta$, that is, $\forall\ \epsilon > 0$, $\exists\ k \in \mathbb{N}$, so that, $\forall \quad c,d > k$, $\|G(\eta_c,\eta_d,\eta)\| < \epsilon$, especially $\|G(\eta_c,\eta_c,\eta)\| < \epsilon$.
Hence $G(\eta_c,\eta_c,\eta) \to 0$ as $c \to \infty$.

(2) $\Rightarrow$ **(3)** if $\forall\ \epsilon > 0$, there exists $k \in \mathbb{N}$ so that $\forall\ c > \mathbb{N}$, we have $G(\eta_c,\eta_c,\eta) < \epsilon/2$, when $c > \mathbb{N}$. Then,

$$\begin{aligned} \|G(\eta_c,\eta,\eta)\| &= \|G(\eta,\eta_c,\eta)\|, \\ &\leq \|G(\eta,\eta_c,\eta_c)\| + \|G(\eta_c,\eta_c,\eta)\| < \epsilon, \end{aligned}$$

that is $G(\eta_c,\eta,\eta) \to 0$ as $c \to \infty$.

(3) $\Rightarrow$ **(1)** if $G(\eta_c,\eta,\eta) \to 0$ as $c \to \infty$, then for $\epsilon > 0$, $\exists\ k_1 \in \mathbb{N}$, so that $\forall\ c > k_1$, $\|G(\eta_c,\eta,\eta)\| < \epsilon/2$, there exists $k_2 \in \mathbb{N}$ so that $\forall\ d > k_2$, $\|G(\eta_d,\eta,\eta)\| < \epsilon/2$. Let $k = k_1 + k_2$, $\forall\ c,d > \mathbb{N}$,

$$\|G(\eta_c,\eta_d,\eta)\| \leq \|G(\eta_c,\eta,\eta)\| + \|G(\eta,\eta_d,\eta)\| < \epsilon,$$

i.e. $\eta_c \xrightarrow{G} \eta$ as $c \to \infty$. ∎

Definition 1.6. A sequence $\{\eta_c\}$ in $(Z,\mathbb{M},G)$ is called G-Cauchy iff

$$\lim_{c,d,e\to\infty} G(\eta_c,\eta_d,\eta_e) = 0, \tag{4}$$

that is, $\forall\ \epsilon > 0$, $\exists\ k \in \mathbb{N}$ so that

$$\|G(\eta_c,\eta_d,\eta_e)\| < \epsilon, \qquad \forall \quad c,d,e > k.$$

The triplet $(Z,\mathbb{M},G)$ is known as G-complete if all G-Cauchy $\{\eta_c\}$ in $\mathbb{M}$ is G-convergent.

The above (Example 1.4) is complete. See the following proof.

Proof. For completeness, let $\{\eta_c\} \subseteq (Z, \mathbb{M}, G)$ be a G-Cauchy. Then $\forall\ \epsilon > 0$, $\exists$ $k \in \mathbb{N}$ so that

$$\|G(\eta_d, \eta_c, \eta_c)\| = \max\ \{g(\eta_d, \eta_c, \eta_c), \alpha g(\eta_d, \eta_c, \eta_c), \beta g(\eta_d, \eta_c, \eta_c)\} < \epsilon.$$

So $g(\eta_d, \eta_c, \eta_c) = 2|\eta_d - \eta_c| < \epsilon$. Since Z is complete, $\exists\ \eta \in Z$ so that $\eta_c \to \eta$. Then there exits $k_\circ \in \mathbb{N}$ so that $|\eta_c - \eta| < \epsilon/2$, for any $c > k_\circ$. It follows that

$$\begin{aligned} \|G(\eta_c, \eta, \eta)\| &= \max\ \{2|\eta - \eta_c|, 2\alpha|\eta - \eta_c|, 2\beta|\eta - \eta_c|\} \\ &= 2\ \max\ \{1, \alpha, \beta\}|\eta - \eta_c| \\ &< \max\ \{1, \alpha, \beta\}\epsilon. \end{aligned}$$

Therefore, $\eta_c \xrightarrow{G} \eta$ and $(Z, \mathbb{M}, G)$ is complete. □

Proposition 1.2. *A sequence $\{\eta_c\}$ in $(Z, \mathbb{M}, G)$ is known as G-Cauchy iff $\forall\ \epsilon > 0$, there exists $k \in \mathbb{N}$, so that $\|G(\eta_d, \eta_c, \eta_c)\| < \epsilon$, $\forall\ c, d > k$.*

Proof. For $\epsilon > 0$, $\exists\ k_1 \in \mathbb{N}$ so that $\|G(\eta_d, \eta_c, \eta_c)\| < \epsilon/2$, $\forall\ c, d > k_1$; $\exists\ k_2 \in \mathbb{N}$ so that $\|G(\eta_e, \eta_c, \eta_c)\| < \epsilon/2$, $\forall\ c, e > k_2$. So for the above ϵ, let $k = k_1 + k_2$, where $c, d, e > k$. Then,
$\|G(\eta_d, \eta_c, \eta_e)\| \leq \|G(\eta_d, \eta_c, \eta_c)\| + \|G(\eta_c, \eta_c, \eta_e)\| < \epsilon$, i.e. $\{\eta_c\}$ is a G-Cauchy sequence. □

Definition 1.7. Let the triplet $(Z, \mathbb{M}, G)$ be a C^*-valued generalized metric space. A map $S : Z \to Z$ is known as contractive mapping if $\exists\ A \in \mathbb{M}$, $\|A\| < 1$ such that

$$G(S\eta, S\zeta, S\theta) \leq A^* G(\eta, \zeta, \theta) A, \qquad \text{for all } \eta, \zeta, \theta \in Z.$$

2. Fixed Point Results via Ψ-Contraction

Further we suppose that let Ψ be the set of all functions ψ such that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing map with $\lim_{m\to+\infty} \psi^m(s) = 0$, $\forall\ s \in (0, +\infty)$. A ψ is said to be a Ψ-map if $\psi \in \Psi$. If ψ is Ψ-map, then

$$\begin{aligned} \psi(s) &< s \qquad \forall\ s \in (0, +\infty), \\ \psi(0) &= 0. \end{aligned}$$

Theorem 2.1. *Suppose $(Z, \mathbb{M}, G)$ is a complete C^*-valued generalized metric space. Let the map $S : Z \to Z$ satisfies*

$$G(S\eta, S\zeta, S\theta) \leq \psi^*(G(\eta, \zeta, \theta))\psi,$$

$\forall\ \eta, \zeta, \theta \in Z$. Then there exists a unique $y \in Z$ such that $Sy = y$.

Proof. Choose one arbitrary point $\eta_\circ \in Z$ and $\eta_c = S\eta_{c-1}$, $\quad c = 1, 2, 3, \cdots$, i.e.

$$\begin{aligned} \eta_1 &= S\eta_\circ, \\ \eta_2 &= S\eta_1 = S^2\eta_\circ, \\ \eta_3 &= S\eta_2 = S^3\eta_\circ, \\ &\vdots \\ \eta_c &= S\eta_{c-1} = S^c\eta_\circ. \end{aligned}$$

We have

$$\begin{array}{rcl} G(\eta_c,\eta_{c+1},\eta_{c+1}) & = & G(S\eta_{c-1},S\eta_c,S\eta_c) \\ & \leq & \psi^*(G(\eta_{c-1},\eta_c,\eta_c))\psi = \psi^*(G(S\eta_{c-2},S\eta_{c-1},S\eta_{c-1}))\psi \\ & \leq & \psi^*\cdot\psi^*(G(\eta_{c-2},\eta_{c-1},\eta_{c-1}))\psi\cdot\psi = (\psi^*)^2(G(\eta_{c-2},\eta_{c-1},\eta_{c-1}))\psi^2 \\ & \leq & (\psi^*)^2(G(\eta_{c-2},\eta_{c-1},\eta_{c-1}))\psi^2 \\ & \leq & (\psi^*)^3(G(\eta_{c-3},\eta_{c-2},\eta_{c-2}))\psi^3 \\ & \vdots & \\ G(\eta_c,\eta_{c+1},\eta_{c+1}) & \leq & (\psi^*)^c(G(\eta_\circ,\eta_1,\eta_1))\psi^c. \end{array}$$

For $\epsilon > 0$,

$$\lim_{c\to\infty}(\psi^*)^c(G(\eta_\circ,\eta_1,\eta_1))\psi^c = 0,$$

and $\psi(\epsilon) < \epsilon$, $\exists\, q_\circ$, so that

$$(\psi^*)^c(G(\eta_\circ,\eta_1,\eta_1))\psi^c < \epsilon - \psi(\epsilon), \quad \forall\ c \geq q_\circ.$$

Thus

$$G(\eta_c,\eta_{c+1},\eta_{c+1}) < \epsilon - \psi(\epsilon). \tag{5}$$

Now, for any two positive integer c, d and $d > c$, then

$$G(\eta_c,\eta_d,\eta_d) < \epsilon, \quad \forall\ \ d \geq c \geq q_\circ. \tag{6}$$

We prove (6) by induction. Inequality (6) holds for $d = c+1$ by using (5) and $\epsilon - \phi(\epsilon) < \epsilon$. Suppose that (6) holds for $d = q$. Now we check for $d = q+1$. By rectangle inequality we get

$$\begin{array}{rcl} G(\eta_c,\eta_{q+1},\eta_{q+1}) & \leq & G(\eta_c,\eta_{c+1},\eta_{c+1}) + G(\eta_{c+1},\eta_{q+1},\eta_{q+1}) \\ & = & G(\eta_c,\eta_{c+1},\eta_{c+1}) + G(S\eta_c,S\eta_q,S\eta_q) \\ & \leq & G(\eta_c,\eta_{c+1},\eta_{c+1}) + \psi^*(G(\eta_c,\eta_q,\eta_q))\psi \\ & = & G(\eta_c,\eta_{c+1},\eta_{c+1}) + \psi^*\{(G(\eta_c,\eta_q,\eta_q))^{1/2}(G(\eta_c,\eta_q,\eta_q))^{1/2}\}\psi \\ & = & G(\eta_c,\eta_{c+1},\eta_{c+1}) + \{\psi(G(\eta_c,\eta_q,\eta_q))^{1/2}\}^*\{\psi(G(\eta_c,\eta_q,\eta_q))^{1/2}\} \\ & = & G(\eta_c,\eta_{c+1},\eta_{c+1}) + |\psi(G(\eta_c,\eta_q,\eta_q))^{1/2}|^2. \end{array}$$

Since $|G(\eta_c,\eta_q,\eta_q)|$ is always positive, then

$$\begin{array}{rcl} G(\eta_c,\eta_{q+1},\eta_{q+1}) & = & G(\eta_c,\eta_{c+1},\eta_{c+1}) + \psi^2(G(\eta_c,\eta_q,\eta_q)) \\ & < & \epsilon - \psi(\epsilon) + \psi(G(\eta_c,\eta_q,\eta_q)) \\ & < & \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon. \end{array}$$

We conclude that (6) holds $\forall\, d \geq c \geq q_\circ$. It means that $\{\eta_c\}$ is a G-Cauchy in respect of $\mathbb{M}$. As $(Z,\mathbb{M},G)$ is complete, then $\exists\ y \in Z$ so that $\eta_c \longrightarrow y$ as $c \longrightarrow \infty$.

For existence of fixed point we have

$$\begin{aligned} G(y,y,Sy) &\leq G(y,y,\eta_{c+1}) + G(\eta_{c+1},\eta_{c+1},Sy) \\ &= G(y,y,\eta_{c+1}) + G(S\eta_c,S\eta_c,Sy) \\ &\leq G(y,y,\eta_{c+1}) + \psi^*(G(\eta_c,\eta_c,y))\psi \\ &= G(y,y,\eta_{c+1}) + \psi^*\{(G(\eta_c,\eta_c,y))^{1/2}(G(\eta_c,\eta_c,y))^{1/2}\}\psi \\ &= G(y,y,\eta_{c+1}) + \{\psi(G(\eta_c,\eta_c,y))^{1/2}\}^*\{\psi G(\eta_c,\eta_c,y))^{1/2}\} \\ &= G(y,y,\eta_{c+1}) + |\psi(G(\eta_c,\eta_c,y)^{1/2}|^2. \end{aligned}$$

Since $|G(\eta_c,\eta_c,y)|$ is always positive, then

$$\begin{aligned} G(y,y,Sy) &= G(y,y,\eta_{c+1}) + \psi^2(G(\eta_c,\eta_c,y), \\ &< G(y,y,\eta_{c+1}) + G(\eta_c,\eta_c,y), \\ &\longrightarrow 0 \quad \text{as} \quad c \longrightarrow \infty. \end{aligned}$$

Thus $G(y,y,Sy) = 0$ and this shows that $Sy = y$.
Now for the uniqueness, suppose z be other fixed point of S, then $y \neq z$. As ψ is a ψ-map, then

$$\begin{aligned} G(y,y,z) &= G(Sy,Sy,Sz) \\ &\leq \psi^*(G(y,y,z))\psi \\ &= \{\psi(G(y,y,z))^{1/2}\}^*\{\psi(G(y,y,z))^{1/2}\} \\ &= |\psi(G(y,y,z)^{1/2}|^2 \\ &= \psi^2(G(y,y,z)) \\ &< G(y,y,z), \end{aligned}$$

which is impossible. It shows that S has only one fixed point. □

The statement of Theorem 2.1 has been applied to a specific context, and as a consequence, the following results have been derived.

Corollary 2.1. *A map $S : Z \longrightarrow Z$ satisfies the following condition on a complete $(Z,\mathbb{M},G)$:*

$$G(S\eta,S\zeta,S\zeta) \leq \psi^*(G(\eta,\zeta,\zeta))\psi,$$

for all $\eta,\zeta \in Z$. Then $\exists$ a unique $z_\circ \in Z$ such that $Sz_\circ = z_\circ$.

Proof. Same as Theorem 2.1, by substituting $\theta = \zeta$. □

Corollary 2.2. *A map $S : Z \to Z$ satisfies the following condition on a complete* $(Z,\mathbb{M},G)$

$$G(S^m\eta,S^m\zeta,S^m\theta) \leq \psi^*(G(\eta,\zeta,\theta))\psi,$$

$\forall\ \eta,\zeta,\theta \in Z$ and $m \in \mathbb{N}$. Then $\exists$ a unique $p \in Z$ such that $Sp = p$.

Proof. We derived from Theorem 2.1 that S^m has only one p such that $Sp = p$. Since we have

$$S(p) = S(S^m p) = S^{m+1}p = S^m(Sp),$$

it shows that Sp is a fixed point of S^m. By uniqueness, we get $Sp = p$. □

Theorem 2.2. *Suppose $(Z, \mathbb{M}, G)$ be a complete C^*-valued generalized metric space. Let the map $S : Z \to Z$ satisfies*

$$G(S\eta, S\zeta, S\theta) \leq \psi^*(max\ \{G(\eta,\zeta,\theta), G(\eta, S\eta, S\eta), G(\zeta, S\zeta, S\zeta), G(S\eta,\zeta,\theta)\})\psi$$

for all $\eta, \zeta, \theta \in Z$. Then $\exists$ a unique $q \in Z$ such that $Sq = q$.

Proof. Let we take one arbitrary point $\eta_\circ \in Z$ and define a sequence $\{\eta_c\}$ by taking $\eta_c = S\eta_{c-1}, \quad c = 1, 2, 3, \cdots$
i.e.

$$\begin{aligned}
\eta_1 &= S\eta_\circ,\\
\eta_2 &= S\eta_1 = S^2\eta_\circ,\\
\eta_3 &= S\eta_2 = S^3\eta_\circ,\\
&\vdots\\
\eta_c &= S\eta_{c-1} = S^c\eta_\circ.
\end{aligned}$$

From given contraction we have

$$\begin{aligned}
G(\eta_c, \eta_{c+1}, \eta_{c+1}) &= G(S\eta_{c-1}, S\eta_c, S\eta_c)\\
&\leq \psi^*(\max\{G(\eta_{c-1}, \eta_c, \eta_c), G(\eta_{c-1}, \eta_c, \eta_c), G(\eta_c, \eta_{c+1}, \eta_{c+1}), G(\eta_c, \eta_c, \eta_c)\})\psi\\
&\leq \psi^*(\max\{G(\eta_{c-1}, \eta_c, \eta_c), G(\eta_c, \eta_{c+1}, \eta_{c+1})\})\psi.
\end{aligned}$$

If

$$\max\{G(\eta_{c-1}, \eta_c, \eta_c), G(\eta_c, \eta_{c+1}, \eta_{c+1})\} = G(\eta_c, \eta_{c+1}, \eta_{c+1}),$$

then

$$\begin{aligned}
G(\eta_c, \eta_{c+1}, \eta_{c+1}) &\leq \psi^*(G(\eta_c, \eta_{c+1}, \eta_{c+1}))\psi\\
&< G(\eta_c, \eta_{c+1}, \eta_{c+1}),
\end{aligned}$$

which is contradiction. Therefore

$$\max\{G(\eta_{c-1}, \eta_c, \eta_c), G(\eta_c, \eta_{c+1}, \eta_{c+1})\} = G(\eta_{c-1}, \eta_c, \eta_c),$$

then

$$G(\eta_c, \eta_{c+1}, \eta_{c+1}) \leq \psi^*(G(\eta_{c-1}, \eta_c, \eta_c))\psi.$$

For $c \in \mathbb{N}$,

$$\begin{aligned}
G(\eta_c, \eta_{c+1}, \eta_{c+1}) &= G(S\eta_{c-1}, S\eta_c, S\eta_c)\\
&\leq \psi^*(G(\eta_{c-1}, \eta_c, \eta_c))\psi\\
&\leq (\psi^*)^2(G(\eta_{c-2}, \eta_{c-1}, \eta_{c-1}))\psi^2\\
&\leq (\psi^*)^3(G(\eta_{c-3}, \eta_{c-2}, \eta_{c-2}))\psi^3\\
&\vdots\\
G(\eta_c, \eta_{c+1}, \eta_{c+1}) &\leq (\psi^*)^c(G(\eta_\circ, \eta_1, \eta_1))\psi^c.
\end{aligned}$$

The same argument can be found in Theorem 2.1, and it is easy to find that $\{\eta_c\}$ is a G-Cauchy in Z. As Z is complete, then $\{\eta_c\}$ is G-convergent to some $q \in Z$.

For existence of fixed point we have

$$\begin{aligned}
G(q,q,Sq) &\leq G(q,q,\eta_c)+G(\eta_c,\eta_c,Sq)\\
&\leq G(q,q,\eta_c)+G(S\eta_{c-1},S\eta_{c-1},Sq)\\
&\leq G(q,q,\eta_c)+\psi^*(\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},S\eta_{c-1},S\eta_{c-1}),G(\eta_{c-1},S\eta_{c-1},S\eta_{c-1})\\
&\quad G(S\eta_{c-1},\eta_{c-1},q)\})\psi\\
&\leq G(q,q,\eta_c)+\psi^*(\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_c,\eta_{c-1},q)\})\psi\\
&\leq G(q,q,\eta_c)+\psi^*(\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_c,\eta_{c-1},q)\})\psi.
\end{aligned}$$

Case 1. If

$$\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_c,\eta_{c-1},q))\}=G(\eta_{c-1},\eta_c,\eta_c),$$

then

$$\begin{aligned}
G(q,q,Sq) &\leq G(q,q,\eta_c)+\psi^*(G(\eta_{c-1},\eta_c,\eta_c))\psi\\
&< G(q,q,\eta_c)+G(\eta_{c-1},\eta_c,\eta_c).
\end{aligned}$$

By taking $c\to+\infty$, we conclude that $G(q,q,Sq)=0$, and thus $Sq=q$.

Case 2. If

$$\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_c,\eta_{c-1},q))\}=G(\eta_{c-1},\eta_{c-1},q),$$

then

$$\begin{aligned}
G(q,q,Sq) &\leq G(q,q,\eta_c)+\psi^*(G(\eta_{c-1},\eta_{c-1},q))\psi\\
&< G(q,q,\eta_c)+G(\eta_{c-1},\eta_{c-1},q).
\end{aligned}$$

By taking $c\to+\infty$, we conclude that $G(q,q,Sq)=0$, and thus $Sq=q$.

Case 3. If

$$\max\{G(\eta_{c-1},\eta_{c-1},q),G(\eta_{c-1},\eta_c,\eta_c),G(\eta_c,\eta_{c-1},q))\}=G(\eta_c,\eta_{c-1},q),$$

then

$$\begin{aligned}
G(q,q,Sq) &\leq G(q,q,\eta_c)+\psi^*(G(\eta_c,\eta_{c-1},q))\psi\\
&< G(q,q,\eta_c)+G(\eta_c,\eta_{c-1},q)\\
&\leq G(q,q,\eta_c)+G(\eta_c,\eta_{c-1},\eta_{c-1})+G(\eta_{c-1},\eta_{c-1},q).
\end{aligned}$$

Letting $c\to+\infty$, we have $G(q,q,Sq)=0$, and thus $Sq=q$. In all three cases we deduced that $Sq=q$. Suppose r be other fixed point of S, then

$$\begin{aligned}
G(q,r,r) &= G(Sq,Sr,Sr),\\
&\leq \psi^*(\max\{G(q,r,r),G(q,Sq,Sq),G(r,Sr,Sr),G(Sq,r,r)\})\psi\\
&\leq \psi^*(\max\{G(q,r,r),G(q,q,q),G(r,r,r),G(q,r,r)\})\psi\\
&= \psi^*(G(q,r,r))\psi\\
&< G(q,r,r),
\end{aligned}$$

which is impossible. This shows that S has only one fixed point. □

The statement of Theorem 2.2 has been applied to a specific context, and as a consequence, the following results have been derived.

Corollary 2.3. *A map $S : Z \to Z$ satisfies the following condition on a complete $(Z, \mathbb{M}, G)$:*

$$G(S\eta, S\zeta, S\theta) \leq k \ max\{G(\eta, \zeta, \theta), G(\eta, S\eta, S\eta), G(\zeta, S\zeta, S\zeta), G(S\eta, \zeta, \theta)\}$$

$\forall\ \eta, \zeta, \theta \in Z$ and $k \in [0, 1)$. Then $\exists$ a unique $p \in Z$ such that $Sp = p$.

Proof. Define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi(\sigma) = k\sigma$. Then clearly ψ is a nondecreasing map so that $\lim_{n\to+\infty} \psi^n(t) = 0$, $\forall\ t > 0$. Then,

$$G(S\eta, S\zeta, S\theta) \leq \psi^*(\max\{G(\eta, \zeta, \theta), G(\eta, S\eta, S\eta), G(\zeta, S\zeta, S\zeta), G(S\eta, \zeta, \theta)\})\psi$$

for all $\eta, \zeta, \theta \in Z$. The proof for the statement being discussed is identical to the proof provided for Theorem 2.2. □

Corollary 2.4. *A map $S : Z \to Z$ satisfies the following condition on a complete $(Z, \mathbb{M}, G)$:*

$$G(S\eta, S\zeta, S\zeta) \leq \psi^*(max\{G(\eta, \zeta, \zeta), G(\eta, S\eta, S\eta), G(\zeta, S\zeta, S\zeta), G(S\eta, \zeta, \zeta)\})\psi$$

for all $\eta, \zeta \in Z$. Then $\exists$ a unique $p \in Z$ such that $Sp = p$.

Proof. Theorem 2.2 provides proof, by substituting $\theta = \zeta$. □

3. Application to integral equation

As we know that integral equations have a lot of applications in applied sciences (see, for example, Bohner et al. [37]). In this part, we derive sufficient conditions for the solutions of integral equation.

Let $W = L^\infty(E)$ and $H = L^2(E)$, where E is Lebesgue measurable set. By $L(H)$ we denote the set of bounded linear operators on Hilbert space H. Clearly $L(H)$ is a C^*- algebra with the usual operator norm. Define $G : W \times W \times W \to L(H)$, by

$$G(f, g, h) = \pi_{|f-g|} + \pi_{|g-h|} + \pi_{|h-f|}\ (f, g, h \in W), \tag{7}$$

where $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_h(\varphi) = h.\varphi, \tag{8}$$

for $\varphi \in H$. Then G is a C^*- algebra-valued generalized metric and $(W, L(H), G)$ is a complete C^*- algebra-valued generalized metric space. Consider the integral equation

$$\eta(t) = g(t) + \int_E Q(t, s, \eta(s))ds, t \in E, \tag{9}$$

here $Q : E \times E \times \mathbf{R} \to \mathbf{R})$ and $g \in L^\infty(E)$. Define $T : W \to W$ by

$$T\eta(t) = g(t) + \int_E Q(t, s, \eta(s))ds, t \in E.$$

Theorem 3.1. *We suppose that the following conditions hold:*

(A_1) *there exists a continuous function $\varphi : E \times E \to \boldsymbol{R}$ and $k \in (0, 1)$ such that*

$$|Q(t, s, u) - Q(t, s, v)) \leq k|\varphi(t, s)(u - v)|$$

for $t, s \in E$ and $u, v \in \boldsymbol{R}$,

(A_2) $\sup_{t\in E} \int_E |\varphi(t, s)|ds \leq 1.$

Then the integral equation has a unique solution x^* in $L^\infty(E)$.

Proof. Set $M = kI$, then $M \in L(H)$ and $\|M\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
&\|d(T\eta, T\zeta)\| + \|d(T\zeta, T\theta)\| + \|d(T\theta, T\eta)\| \\
&= \sup_{\|h\|=1} (\pi_{|T\eta - T\zeta|}, h) + \sup_{\|h\|=1} (\pi_{|T\zeta - T\theta|}, h) + \sup_{\|h\|=1} (\pi_{|T\theta - T\eta|}, h) \\
&\leq \sup_{\|h\|=1} \int_E \left[\left| \int_E (Q(t,s,\eta(s)) - Q(t,s,\zeta(s))ds) \right| \right] h(t)\overline{h(t)} dt \\
&+ \sup_{\|h\|=1} \int_E \left[\left| \int_E (Q(t,s,\zeta(s)) - Q(t,s,\theta(s))ds) \right| \right] h(t)\overline{h(t)} dt \\
&+ \sup_{\|h\|=1} \int_E \left[\left| \int_E (Q(t,s,\theta(s)) - Q(t,s,\eta(s))ds) \right| \right] h(t)\overline{h(t)} dt \\
&\leq \sup_{\|h\|=1} \int_E \left[\int_E |Q(t,s,\eta(s)) - Q(t,s,\zeta(s))|ds) \right] |h(t)|^2 dt \\
&+ \sup_{\|h\|=1} \int_E \left[\int_E |Q(t,s,\zeta(s)) - Q(t,s,\theta(s))|ds) \right] |h(t)|^2 dt \\
&+ \sup_{\|h\|=1} \int_E \left[\int_E |Q(t,s,\theta(s)) - Q(t,s,\eta(s))|ds) \right] |h(t)|^2 dt
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\|h\|=1} \int_E \left[\int_E |k\varphi(t,s)(\eta(s) - \zeta(s)|ds \right] |h(t)|^2 dt \\
&+ \sup_{\|h\|=1} \int_E \left[\int_E |k\varphi(t,s)(\zeta(s) - \theta(s)|ds \right] |h(t)|^2 dt \\
&+ \sup_{\|h\|=1} \int_E \left[\int_E |k\varphi(t,s)(\theta(s) - \eta(s)|ds \right] |h(t)|^2 dt \\
&\leq k \sup_{\|h\|=1} \int_E \left[\int_E |\varphi(t,s)|ds \right] |h(t)|^2 dt . \|\eta - \zeta\|_\infty \\
&+ k \sup_{\|h\|=1} \int_E \left[\int_E |\varphi(t,s)|ds \right] |h(t)|^2 dt . \|\zeta - \theta\|_\infty \\
&+ k \sup_{\|h\|=1} \int_E \left[\int_E |\varphi(t,s)|ds \right] |h(t)|^2 dt . \|\theta - \eta\|_\infty \\
&\leq k \sup_{t \in E} \int_E |\varphi(t,s)|ds . \sup_{\|h\|=1} \int_E |h(t)|^2 dt . \|\eta - \zeta\|_\infty \\
&+ k \sup_{t \in E} \int_E |\varphi(t,s)|ds . \sup_{\|h\|=1} \int_E |h(t)|^2 dt . \|\eta - \zeta\|_\infty \\
&+ k \sup_{t \in E} \int_E |\varphi(t,s)|ds . \sup_{\|h\|=1} \int_E |h(t)|^2 dt . \|\eta - \zeta\|_\infty
\end{aligned}$$

$$\begin{aligned}
&\leq \quad k\|\eta - \zeta\|_\infty + k\|\zeta - \theta\|_\infty + k\|\theta - \eta\|_\infty, \\
&\leq \quad \|M\| \|d(\eta, \zeta)\|_\infty + \|M\| \|d(\zeta, \theta)\|_\infty + \|M\| \|d(\theta, \eta)\|_\infty
\end{aligned}$$

but $\|M\| < 1$, by taking $\psi(t) = kt$, from Theorem 2.1 the integral equation has a unique solution η^* in $L^\infty(E)$. □

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