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FUZZY CLASSES OF ANALYTIC FUNCTIONS DEFINED BY FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract. Recently the concept of fuzzy set theory is used to introduce the notion of fuzzy differentiation by several authors in the field of geometric function theory. In this paper, fuzzy differential subordination and fractional differential operator are employed to define certain new classes of analytic functions in the unit disc. Several new interesting results, such as some inclusion relations, convolution and integral preserving properties, for these classes are discussed. Some significant special cases of our main results are also pointed out.

1. Introduction

The classical theory of differential subordination has been introduced and developed by Miller and Mocanu [5, 12] in geometric function theory, which is vibrant field of research of complex analysis of one variable. The concept of fuzzy differential subordination is one of the numerous application of the fuzzy set theory, which was introduced by Zadeh [23]. The research in this direction combines the fuzzy subordination with different types of operators. Fractional calculation has seen a tremendous recently there have been many applications in several domains such as physics, engineering, electric networks, biological systems with memory, computer graphics and others.
In Section2, we present the definitions and preliminaries which makes the foundation of this study. Section 3 includes lemmas that are required to prove the main results in Section 4. Special cases are also discussed.

2. Definitions and preliminaries

Let $\mathbb{C}$ denote the set of all complex numbers and $\mathbb{N}$ be the set of positive integers. For $D \subset \mathbb{C}$, let $\mathcal{H}(D)$ be the class of analytic functions in D.

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For $n \in \mathbb{N}, \mathcal{A}_n$ denotes the class of analytic functions defined as

$$\mathcal{A}_n = \{f: \quad f \in \mathcal{H}(E) \quad \text{and} \quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in E\},$$

where E is the open unit disc given by

$$E = \{z: \quad f \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We write $\mathcal{A} = \mathcal{A}_1$.

Let $S, S^\star$ and C be the classes of functions in $\mathcal{A}$, which are, respectively, univalent, starlike and convex in E with definitions as given below

$$\begin{aligned} S &= \big\{f \in \mathcal{A}: \quad f \text{ is a univalent function in } E\big\} \\ S^\star &= \big\{f \in \mathcal{A}: \quad Re\big(\frac{zf'(z)}{f(z)}\big) > 0, \quad z \in E\big\} \\ C &= \big\{f \in \mathcal{A}: \quad Re\big(1 + \frac{zf''(z)}{f'(z)}\big) > 0, \quad z \in E\big\}. \end{aligned}$$

The classes $S, \quad S^\star$ and C are the extensively studied subclasses of univalent functions in the area of geometric function theory.

Also, given two analytic functions f_1 and f_2 in the open unit disc E, we say that f_1 is subordinate to f_2, denoted by $f_1 \prec f_2 \quad$ or $\quad f_1(z) \prec f_2(z), \quad z \in E$, if there exists a Schwartz function w, which is analytic in E and satisfies the conditions $w(0) = 0, \quad |w(z)| < 1, \quad z \in E$, such that

$$f_1(z) = f_2(w(z)).$$

Moreover, in the case f_2 is univalent in E, then the following equivalence holds true:

$$f_1(z) \prec f_2(z) \quad \Leftrightarrow \quad f_1(0) = f_2(0) \quad \text{and} \quad f_1(E) \subset f_2(E).$$

Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) of these power series is defined as the power series,

$$(f \star g)(z) = f(z) \star g(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Recently the theory of fractional calculus has found interesting applications in the geometric function theory. The classical definitions of fractional operators have been applied to study various subclasses of $\mathcal{A}$.
The fractional derivative of order $\alpha, \quad 0 \le \alpha < 1$, is defined as follows:

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\alpha} dt, \quad 0 \le \alpha < 1, \quad z \in E, \tag{2.1}$$

where Γ represents the Gamma function and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t) \in \mathbb{R}$, whenever $(z-t) > 0$. It is obvious that $D_z^0 = f(z)$.
In [15], the operator $L_\alpha : \mathcal{A} \to \mathcal{A}$ has been defined as

$$\begin{aligned} L_\alpha(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \quad 0 \le \alpha < 1, \quad f \in \mathcal{A} \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k, \quad z \in E \\ &= z + \sum_{k=2}^{\infty} \sigma_k(\alpha) a_k z^k, \quad z \in E \qquad (2.2) \\ &= \phi(2, 2-\alpha; z) \star f(z), \end{aligned}$$

where

$$\phi(a, c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+a)\Gamma(c)}{\Gamma(k+c)\Gamma(a)}, \quad a, c \neq 0, -1, -2, \ldots$$

is the incomplete beta function and Γ denotes gamma function.
Using (2.1), a generalized fractional differential operator

$$D_\lambda^{n,\alpha} : \mathcal{A} \to \mathcal{A}$$

is defined in [2] as follows:
Let $n \in \mathbb{N}, \quad 0 \le \lambda \le 1$ and $f \in \mathcal{A}$.

$$D_\lambda^{n,\alpha} f(z) = D\big(D_\lambda^{n-1,\alpha} f(z)\big).$$

Note that

$$\begin{aligned} D_0^{0,0} f(z) &= f(z) \\ D_\lambda^{1,\alpha} f(z) & \quad (1-\lambda) L_\alpha f(z) + \lambda z (L_\alpha f(z))'. \end{aligned}$$

and, in general, as a power series, from (2.2), we have

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \big[\sigma_{k,}(\alpha, \sigma)\big]^n a_k z^k, \qquad (2.3)$$

where

$$\sigma_k(\alpha, \lambda) = \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \big(1 + \lambda(k-1)\big). \qquad (2.4)$$

(i). If $\alpha = 0, \quad \lambda = 1$, then

$$D_1^{n,0} f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (2.5)$$

is the Salagean differential operator defined in [17].
(ii). If $n = 1$ and $\lambda = 0$, the operator $D_\lambda^{n,\alpha}$ reduces to the operator L_α defined by (2.2).

Let $P[\beta; A, B], \quad 0 < \beta \le 1, \quad -1 \le B < A \le 1$ consist of functions p, analytic in E, $p(0) = 1$ and

$$p(z) \prec \big(\frac{1+Az}{1+Bz}\big)^\beta = h(\beta, A, B).$$

(iii). If $\alpha = 0$, then

$$D_z^\alpha f(z) = z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]^n a_k z^k$$

is the Al-Oboudi differential operator [1].

The function $\left(\frac{1+Az}{1+Bz}\right)^\beta$ can easily shown to be convex and univalent in E and

$$Re\left[\left(\frac{1+Az}{1+Bz}\right)^\beta\right] > \rho = \left(\frac{1-A}{1-B}\right)^\beta, \quad z \in E.$$

We note the following special cases:
(i). $h(\beta,1,-1) = h_\beta$; $|\arg h_\beta| \le \frac{\beta\pi}{2}, z \in E$.
(ii). $h(1,2\gamma-1,-1) = h(\gamma,z)$; $Re[h(\gamma,z)] > \gamma$, $z \in E$.
(iii). $h(1,1,-1) = h$, $Re[h(z)] > 0$, $z \in E$.

We have the followings:

Definition 2.1. [23] *Let X be a nonempty set. A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : \ 0 < F_A(x) \le 1\}$is called the fuzzy subset of X. The set A is called the support of (A, F_A) and F_A is called the membership of (A, F_A). we can write*

$$A = Supp(A, F_A).$$

Remark 2.1. [23] *If $A \subset X$, then*

$$F_A(x) = \begin{cases} 1, & if \quad x \in A \\ 0, & otherwise. \end{cases}$$

For empty set ϕ, $F_\phi(x) = 0$, $x \in X$ and $F_X(x) = 1$, $x \in X$.

Definition 2.2. [14] *Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, $z \in E$, for any function $\mathcal{F}$ analytic in E, if they satisfy the following conditions.*
(i). $f(z_0) = g(z_0)$
(ii). $\mathcal{F}_{f(D)} f(z) \le \mathcal{F}_{g(D)} g(z)$, $z \in D$.

For the general theory of fuzzy differential subordination, convolution, applications and related areas, see [2, 3, 6, 7, 8, 9, 10, 11, 14, 15] and the references therein.

Definition 2.3. *A function $f \in \mathcal{A}$ is said to be belong to the fuzzy class $T_{\lambda,\delta}^{n;\alpha}(h)$, if it satisfies the following condition*

$$\left\{\left(D_\lambda^{n,\alpha} f\right)' + \delta z \left(D_\lambda^{n,\alpha} f\right)''\right\} \prec_{\mathcal{F}} h(z), \quad \delta \ge 0, \quad 0 \le \delta \le 1, \quad \forall z \in E.$$

If

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^\beta, \quad 0 \le \beta \le 1, \quad -1 \le B < A \le 1,$$

then we obtain the class $T_{\lambda,\delta}^{n;\alpha}(\beta, A, B)$.

3. Set of important lemmas

To prove our main results, we need the following Lemmas.

Lemma 3.1. [1] *If $p(z)$ is analytic in E with $p(0)=1$ and $Re(p(z)) > \frac{1}{2}, z \in E$, then the function $(p \star F)(z)$ takes the values in the convex hull of $F(E)$. That is, $(p \star F)(E) \subset F(E)$ in E.*

Lemma 3.2. [16] *Let h be analytic in E with $h(0)=1$ and let $Re\{h(z)\} > \gamma, z \in E$. Then*

$$Re\{h(z)\} \geq (2\gamma - 1) + \frac{2(1-\gamma)}{1+|z|}, \quad 0 \leq \gamma < 1, \quad z \in E.$$

The following Lemma is a generalization of one given in [22].

Lemma 3.3. [19] *Let $P(\beta_i), \quad i = 0, 1, 2, \ldots, m$ be the class of analytic functions p with $p(0) = 1$ and $Re\{p(z))_i\} > \beta_i, \quad \beta_i \leq 1$. Then*

$$P(\beta_1) \star P(\beta_2) \star \ldots \star P(\beta_m) = P(\eta),$$

where

$$\eta = 1 - 2^{m-1}(1-\beta_1)(1-\beta_2)\ldots(1-\beta_m).$$

Lemma 3.4. [13] *Assume that h is a convex function with $h(0) = 1$ and $\nu \in \mathbb{C}^\star = \mathbb{C}\ \{0\}$ with $Re(\nu) \geq 0$. If p is analytic in E with $p(0) = 1$,*

$$\phi : \mathbb{C}^2 \times E \to \mathbb{C}, \quad \phi\big(p(z(, zp'(z); z\big) = p(z) + \nu^{-1} zp'(z)$$

is analytic in E and

$$\mathcal{F}_{\phi(\mathbb{C}^2 \times E)}\big(p(z) + \frac{1}{\nu} zp'(z)\big) \leq \mathcal{F}_{h(E)}\big(h(z)\big),$$

that is,

$$\big(p(z) + \frac{1}{\nu} zp'(z)\big) \prec_{\mathcal{F}} h(z), \quad z \in E,$$

then

$$\mathcal{F}_{p(E)}(p(z)) \leq \mathcal{F}_{q(E)}(q(z)) \leq \mathcal{F}_{h(E)}(h(z)),$$

that is,

$$p(z) \prec_{\mathcal{F}} q(z), \quad z \in E,$$

where

$$q(z) = \frac{\nu}{z^\nu} \int_0^z h(t) t^{\nu-1} dt, \quad z \in E.$$

The function q is convex and it is the fuzzy best dominant.

Lemma 3.5. [16] *Let $\{c_k\}_0^\infty$ be a convex null sequence. Then the function*

$$p(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k, \quad z \in E,$$

is analytic in E and $Re\{p(z)\} > 0$ in E.

Remark 3.1. *A $\{a_k\}_0^\infty$ of non-negative numbers is called a null sequence of $a_n \to 0$, as $\quad n \to \infty$ and $\quad a_0 - a_1 \geq a_1 - a_2 \geq \ldots a_n - a_{n+1}$.*

Lemma 3.6. [2] *Let ${}_2F_1$ represent the Gauss hypergeometric function. For complex numbers a, b, c different from $0, -1, -2, \dots$ we have*

$$(i).\quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \quad {}_2F_1(a,b;c;z),$$
$$Re(c) > Re(b) > 0.$$

$$(ii).\quad (a+1) \quad {}_2F_1(1,a;a+1;z) = (a+1) + az \quad {}_2F_1(1,a+1;a+2;z).$$

$$(iii).\quad {}_2F_1(a,b;c;z) = \quad {}_2F_1(a,c-b;c;\frac{z}{1-z}).$$

4. Main Results

In this section, we prove our main results for the class $T^{n,\alpha}_{\lambda,\delta}(\gamma)$. Throughout our discussion, we assume $0 \le \alpha < 1$, $0 \le \lambda, \delta \le 1$ and n is a positive integer unless stated otherwise.

Theorem 4.1. *Let $h(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $\delta = 0$. Then*

$$T^{n+1,\alpha}_{\lambda,0}(\gamma) \subset_{\mathcal{F}} T^{n,\alpha}_{\lambda,0}(\gamma), \quad \gamma = \frac{1-A}{1-B}.$$

Proof. Let $f \in T^{n,\alpha}_{\lambda,0}(\gamma)$. Then

$$\big(D^{n+1,\alpha}_{\lambda} f\big)'(z) \prec_{\mathcal{F}} \frac{1-(2\gamma-1)z}{1-z} = h_\gamma(z).$$

Note that by (2.3), we have

$$\big(D^{n+1,\alpha}_{\lambda} f\big)'(z) = 1 + \sum_{k=2}^{\infty} \big[\sigma(\alpha,\lambda)\big]^{n+1} k a_k z^{k-1}. \tag{4.1}$$

Thus, by given hypothesis, and (4.1), we have

$$\big(D^{n+1,\alpha}_{\lambda} f(z)\big)' = \big\{1 + \sum_{k=2}^{\infty} \big[\sigma(\alpha,\lambda)\big]^{n+1} k a_k z^{k-1}\big\} \prec_{\mathcal{F}} h_\gamma(z). \tag{4.2}$$

Now, with $\gamma = \frac{1-A}{1-B}$ and (4.2), we have

$$\big(D^{n,\alpha}_{\lambda} f\big)'(z) = \big\{1 + \sum_{k=2}^{\infty} \big[\sigma(\alpha,\lambda)\big]^{n} k a_k z^{k-1}\big\}$$
$$= \big[1 + \frac{1-B}{2(A-B)} \sum_{k=2}^{\infty} k\big\{\sigma_k(\alpha,\lambda)\big\}^{n+1} a_k z^{k-1}\big\}\big] \star \big[1 + \frac{2(A-B)}{1-B} \sum_{k=2}^{\infty} \frac{z^{k-1}}{\sigma_k(\alpha,\lambda)}\big] \tag{4.3}$$

From which, it follows that

$$\big[1 + \frac{1-B}{2(A-B)} \sum_{k=2}^{\infty} k\big\{\sigma_k(\alpha,\lambda)\big\}^{n+1} a_k z^{k-1}\big] \prec_{\mathcal{F}} \frac{1}{1-z}, \quad z \in E. \tag{4.4}$$

It is known [18] that the function

$$\sigma(\alpha) = \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}$$

is a decreasing function of k and

$$0 < \sigma_k \leq \sigma_2 = \frac{1}{2-\alpha}.$$

In [1], it is proved that $d_k = \frac{1}{1+\lambda k}$, $k = 1, 2, 3, \ldots$ is a convex null sequence. This implies that the sequence

$$c_{k-1} \quad = \quad \Big[\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(2-\alpha)}\Big]\Big[\frac{1}{(1+\lambda(k-1))}\Big]$$

with $c_0 = 1$ is a convex null sequence. Therefore, using Lemma 3.1, we have

$$\Big[1 + \frac{2(A-B)}{(1-B)}\sum_{k=2}^{\infty} c_{k-1}z^{k-1}\Big] \prec_{\mathcal{F}} h_\gamma(z), \quad \gamma = \frac{1-A}{1-B}, \quad z \in E. \tag{4.5}$$

Now, using (4.3), (4.4), (4.5) and Lemma 3.5, we obtain

$$\big(D_\lambda^{n,\alpha} f\big)' \prec_{\mathcal{F}} h_\gamma(z), \quad \gamma = \frac{1-A}{1-B}, \quad z \in E.$$

This completes the proof that $f \in T_{\lambda,0}^{n,\alpha}(\gamma)$, $z \in E$. □

We now prove a more general form of Theorem 4.1 as follows.

Theorem 4.2. *Let* $h(z) = \frac{1+Az}{1+Bz}$, $\delta = 0$. *Then, for all* $z \in E$,

$$T_\lambda^{n+1,\alpha}(h) \subset_{\mathcal{F}} T_\lambda^{n,\alpha}(h_\rho), \quad h_\rho = \frac{1+(1-2\rho)z}{1-z}$$

and

$$\rho = \begin{cases} \frac{A}{B} + \frac{B-A}{B(1-B)} \quad {}_2F_1\big(1, \frac{B-A}{B}; 1 + \frac{1}{\sigma_k(\alpha,\lambda)}; \frac{B}{B-1}\big), & \text{if} \quad B \neq 0, \quad k = 2, 3, 4, \ldots \\ 1 + \frac{A}{1+\sigma_k(\alpha,\lambda)}, \quad \text{if} \quad B = 0, \quad k = 2, 3, 4, \ldots \end{cases}$$

The value of ρ *is best possible.*

Proof. Let

$$p(z) = \big(D_\lambda^{n,\alpha} f\big)'(z), \tag{4.6}$$

where p is analytic in E with $p(0) = 1$.
From (2.3), (2.4), (4.1) and (4.3), we have

$$z\big(D_\lambda^{n,\alpha} f\big)'(z) = \frac{1}{\sigma_k(\alpha,\lambda)}\big(D_\lambda^{n+1,\alpha} f\big)(z) + \big(1 - \frac{1}{\sigma_k(\alpha,\lambda)}\big)(D_\lambda^{n,\alpha} f)(z). \tag{4.7}$$

Differentiating (4.6) and using (4.7), we obtain

$$\begin{aligned} \big(D_\lambda^{n+1,\alpha} f\big)'(z) &= p(z) + \sigma_k(\alpha,\lambda)\big[zp'(z)\big] \\ &= p(z) + \frac{1}{\nu} zp'(z) \end{aligned} \tag{4.8}$$

with $\nu = \frac{1}{\sigma_k(\alpha,\lambda)}$.
Since

$$f \in T_\lambda^{n+1,\alpha}(h), \quad h(z) = \frac{1+Az}{1+Bz},$$

this means that

$$\{p(z)+\sigma_k(\alpha,\lambda)[zp'(z)]\} \prec_{\mathcal{F}} \frac{1+Az}{1+Bz} \quad \text{in} \quad E.$$

Now by using Lemma 3.4 with $\nu \neq 0$ and $Re(\nu) \geq 0$, we have

$$p(z) \prec_{\mathcal{F}} q(z) = \frac{1}{\sigma_k(\alpha,\lambda)} z^{(-\frac{1}{\sigma_k(\alpha,\lambda)})} \int_0^z t^{(\frac{1}{\sigma_k(\alpha,\lambda)}-1)}(1+At)(1+Bt)^{-1}dt. \quad (4.9)$$

Using $(i),(ii)$ and (iii) of Lemma 3.6 together with some computation, we obtain

$$q(z) = \frac{A}{B} + \frac{(B-A)}{B(1-B)} \quad {}_2F_1\big(1,1;\frac{1}{\sigma_k(\alpha,\lambda)}+1;\frac{Bz}{Bz+1}\big). \quad B \neq 0. \quad (4.10)$$

To show that $q(z)$ given by (4.10) is best dominant, it is sufficient to prove that

$$\inf\{Re(q(z))\} = q(-1).$$

For $|z| \leq r < 1$, $Re\big(\frac{1+Az}{1+Bz}\big) \geq \frac{1-Ar}{1-Br}$, see [2],
we can write

$$q(z) = \int_0^1 H(t,z)d\mu(t), \quad H(t,z) = \frac{1+Atz}{1+Btz},$$

and

$$d\mu(t) = \frac{1}{\sigma_k(\alpha,\lambda)} t^{(\frac{1}{\sigma_k(\alpha,\lambda)}-1)}dt$$

which is positive measure on $[0,1]$. So

$$Re(q(z)) \geq \int_0^1 \frac{1-Atr}{1-Btr} d\mu(t) = q(-r), \quad |z| < r < 1,$$

and it implies that $\inf\{Re(q(z))\} = q(-1)\}$ by letting $r \to -1$.
This proves $q(z)$ given by (4.10) is fuzzy best dominant and the result is best dominant. □

Remark 4.1. *Let*

$$h(z) = \big(\frac{1+Az}{1+Bz}\big)^{\beta}, \quad \beta \in (0,1]$$

and

$$f \in T_{\lambda,o}^{n,\alpha}(h), \quad \delta \geq 0.$$

Then $f \in T_{\lambda,0}^{n,\alpha}(h_\rho)$, *where*

$$h_\rho(z) = \frac{1+(1-2\rho)z}{1-z}, \quad \rho = \big(\frac{1-A}{1-B}\big)^{\beta}. \quad (4.11)$$

For $A=0$, $B=-1$, $\beta=1$, *we have* $\rho=\frac{1}{2}$ *and* $h_{\frac{1}{2}}(z) = \frac{1}{1-z}$, $z \in E$.

Theorem 4.3. *Let*

$$f_i \in T_{\lambda,\delta}^{n,\alpha}(h) = \quad h(z) = \big(\frac{1+A_iz}{1+B_iz}\big)^{\beta}, \quad -1 \leq \beta_i < A_i \leq 1, \quad i=1,2,3,\ldots,m.$$

Let $g \in \mathcal{A}$ *be defined by*

$$\big(D_\lambda^{n,\alpha}g\big)' = \int_0^z \big[\big(D_\lambda^{n,\alpha}f_1\big)' \star \big(D_\lambda^{n,\alpha}f_2\big)' \star \ldots \star \big(D_\lambda^{n,\alpha}f_m\big)'\big](t)dt. \quad (4.12)$$

Then

$$g \in T_{\lambda,\delta}^{n,\alpha}(h_\zeta), \quad h_\zeta = \frac{1+(1-2\zeta)z}{1-z}, \quad z \in E,$$

where

$$\zeta = \begin{cases} 1-2^{m-1}(1-\rho_1)(1-\rho_2)\dots(1-\rho_m)\big[1-\frac{1}{2}\quad {}_2F_1(1,1;1+\frac{1}{\delta};\frac{1}{2}\big], & \textit{for} \quad \delta>0 \\ 1-2^{m-1}\big[(1-\rho_1)(1-\rho_2)\dots(1-\rho_m)\big], & \textit{for} \quad \delta=0, \end{cases}$$

and

$$\rho_i = \big(\frac{1-A_i}{1-B_i}\big)^{\beta}, \quad i=1,2,3,\dots,m.$$

Proof. Let $\delta = 0$. Then, from Remark 4.1, $f_i \in T^{n,\alpha}_{\lambda,o}(h_{\rho_i})$, where ρ_i is given in (4.1) for each $i = 1,1,\dots,m$. Now, using Lemma 3.3. we have

$$\big(h_{\rho_1} \star h_{\rho_2} \star \dots \star h_{\rho_m}\big)(z) \prec_{\mathcal{F}} h_{\xi}(z) = \frac{1+(1-2\xi)z}{1-z}. \tag{4.13}$$

From (4.12) and (4.13), we have

$$\begin{aligned} \big(D^{n,\alpha}_{\lambda} g\big)'(z)\big) &\prec_{\mathcal{F}} h_{\zeta}(z), \\ \zeta &= 1-2^{m-1}(1-\rho_1)(1-\rho_2)\dots(1-\rho_m), \quad \rho_i = \big(\frac{1-A_i}{1-B_i}\big)^{\beta}. \end{aligned}$$

Let $\delta > 0$ and $f_i \in T^{n,\alpha}_{\lambda,\delta}(h_{\rho_i}), \quad i = 1,2,\dots,m.$
Let

$$\big(D^{\lambda,\alpha}_{\lambda} f_i)(z)\big)'(z) = h_{\rho_i}(z).$$

Then

$$\big[(D^{\lambda,\alpha}_{\lambda} f_i)' + \delta z(D^{\lambda,\alpha}_{\lambda} f_i))'\big](z) = h_{\rho_i}(z) + \frac{1}{\delta} z h'_{\rho_i}(z),$$

and

$$h_{\rho_i}(z) + \frac{1}{\delta} z h'_{\rho_i}(z) \prec_{\mathcal{F}} \big(\frac{1-A_i}{1-B_i}\big)^{\beta}. \tag{4.14}$$

From (4.12), (4.14) and Lemma 3.4. we obtain

$$\begin{aligned} &\big(D^{n,\alpha}_{\lambda} g\big)'(z) \\ = &\big[\frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z t^{(\frac{1}{\delta}-1)} h_{\rho_1}(t)dt\big] \star \big[\frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z t^{(\frac{1}{\delta}-1)} h_{\rho_2}(t)dt\big] \star \dots \big[\frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z t^{(\frac{1}{\delta}-1)} h_{\rho_m}(t)dt\big] \\ = &\frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z t^{(\frac{1}{\delta}-1)} \big(h_{\rho_1} \star h_{\rho_2} \star \dots \star h_{\rho_m}\big)(t)dt \\ = &\frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z t^{(\frac{1}{\delta}-1)} h_{\rho_0}(t)dt. \end{aligned}$$

This means that

$$(D^{\lambda,\alpha}_{\lambda} g)'(z) = \frac{1}{\delta} z^{\frac{-1}{\delta}} \int_0^z u^{\frac{1}{\delta}-1} h_{\rho_0}(uz)du,$$

where

$$\begin{aligned} h_{\rho_o} &= \big[(D^{\lambda,\alpha}_{\lambda} g)' + \delta z(D^{\lambda,\alpha}_{\lambda} g))'\big](z) \\ &= \frac{1}{\delta} \int_0^z t^{(\frac{1}{\delta}-1)} \big(h_{\rho_1} \star h_{\rho_2} \star \dots h_{\rho_m}\big)dt. \end{aligned} \tag{4.15}$$

Now applying Lemma 3.2 and (4.13), we have

$$\begin{aligned} Re\big(h_{\rho_0}(z)\big) &\geq \frac{1}{\delta}\int_0^z t^{(\frac{1}{\delta}-1)}\big[(2\xi-1)+\frac{2(1-\xi)}{1+u|z|}\big]du \\ &> \frac{1}{\delta}\int_0^1 \big[(2\xi-1)+\frac{2(1-\xi)}{1+u}\big]du. \end{aligned}$$

Using (4.13) and some computation, we obtain

$$\begin{aligned} Re\big(h_{\rho_0}(z)\big) &> 1-2^{m-1}\big[(1-\rho_1)(1-\rho_2)\dots(1-\rho_m)\big]\big[1-\frac{1}{\delta}\int_0^1 t^{(\frac{1}{\delta}-1)}\frac{1}{1+ut}du\big] \\ &> 1-2^{m-1}\big[(1-\rho_1)(1-\rho_2)\dots(1-\rho_m)\big]\big[1-\frac{1}{2}\quad {}_2F_1(1,1;1+\frac{1}{\delta};\frac{1}{2})\big] \\ &= \zeta, \quad z\in E. \end{aligned}$$

This shows that

$$g\in T^{n,\alpha}_{\lambda,\delta}(h_\xi),\quad h_\xi(z)=\frac{1+(1-2\xi)z}{1-z},\quad \text{for}\quad z\in E$$

and the proof is complete. □

Next, we prove that the fuzzy class $T^{n,\alpha}_{\lambda,\delta}(h)$ is closed under convolution.

Theorem 4.4. *Let $f\in T^{n,\alpha}_{\lambda,\delta}(h)$ and $\psi\in C$. Then $(f\star\psi)\in T^{n,\alpha}_{\lambda,\delta}(h)$.*

Proof. Since $\psi\in C$, it is known [2] that

$$Re\{\frac{\psi(z)}{z}\}>\frac{1}{2},\quad z\in E.$$

Let

$$Q(z)=\big(D^{n,\alpha}_\lambda f\big)'(z)+\delta z\big(D^{n,\alpha}_\lambda f\big)''(z)$$

and

$$S(z)=\frac{\psi(z)}{z},$$

where Q and S are analytic functions in E with $Q(0)=S(0)=1$.
Consider

$$\begin{aligned} (1-\delta)D^{n,\alpha}_\lambda\big(f\star\psi\big)'(z) &+ \delta\big(zD^{n,\alpha}_\lambda(f\star\psi)'\big)'(z) \\ &= \big[\big(D^{n,\alpha}_\lambda f\big)'(z)+\delta z\big(D^{n,\alpha}_\lambda f\big)''(z)\big]\star\frac{\psi(z)}{z} \\ &= Q(z)\star S(z). \end{aligned}$$

Now, by given hypothesis,

$$Q(z)\prec_{\mathcal{F}}\big(\frac{1+Az}{1+Bz}\big)^\beta,\quad S(z)\prec_{\mathcal{F}}\frac{1}{1-z}.$$

The required result follows by using Lemma 3.1 that $(f\star\psi)\in T^{n,\alpha}_{\lambda,\delta}(h)$.

□

Theorem 4.5. *Let, for each $i=1,2,3,\dots m$ and $\delta=0$, $f_i\in T^{n,\alpha}_\lambda(h_{\rho_i})$, where*

$$h_{\rho_i}=\frac{1+(1-2\rho_i)z}{1-z},\quad \rho_i=\big(\frac{1-A_i}{1-B_i}\big)^\beta.$$

Let

$$G(z)=\big[D^{n,\alpha}_\lambda\big(f_1\star f_2\star\dots\star f_m\big)\big](z).$$

Then

$$G'(z) \prec_{\mathcal{F}} H(z),$$

where

$$H(z) = 1 + \sum_{k=1}^{\infty} \left[\Pi_{i=1}^{n}(1-\rho_i)\right] \frac{z^k}{k+1}, \tag{4.16}$$

is convex univalent in E.

Proof. Since $f_i \in T_{\lambda,0}^{n,\alpha}(h_{\rho_i})$, we have

$$\left(D_\lambda^{n,\alpha} f_i\right)'(z) \prec_{\mathcal{F}} \frac{1+(1-2\rho_i)z}{1-z}, \quad \rho_i = \big(\frac{1-A_i}{1-B_i}\big)^{\beta}.$$

Using Lemma (3.3, we have

$$\left\{\left(D_\lambda^{n,\alpha} f_1\right)' \star \left(D_\lambda^{n,\alpha} f_2\right)' \star \ldots \star \left(D_\lambda^{n,\alpha} f_m\right)'\right\}(z) \prec_{\mathcal{F}} \left(h_{\rho_1} \star h_{\rho_2} \star \ldots \star h_{\rho_m}\right)(z). \tag{4.17}$$

Now

$$\left(h_{\rho_1} \star h_{\rho_2} \star \ldots \star h_{\rho_m}\right)(z) = 1 + \sum_{k=1}^{\infty} \left[\Pi_{i=1}^{m}(1-\rho_i)\right] z^k.$$

We note that

$$H_1(z) = \frac{-2}{z}\left[z + \log(1-z)\right] = \sum_{k=1}^{\infty} \big(\frac{2}{k+1}\big) z^k$$

is convex and so, for $f \in \mathcal{A}$,

$$\left(f \star H_1\right)(z) = \frac{2}{z}\int_0^z f(t)dt.$$

Therefore the function $H_2(z) = 1 + H_1(z), \quad z \in E$ is convex univalent in E, and we can write for $Re\{p(z)\} > 0$,

$$\left(p \star H_2\right)(z) = -1 + \frac{2}{z}\int_0^z p(t)dt, \quad z \in E. \tag{4.18}$$

Applying a result given in [22] to (4.17), (m-1) times with $H_2 \prec_{\mathcal{F}} H_2$, we have

$$\begin{aligned}&\left(D_\lambda^{n,\alpha} f_1\right)' \star \left(D_\lambda^{n,\alpha} f_2\right)' \star \ldots \star \left(D_\lambda^{n,\alpha} f_m\right)' \star H_2 \star H_2 \star \ldots \star H_2 \\ &\prec_{\mathcal{F}} h_{\rho_1} \star h_{\rho_1} \star \ldots \star h_{\rho_m} \star H_2 \star H_2 \star \ldots \star H_2.\end{aligned}$$

This implies

$$\begin{aligned}&\left[D_\lambda^{n,\alpha} f_1\right)' \star H_2\right] \star \left[D_\lambda^{n,\alpha} f_2\right)' \star H_2\right] \ldots \star \left[D_\lambda^{n,\alpha} f_{m-1}\right)' \star H_2\right] \star \left(D_\lambda^{n,\alpha} f_m\right)' \\ &\prec_{\mathcal{F}} \left[h_{\rho_1} \star H_2\right] \star \left[h_{\rho_2} \star H_2\right] \ldots \star \left[h_{\rho_{m-1}} \star H_2\right] \star h_{\rho_m}.\end{aligned} \tag{4.19}$$

Now, using 4.18), we can rewrite (4.19) as follows

$$\begin{aligned}
&\Big(\frac{D_\lambda^{n,\alpha} f_1}{z}\Big) \star \Big(\frac{D_\lambda^{n,\alpha} f_2}{z}\Big) \star \ldots \star \Big(\frac{D_\lambda^{n,\alpha} f_{m-1}}{z}\Big) \star \Big(\frac{D_\lambda^{n,\alpha} f_1}{z}\Big)' \\
\prec_{\mathcal{F}} \quad & \frac{1}{z}\int_0^z h_{\rho_1}(t)dt \star \frac{1}{z}\int_0^z h_{\rho_2}(t)dt \star \ldots \star \frac{1}{z}\int_0^z h_{\rho_{m-1}}(t)dt \star h_{\rho_m}(z) \\
= \quad & 1 + \sum_{k=1}^{\infty} \big[\Pi_{i=1}^m (1-\rho_i)\big] \frac{z^k}{k+1}. \qquad (4.20)
\end{aligned}$$

It can easily be seen that left hand side of (4.20) is

$$\begin{aligned}
\big(D_\lambda^{n,\alpha} f_1 \star D_\lambda^{n,\alpha} f_2 \star \ldots \star D_\lambda^{n,\alpha} f_m\big)'(z) &= \big[D_\lambda^{n,\alpha}\big(f_1 \star f_2 \star \ldots \star f_m\big)\big]'(z) \\
&= G'(z), \quad z \in E. \qquad (4.21)
\end{aligned}$$

Hence, from (4.20) and (4.21, we obtain that

$$G'(z) \prec_{\mathcal{F}} H(z), \quad z \in E.$$

This completes the proof. □

Special cases

We now discuss some special cases of Theorem 4.5 as

(i). We take $\beta = 1, \quad B_i = -1, \quad A_i = 1 - 2\beta_i, \quad 0 \le \beta_i < 1$ for $i = 1, 2, \ldots, m$. Then $\rho_i = \beta_i$ and $h_{\rho_i} = h_{\beta_i}$. Consequently $H(z)$ in (4.16) can be written as

$$H_\beta = 1 + 2^m \big(1\beta_1)(1-\beta_2)\ldots(1-\beta_{m-1}\big) \sum_{k=1}^{\infty} \frac{z^k}{(k+1)^m}.$$

For $n = 0, \quad \alpha = 0, \lambda = 0, \quad \delta = 0$ and $Re\{f_i'(z)\} > \beta_i$ with $\beta_i \in [0, 1)$ for each $i = 1, 2 \ldots, m$, we have from Theorem 4.5 that $G' \prec_{\mathcal{F}} H_{\beta_i}$ in E. In this case,

$$G(z) = \big(f_1 \star f_2 \star \ldots, \star f_m\big)(z).$$

(ii). $H(z)$ defined by (4.16) is convex univalent in function with real coefficients . So, from Theorem 4.5. we have

$$H(-1) \le Re\{G'(z)\} \le H(1).$$

(iii). For

$$\beta = 0, \quad h_{\rho_i} \prec_{\mathcal{F}} \frac{1 + A_i}{1 + B_i}, \quad G(z) = \big[D_\lambda^{n,\alpha}\big(f_1 \star f_2 \star \ldots, \star f_m\big)\big](z),$$

then

$$Re\{G'(z)\} \ge 1 + \sum_{k=1}^{\infty} \big[\Pi_{i=i}^m \big(B_i^k - A_i B_i^{k-1}\big)\big]\big\{(1 - 2^{2-m})\eta(m-1) - 1\big\},$$

where η is the well known zeta function in [5].
By taking $n = 0, \quad m = 3, \quad \alpha = \lambda = \delta = 0, \quad B_i = -1, \quad A_i = 1 - \beta_i, \quad i = 1, 2, \ldots,$ writing $\eta(2) = \frac{\pi^2}{6}$, we obtain a result proved in [21].

Conclusion. In this paper, some new fuzzy classes of analytic functions are defined using the fuzzy differential subordination and fractional differential operator in the unit disc. Inclusion relations, convolution and integral preserving properties for these new classes are derived. Several important special are also highlighted. The ideas and techniques of this paper be starting point for future results. It is an interesting problem to explore the applications of the fuzzy analytic classes in various branches of pure and applied sciences.

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