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SOME RESULTS ON ROUGH STATISTICAL ϕ-CONVERGENCE FOR DIFFERENCE DOUBLE SEQUENCES

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Abstract. In this paper, we put forward rough statistical ϕ-convergence of difference sequences as a generalization of rough statistical convergence as well as statistical ϕ-convergence. We study some of its fundamental properties. We obtain some results for rough statistical ϕ-convergence for difference double sequences by introducing the rough statistical-ϕ limit set. So our main objective is to find out the different behaviour of the new convergence concept based on rough statistical-ϕ limit set.

In this study, since the concept of rough statistical ϕ-convergence will be worked for difference double sequences, it is important to present some literature knowledge about difference sequences. Kizmaz [19] investigated the concept of difference sequence such that $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$. After this study, which can be accepted as a base about difference sequences, Aydın and Başar [1], Başarır [4], Bektaş et al. [5], Demir and Gümüş [8, 9], Et [10], Et and Çolak [11], Et and Nuray [12], Et and Esi [13], Savaş [32] and many others researched significant properties of this concept. Et and Çolak [11] generalized Kızmaz's results for generalized difference sequences. The notion of statistical convergence was first presented by Fast [16]. The main idea behind statistical convergence was the notion of natural density. The natural density of a set $A \subseteq \mathbb{N}$ is denoted and defined by

$$d(A) = \lim_n \frac{1}{n} \left| \{k \in A : k \leq n\} \right|,$$

where the vertical bars indicate the cardinality of the enclosed set. Clearly, $d(\mathbb{N} \setminus A) + d(A) = 1$ and $A \subseteq B$ implies $d(A) \leq d(B)$. It is obvious that when A is a finite set then $d(A) = 0$. A real-valued sequence $w = (w_k)$ is said to be statistically convergent to the number w_0 if for each $\varepsilon > 0$,

$$d(\{k \in \mathbb{N} : |w_k - w_0| \geq \varepsilon\}) = 0.$$

Mursaleen and Edely [26] put forward the statistical convergence for double sequences. Later on, statistical convergence was further investigated and worked from the sequence space point of view by Braha et al. [3], Debnath and Choudhury

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[7], Mohiuddine et al. [17, 23, 24], Mursaleen and Başar [25], Nuray and Ruckle [27], Temizsu and Et [34], Yaying and Saikia [35], Yaying and Hazarika [36], Yaying [37], and many others.

In another direction, the concept of rough convergence was first introduced independently by Burgin [6] and H. X. Phu [29]. Although their idea was very much similar but Burgin worked it in the fuzzy setting whereas Phu developed it for finite dimensional normed spaces.

Let r be a non-negative real number. A sequence $w=(w_k)$ in a normed linear space $(Y,\|\cdot\|)$ is said to be rough convergent to $w_0 \in Y$ with roughness degree r, if for each $\varepsilon>0$, there is an $N=(N_\varepsilon)$ so that for all $k \geq N$, $\|w_k - w_0\| < r+\varepsilon$. Symbolically, it is represented as $w_k \overset{r-\|.\|}{\longrightarrow} w_0$.

Phu [29] mainly proved that the set $LIM^r w$ is bounded, closed, and convex and studied the basic properties of this interesting concept. It should be mentioned that the idea of rough convergence occurs quite naturally in numerical analysis and has interesting applications there. In [30], Phu further investigated the notion of rough convergence in an infinite dimensional normed space setting. Combining the notion of rough convergence and statistical convergence, in 2008, Aytar [2] developed rough statistical convergence. Rough convergence of double sequences was examined by Malik and Maity [21] and obtained some significant properties of this type of convergence and also established relation between rough convergence and Pringsheim convergence for double sequences. In [22], rough statistical convergence of double sequences in finite dimensional normed linear spaces was studied. For extensive study in this direction, one may refer to [14, 15, 20], where many more references can be found.

An Orlicz function [31] is a function $\phi : \mathbb{R} \to \mathbb{R}$ such that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \iff x = 0 \text{ and } \phi(x) \to \infty \text{ as } x \to \infty,$$

where $\mathbb{R}$, $\mathbb{R}^+$, and ϕ stands for the set of all real numbers, set of all positive real numbers, and Orlicz function respectively.

Rao and Ren [31] describe that Orlicz functions have important roles and applications in many areas such as economics, stochastic problems etc.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Δ_2 condition, if there exists a $M > 0$ such that $\phi(2x) \leq M\phi(x)$, for every $x \in \mathbb{R}^+$.

Example 1. *(i) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = |x|$ is an Orlicz function.*
(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^3$ is not an Orlicz function.
(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is an Orlicz function satisfying the Δ_2 condition.
(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the Δ_2 condition.

In 2019, Khusnussaadah and Supama [18] introduced the concept of $\phi-$convergence and later on Supama introduced the concept of statistical ϕ-convergence using the Orlicz function ϕ.

In this paper, by using rough statistical convergence and ϕ-convergence, we present a concept of rough statistically ϕ-convergence on $\mathbb{R}$, as a generalization of rough statistical convergence and statistical ϕ-convergence. We first define the rough statistical ϕ-limit set and then we prove that it is convex if ϕ is convex. At the end, we find various results related to the rough statistical ϕ-limit set which seems to be different from rough statistical limit set, already studied by Nihal and Demir [9].

1. Definitions and Preliminaries

We recall the concepts of rough convergence, rough statistical convergence, ϕ-convergence are as follows:

Definition 1. *([29]) Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. A sequence (w_k) is said to be rough convergent to $w_0 \in \mathbb{R}$, denoted by $w_k \xrightarrow{r} w_0$, if*

$$\forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N} : k \geq k_\varepsilon \Longrightarrow |w_k - w_0| < r + \varepsilon$$

Definition 2. *([2]) Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. A sequence (w_k) is said to be rough statistically convergent to $w_0 \in \mathbb{R}$, denoted by $w_k \overset{r-st}{\to} w_0$ if*

$$\forall \varepsilon > 0, d\left(\{k \in \mathbb{N} : |w_k - w_0| \geq r + \varepsilon\}\right) = 0$$

Definition 3. *([2]) Let $w = (w_k)$ be a sequence of real numbers and r be a non-negative real number. Then the set*

$$st - LIM^r w = \left\{w_0 \in \mathbb{R} : w_k \xrightarrow{r} w_0\right\}$$

is known as the rough statistical limit set of $w = (w_k)$.

A double sequence $w = (w_{kl})$ of real numbers is said to be convergent to $w_0 \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $w_0 \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|w_{kl} - w_0| < \varepsilon$, whenever $k, l > N_\varepsilon$. In this case, we write

$$\lim_{k,l\to\infty} w_{kl} = w_0.$$

We recall that a subset K of $\mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{k,l\to\infty} \frac{K(k,l)}{k.l},$$

where $K(k,l) = |\{(m,n) \in \mathbb{N} \times \mathbb{N} : m \leq k, n \leq l\}|$.

Let $w = (w_{kl})$ be a double sequence in a normed space $(X, \|.\|)$ and r be a non negative real number. w is said to be r-statistically convergent to w_0, denoted by $w \overset{r-st_2}{\longrightarrow} w_0$, if for $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|w_{kl} - w_0\| \geq r + \varepsilon\}$. In this case, w_0 is called the r-statistical limit of w.

A double sequence $w = (w_{kl})$ is said to be rough convergent (r-convergent) to w_0 with the roughness degree r, denoted by $w_{kl} \xrightarrow{r} w_0$ provided that

$$\forall \varepsilon > 0 \;\; \exists k_\varepsilon \in \mathbb{N} : k, l \geq k_\varepsilon \Rightarrow \|w_{kl} - w_0\| < r + \varepsilon,$$

or equivalently, if

$$\limsup \|w_{kl} - w_0\| \leq r.$$

A double sequence (Δw_{kl}) is said to be bounded if there exists a positive real number K such that $\|\Delta w_{kl}\| < K$ for all $(k,l) \in \mathbb{N} \times \mathbb{N}$.

A double sequence (Δw_{kl}) is said to be statistically bounded if there exists a positive real number K

$$d\left(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta w_{kl}\|\geq K\}\right)=0.$$

A point $c\in X$ is said to be a statistical cluster point of a double sequence (Δw_{kl}) if for any $\varepsilon>0$, the set

$$d\left(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta w_{kl}-c\|<\varepsilon\}\right)\neq 0.$$

We use the notation $\Gamma^2(\Delta w_{kl})$ to denote the set of all statistical cluster points of (Δw_{kl}).

Definition 4. *([18]) Let $\phi:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $w=(w_k)$ is said to be ϕ-convergent to w_0 if $\lim_k \phi(w_k-w_0)=0$. In this case, w_0 is called the ϕ-limit of (w_k) and denoted by $\phi-\lim w=w_0$. If a sequence (w_k) is ϕ-convergent to w_0, we denote it by $w_k\overset{\phi}{\to}w_0$.*

Definition 5. *([33]) Let $\phi:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A sequence $w=(w_k)$ is said to be statistically ϕ-convergent to $w_0\in\mathbb{R}$ denoted by $w_k\overset{st-\phi}{\to}w_0$, if*

$$\forall\varepsilon>0,\ d\left(\{k\in\mathbb{N}:\phi(w_k-w_0)\geq\varepsilon\}\right)=0.$$

2. Main Results

Definition 6. *Assume $\phi:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $r\geq 0$ be a real number. A sequence (Δw_{kl}) is called to be rough statistically ϕ-convergent to $w_0\in\mathbb{R}$ if for every ε, the set*

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi(\Delta w_{kl}-w_0)\geq r+\varepsilon\}$$

has natural density zero. w_0 is called rough statistical ϕ-limit of the sequence (Δw_{kl}) and we demonstrate, $\Delta w_{kl}\overset{r-st_2-\phi}{\to}w_0$ or r-st_2-$\phi\lim_{k,l\to\infty}\Delta w_{kl}=w_0$.

If we take $r=0$, then we obtain statistical ϕ-convergence. If we take $\phi(\Delta w_{kl})=|\Delta w_{kl}|$, we get rough statistical convergence [9]. If we take $\phi(\Delta w_{kl})=|\Delta w_{kl}|$ and $r=0$ both, we get the statistical convergence [4]. Therefore, our main interest is to deal with the case $r>0$.

If $\Delta w_{kl}\overset{r-st_2-\phi}{\to}w_0$, w_0 is an r-st_2-ϕ limit point of (Δw_{kl}), which is usually no more unique (for $r>0$). So, we consider the rough statistical ϕ-limit set of (Δw_{kl}) defined by

$$st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})=\left\{w_0\in\mathbb{R}:\Delta w_{kl}\overset{r-st_2-\phi}{\to}w_0\right\}.$$

A sequence (Δw_{kl}) is said to be rough statistically ϕ-convergent if $st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})\neq\emptyset$. In this case, r is called a rough statistical-ϕ convergence degree of (Δw_{kl}).

Let us illustrate by an example:

Example 2. *Let $p\geq 1$ and $u\geq 2$ be any two given natural numbers. Suppose $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(\Delta w_{kl})=|\Delta w_{kl}|^p$, be the Orlicz function. Consider the sequence (Δw_{kl}) defined by*

$$(\Delta w_{kl}):=\begin{cases} kl, & \text{if } k=n^u,\ l=m^u \text{ for some } n,m\in\mathbb{N}\\ (-1)^{k+l}, & \text{if not.}\end{cases}$$

So, we obtain

$$st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})=\begin{cases}\left[1-\sqrt[p]{r},\ \sqrt[p]{r}-1\right], & \text{for } r\geq 1\\ \emptyset, & \text{for } r<1.\end{cases}$$

As a result, (Δw_{kl}) *is rough statistically* ϕ*-convergent for* $r \geq 1$ *but not rough statistically* ϕ*-convergent for* $r < 1$*.*

Also, the considered sequence is neither statistically convergent nor statistically ϕ-convergent.

Hence, we can observe that rough statistical ϕ-convergence is a generalization of statistical convergence and statistically ϕ-convergent both.

But, we state that rough statistical ϕ-convergence and rough statistical convergence are two independent notions. The following two examples will illustrate the fact.

Example 3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$*,* $\phi(\Delta w_{kl}) = \sqrt{|\Delta w_{kl}|}$*, be an Orlicz function. Consider the sequence* (Δw_{kl}) *defined as*

$$(\Delta w_{kl}) = \begin{cases} 0, & \text{if } k,\ l \text{ are odd,} \\ 1, & \text{if } k,\ l \text{ are even.} \end{cases}$$

Then,

$$st_2\text{-}LIM^r(\Delta w_{kl}) = \begin{cases} [1-r, r], & \text{for } r \geq \frac{1}{2} \\ \emptyset, & \text{for } r < \frac{1}{2}. \end{cases}$$

But

$$st_2\text{-}LIM^{r-\phi}(\Delta w_{kl}) = \begin{cases} \left[1-r^2, r^2\right], & \text{for } r \geq \frac{1}{\sqrt{2}} \\ \emptyset, & \text{for } r < \frac{1}{\sqrt{2}}. \end{cases}$$

That is to say, for $\frac{1}{2} \leq r < \frac{1}{\sqrt{2}}$*, the sequence is rough statistically convergent but not rough statistically* ϕ*-convergent and if* $r \geq \frac{1}{\sqrt{2}}$*, the sequence is convergent in both the sense.*

Example 4. *Let* $\phi : \mathbb{R} \to \mathbb{R}$*,* $\phi(\Delta w_{kl}) = (\Delta w_{kl})^2$*, be an Orlicz function. Consider the sequence* (Δw_{kl}) *defined in Example 3. Then, according to the definition,*

$$st_2\text{-}LIM^{r-\phi}(\Delta w_{kl}) = \begin{cases} [1-\sqrt{r}, \sqrt{r}], & \text{for } r \geq \frac{1}{4} \\ \emptyset, & \text{for } r < \frac{1}{4}. \end{cases}$$

But,

$$st_2\text{-}LIM^r(\Delta w_{kl}) = \begin{cases} [1-r, r], & \text{for } r \geq \frac{1}{2} \\ \emptyset, & \text{for } r < \frac{1}{2}. \end{cases}$$

So, for $\frac{1}{4} \leq r < \frac{1}{2}$*, the sequence is rough statistically* ϕ*-convergent but not rough statistically convergent and if* $r \geq \frac{1}{2}$*, the sequence has both type of convergence.*

Definition 7. *A sequence* (Δw_{kl}) *is statistically* ϕ*-bounded if there exists a real number* $B > 0$ *such that*

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl}) \geq B\}) = 0. \tag{2.1}$$

Theorem 2.1. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function which satisfies* Δ_2 *condition. A sequence* (Δw_{kl}) *is statistically* ϕ*-bounded if and only if* $\exists r \in \mathbb{R}$ *with* $r \geq 0$ *such that* $st_2\text{-}LIM^{r-\phi}(\Delta w_{kl}) \neq \emptyset$*.*

Proof. Let's assume that (Δw_{kl}) is statistically ϕ-bounded. From the Definition 7, (2.1) is provided. By taking $A = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl}) \geq B\}$, we get $d(A) = 0$. Let $r' = \sup\{\phi(\Delta w_{kl}) : k, l \in A^c\}$. Then the set $st_2\text{-LIM}^{r'-\phi}(\Delta w_{kl})$ contains the origin of $\mathbb{R}^n$. So, we obtain $st_2\text{-LIM}^{r'-\phi}(\Delta w_{kl}) \neq \emptyset$.

For the converse part, assume $st_2\text{-LIM}^{r-\phi}(\Delta w_{kl}) \neq \emptyset$ for some $r \geq 0$. Let $w_0 \in st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})$. Then, we have

$$d\left(\{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl} - w_0) \geq r + \varepsilon\}\right) = 0 \tag{2.2}$$

for all $\varepsilon > 0$. Since the function ϕ satisfies Δ_2 condition, so there exists $Q \in \mathbb{R}^+$ such that $\phi(2x) \leq Q\phi(x)$ for every $x \in \mathbb{R}^+$. Now, we assert that for every $\varepsilon > 0$,

$$\begin{aligned} &\left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl}) \geq \tfrac{Q}{2}(r + \phi(w_0) + \varepsilon)\right\} \\ &\quad \subseteq \{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl} - w_0) \geq r + \varepsilon\}. \end{aligned} \tag{2.3}$$

So, let's assume

$$(p,s) \in \left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl}) \geq \frac{Q}{2}(r + \phi(w_0) + \varepsilon)\right\}.$$

Then, we obtain

$$\begin{aligned} \frac{Q}{2}(r + \phi(w_0) + \varepsilon) &\leq \phi(\Delta w_{ps}) = \phi(\Delta w_{ps} - w_0 + w_0) \\ &= \phi\left(\frac{1}{2}(2(\Delta w_{ps} - w_0)) + \frac{1}{2}(2w_0)\right) \\ &\leq \frac{1}{2}(\phi(2(\Delta w_{ps} - w_0))) + \frac{1}{2}(\phi(2w_0)) \\ &\leq \frac{Q}{2}(\phi(\Delta w_{ps} - w_0)) + \frac{Q}{2}(\phi(w_0)) \end{aligned}$$

where the third inequality follows from the convexity and the fourth inequality follows from the Δ_2 condition. So, we get

$$r + \phi(w_0) + \varepsilon \leq \phi(\Delta w_{ps} - w_0) + \phi(w_0)$$

namely, $\phi(\Delta w_{kl} - w_0) \geq r + \varepsilon$. To put it another way,

$$(p,s) \in \{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl} - w_0) \geq r + \varepsilon\},$$

indicates that (2.3) is accurate. We may infer from (2.2) and (2.3) that for any $\varepsilon > 0$,

$$d\left(\left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl}) \geq \frac{Q}{2}(r + \phi(w_0) + \varepsilon)\right\}\right) = 0.$$

(Δw_{kl}) is therefore statistically ϕ-bounded. This completes the proof. □

Theorem 2.2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function. If $y_0 \in st_2\text{-}LIM^{r_0-\phi}(\Delta w_{kl})$ and $y_1 \in st_2\text{-}LIM^{r_1-\phi}(\Delta w_{kl})$, then we have*

$$y_\lambda = (1-\lambda)y_0 + \lambda y_1 \in st_2\text{-}LIM^{(1-\lambda)r_0+\lambda r_1-\phi}(\Delta w_{kl})$$

for $\lambda \in [0,1]$.

Proof. Assume $y_0 \in st_2\text{-LIM}^{r_0-\phi}(\Delta w_{kl})$. Then, for each $\varepsilon > 0$, $d(K_1) = 0$, where

$$K_1 = \{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl} - y_0) \geq r_0 + \varepsilon\}.$$

Similarly if $y_1 \in st_2\text{-LIM}^{r_1-\phi}(\Delta w_{kl})$, then we get for every $\varepsilon > 0$, $d(K_2) = 0$, where

$$K_2 = \{(k,l) \in \mathbb{N}\times\mathbb{N} : \phi(\Delta w_{kl} - y_1) \geq r_1 + \varepsilon\}.$$

Now

$$\begin{aligned}\phi(\Delta w_{kl}-y_\lambda)&=\phi(\Delta w_{kl}-(1-\lambda)y_0-\lambda y_1)\\&=\phi((1-\lambda)\Delta w_{kl}+\lambda\Delta w_{kl}-(1-\lambda)y_0-\lambda y_1)\\&=\phi((1-\lambda)(\Delta w_{kl}-y_0)+\lambda(\Delta w_{kl}-y_1))\\&\leq(1-\lambda)\phi(\Delta w_{kl}-y_0)+\lambda\phi(\Delta w_{kl}-y_1)\\&<(1-\lambda)(r_0+\varepsilon)+\lambda(r_1+\varepsilon)\\&=r+\varepsilon,\ r=(1-\lambda)r_0+\lambda r_1.\end{aligned}$$

Namely, $\phi(\Delta w_{kl}-y_\lambda)\geq r+\varepsilon$, $\forall(k,l)\in K_1\cup K_2$. Therefore,

$$d(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi(\Delta w_{kl}-y_\lambda)\geq r+\varepsilon\})=d(K_1\cup K_2)=0.$$

Hence

$$y_\lambda=(1-\lambda)y_0+\lambda y_1\in st_2\text{-LIM}^{(1-\lambda)r_0+\lambda r_1-\phi}(\Delta w_{kl})$$

for $\lambda\in[0,1]$. As a result, st_2-LIM$^{(1-\lambda)r_0+\lambda r_1-\phi}(\Delta w_{kl})$. Then, the desired result has been obtained. □

Remark. *If we take $r_0=r_1=r$, then we have*

$$y_\lambda=(1-\lambda)y_0+\lambda y_1\in st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})\ \text{ for }\lambda\in[0,1].$$

namely, the set st_2-LIM$^{r-\phi}(\Delta w_{kl})$ is convex.

Nihal and Demir [9] stated that diam $(st_2\text{-LIM}^r(\Delta w_{kl}))\leq 2r$ for the sequence (Δw_{kl}). However, this might not be the case in the case of rough statistical ϕ-convergence.

Example 5. *Let $\phi:\mathbb{R}\to\mathbb{R}$, $\phi(\Delta w_{kl})=\sqrt{|\Delta w_{kl}|}$, be an Orlicz function and (Δw_{kl}) be defined as*

$$(\Delta w_{kl})=\begin{cases}0, & \text{if } k,l \text{ are odd}\\ 1, & \text{if } k,l \text{ are even.}\end{cases}$$

Hence, we get

$$\text{diam}\left(st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})\right)=\sup\left\{|p-q|:p,q\in st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})\right\}=2r^2-1>2r,$$

for any $r>\frac{1+\sqrt{3}}{2}$.

We can thus conclude from the example above that the diameter of st_2-LIM$^{r-\phi}(\Delta w_{kl})$ relies on the ϕ function that we have taken into account. The ϕ-image of the diameter of st_2-LIM$^{r-\phi}(\Delta w_{kl})$ does not exceed Qr for some $Q>0$, but, for a convex ϕ function that satisfies the Δ_2 requirement, the following theorem follows.

Theorem 2.3. *Let $\phi:\mathbb{R}\to\mathbb{R}$ be a convex Orlicz function which satisfies Δ_2 condition. Then there exists a real number $Q>0$ such that $\phi\left(\text{diam}\left(st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})\right)\right)\leq Qr$.*

Proof. Since ϕ is even and non decreasing on $\mathbb{R}^+$, we get

$$\begin{aligned}&\phi\left(\text{diam}\left(st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})\right)\right)\\&=\phi\left(\sup\left\{|w_1-w_2|:w_1,w_2\in st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})\right\}\right)\qquad(2.4)\\&=\sup\left\{\phi(w_1-w_2):w_1,w_2\in st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})\right\}.\end{aligned}$$

From Δ_2 condition, there exists a $Q > 0$ in $\mathbb{R}$ such that $\forall x \in \mathbb{R}$, $\phi(2x) \leq Q\phi(x)$. Now, since $w_1, w_2 \in st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})$, so for each $\varepsilon > 0$, $d(K_1) = 0$ and $d(K_2) = 0$ where

$$K_1 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - w_1) \geq r + \tfrac{\varepsilon}{Q}\right\},$$
$$K_2 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - w_2) \geq r + \tfrac{\varepsilon}{Q}\right\}.$$

Obviously, $d(K_1^c \cap K_2^c) = 1$. Let $(p,s) \in K_1^c \cap K_2^c$ then $\phi(\Delta w_{ps} - w_1) < r + \frac{\varepsilon}{Q}$ and $\phi(\Delta w_{ps} - w_2) < r + \frac{\varepsilon}{Q}$. Hence

$$\begin{aligned}\phi(w_1 - w_2) &= \phi(w_1 - \Delta w_{ps} + \Delta w_{ps} - w_2) \\ &\leq \tfrac{Q}{2}\phi(\Delta w_{ps} - w_1) + \tfrac{Q}{2}\phi(\Delta w_{ps} - w_2) < Qr + \varepsilon.\end{aligned}$$

Since ε arbitrary, so we get $\phi(w_1 - w_2) \leq Qr$ and clearly from equation (2.4), we obtain that $\phi\left(\text{diam}\left(st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})\right)\right) \leq Qr$. This finalizes the proof. □

Nihal and Demir [9] also proved that for any statistically bounded sequence (Δw_{kl}), $\Gamma_{\Delta w_{kl}} \subseteq st_2\text{-LIM}^{diam\left(\Gamma_{\Delta w_{kl}}\right)}(\Delta w_{kl})$. However, this conclusion also appears to be false in the situation of rough statistical ϕ-convergence. The example that follows supports our assertion.

Example 6. *Let $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(\Delta w_{kl}) = (\Delta w_{kl})^2$, be an Orlicz function. Consider the sequence (Δw_{kl}) defined as*

$$(\Delta w_{kl}) = \begin{cases} (-1)^{k+l}, & \text{if } k,l \text{ are not a perfect square} \\ kl, & \text{if } k,l \text{ are a perfect square.} \end{cases}$$

Then, $\Gamma_{(\Delta w_{kl})} = \{-1, 1\}$. So, we get $\text{diam}\left(\Gamma_{(\Delta w_{kl})}\right) = 2$. But $st_2\text{-}LIM^{2-\phi}(\Delta w_{kl}) = \left[1 - \sqrt{2}, -1 + \sqrt{2}\right] \not\supseteq \Gamma_{(\Delta w_{kl})}$.

Further, Nihal and Demir [9] demonstrated that for any arbitrary $c \in \Gamma_{(\Delta w_{kl})}$ of a sequence (Δw_{kl}), $|w_1 - c| \leq r$ for all $w_1 \in st_2\text{−LIM}^r(\Delta w_{kl})$. For rough statistical ϕ-convergence, however, it is not true.

Example 7. *Consider Example 3. Then, it is obvious that $\Gamma_{(\Delta w_{kl})} = \{0, 1\}$ and $st_2\text{−}LIM^{5-\phi}(\Delta w_{kl}) = [-24, 25]$. If we take $c = 0$ and $w_1 = 20$, then $|w_1 - c| \leq 5$ does not hold.*

Definition 8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A real number γ is said to be statistically ϕ-cluster point of a sequence (Δw_{kl}), if for every $\varepsilon > 0$,*

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - \gamma) < \varepsilon\}) \neq 0.$$

The set of all statistically ϕ-cluster points of a sequence (Δw_{kl}) is demonstrated by $\phi\left(\Gamma_{(\Delta w_{kl})}\right)$.

Theorem 2.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function which satisfies Δ_2 condition. Then, for any arbitrary $c \in \phi\left(\Gamma_{(\Delta w_{kl})}\right)$ and $w_1 \in st_2\text{-}LIM^{r-\phi}(\Delta w_{kl})$, there exists $Q > 0$ in $\mathbb{R}$ such that $\phi(w_1 - c) \leq Qr$.*

Proof. Since $c \in \phi\left(\Gamma_{(\Delta w_{kl})}\right)$, so for every $\varepsilon > 0$,

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - c) < \varepsilon\}) \neq 0. \tag{2.5}$$

Now as ϕ supplies Δ_2 condition, so there exists $M \in \mathbb{R}^+$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in \mathbb{R}$. We will demonstrate that $Q = \frac{M}{2}$ is the positive real number that works here. If possible suppose

$$\phi(w_1 - c) > \frac{Mr}{2} \tag{2.6}$$

Take $\varepsilon = \frac{2\left(\phi(w_1-c)-\frac{Mr}{2}\right)}{3M}$. We state that the following inclusion holds

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - c) < \varepsilon\} \subseteq \{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - w_1) \geq r + \varepsilon\} \tag{2.7}$$

Suppose

$$(p,s) \in \{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - c) < \varepsilon\}.$$

Then, we get

$$\phi(\Delta w_{ps} - c) < \frac{2\left(\phi(w_1 - c) - \frac{Mr}{2}\right)}{3M}. \tag{2.8}$$

Then, it can be proved by using the convexity, Δ_2 condition and the inequalities (2.8) that

$$2\phi(w_1 - c) < \frac{3M}{2}\phi(\Delta w_{ps} - w_1) - \frac{Mr}{2}.$$

Also, using the the inequalities (2.6), we get

$$\frac{Mr}{2} + \phi(w_1 - c) < \frac{3M}{2}\phi(\Delta w_{ps} - w_1) - \frac{Mr}{2},$$

$$\frac{3M}{2}\phi(\Delta w_{ps} - w_1) > Mr + \frac{3M\varepsilon}{2} + \frac{Mr}{2}$$

and so

$$\phi(\Delta w_{ps} - w_1) > r + \varepsilon.$$

Thus, we have

$$(p,s) \in \{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - w_1) \geq r + \varepsilon\}$$

proving that the inclusion (2.7) is true. Combining (2.5) and (2.7) we conclude that

$$d(\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi(\Delta w_{kl} - w_1) \geq r + \varepsilon\}) \neq 0,$$

which is a contradiction to the fact that $w_1 \in st_2\text{-LIM}^{r-\phi}(\Delta w_{kl})$. This concludes the proof. □

3. Conclusion

The main aim of this paper is to present the notion of rough statistical ϕ-convergence of difference double sequences and study some of its properties. In addition, we obtain some results for rough statistical ϕ-convergence for difference double sequences by defining the rough statistical-ϕ limit set. These ideas and results are expected to be a source for researchers in the area of rough convergence of sequences. Also, these concepts can be generalized and applied for further studies.

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