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IDENTIFYING THE SOURCE TERM IN A SOBOLEV-TYPE EQUATION BY OPTIMIZATION METHOD

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Abstract. This paper is devoted to the study of an inverse problem of recovering a space-dependant source term in a Sobolev-type equation from final measurement data. The aim of this work is twofold, first, to establish some results concerning the stability and local uniqueness of the solution which is done using the optimal control framework, and second, to design an efficient algorithm based on the Landweber iteration method for the numerical identification of the unknown source term. Some typical numerical experiments are performed to verify the effectiveness and validity of the proposed algorithm.

1. Introduction

For several applications in modern sciences and engineering, it is crucial to measure some parameters in the model that describes the physical process. The determination of these unknown parameters from additional measurement data, which is known as inverse problem, has received much attention.

In the present paper, we study the inverse problem of identifying a space-dependent source term in a third order pseudoparabolic equation from the theoretical analysis and numerical computation angles. More precisely, we consider the following system:

$$\partial_t u(x,t) - \partial_t \Delta u(x,t) - \Delta u(x,t) = f(x), \quad (x,t) \in Q_T := \Omega \times (0,T] \tag{1.1}$$

attached by initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad x \in \Omega :, \\ &u(0,t) = u(1,t) = 0, \quad t \in [0,T], \end{aligned} \tag{1.2}$$

where $\Omega := (0,1)$, $T > 0$ represents a final time, the initial data $u_0(x)$ is a given smooth function, and the source term $f(x)$ is an unknown source term in the equation (1.1). Given an overspecified condition at the final time, that is,

$$u(x,T) = g(x), \quad x \in \overline{\Omega}, \tag{1.3}$$

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where the given function g is a known function that satisfies the homogeneous Dirichlet boundary conditions. Then the inverse problem to be investigated is to determine the pair of functions $\{u, f\}$ satisfying (1.1)-(1.2) from the measured data at the final time $u(\cdot, T)$.

The inverse problem of identifying unknown parameters in partial differential equations, from overspecified data on the solution, has received considerable attention from a broad cross-section of researchers. The reason for this interest relates to the theoretical analysis and practical applications of these problems in many disciplines such as geophysics [4], medicine [3, 15], data assimilation [13] and so on.

Pseudo-parabolic equations from a subclass of a general equations of Sobolev type, sometimes referred to as Sobolev-Galpern type. They are characterized by having mixed time and space derivatives appearing in the highest-order terms of the equation. Mathematical models relying on this type of equations arise in various fields of mathematical physics to describe various process, we mention, fluid flow in fissured medium, heat conduction in composite medium and propagation of long waves of small amplitude. A variety of studies have been devoted to the forward problem for the third-order pseudo-parabolic equations (see [1, 7, 11]).

The inverse problems for pseudo-parabolic equations have been much less intensively investigated compared to parabolic problems, which can be seen from the huge amount of papers and works devoted to studying inverse problems for parabolic systems comparing to those devoted to pseudo-parabolic equations. Nevertheless, several works in the literature provide some results on this kind of problem in both aspects, theoretically and numerically. In this context, we mention some recents works in this direction. The inverse problem of identifying a leading coefficient in a pseudo-parabolic equation from an integral-type over-determination condition is studied in [16, 17], where the local existence and uniqueness of strong solution are proved. In a recent paper [2], the authors proved the existence of solutions in a local and global time to the inverse problem of determining the right side of the pseudo-parabolic equation with a p-Laplacian and nonlocal integral overdetermination condition. For the numerical resolution, the paper [12] address the inverse problem of constructing numerically the time-dependent potential term in a third-order pseudo-parabolic equation with initial and Neumann boundary conditions.

Our aim in this paper is to discuss the problem of recovering the source term f in the pseudo-parabolic equation (1.1) from the theoretical analysis and numerical computation angles. Apart from the aforementioned works, we follow in this paper a methodology which initially used for source identification problems for parabolic systems [8, 18]. This method is based on optimal control framework, the basic idea is to transform the inverse source problem into an optimization problem, and then take the minimizer of an adequate cost functional as the general solution. Then, the uniqueness and stability results are established with the aid of a first-order necessary optimality condition satisfied by the optimal solution.

The brief description of our main result is as follows: Let u and $\tilde{u}$ be two solutions of (1.1) with the source terms f and $\tilde{f}$ respectively. Then, for all $T > 0$ there exists a positive constant $C > 0$ satisfying

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C\|g - \tilde{g}\|_{L^2(\Omega)},$$

where g and $\tilde{g}$ are final observation data, that is $u(x, T) = g(x)$ and $\tilde{u}(x, T) = \tilde{g}(x)$.

The remainder of this paper comprises four section. In Section (2), the considered inverse problem is transformed into an optimal control problem and the existence of a minimizer of the cost functional is proved. In section (3) we establish the first-order necessary optimality. Making use of the obtained necessary conditions, we prove the stability result in Section (4). The fifth section (5) is devoted to numerical simulation, in which, we first design an iterative algorithm based on the Landweber iteration method for solving numerically the source identification problem, and then perform some numerical experiments to verify the validity of the proposed method. Finally, further discussion and concluding remarks are offered in the last section.

2. Optimal control problem

It is well-known that the above inverse problem is ill-posed in the sense of Hadamard (see [14] for more details), i.e., its solution depends unstably on the data, therefore, we treat the inverse problem of recovering the coefficient $f(x)$ in the equation (1.1) by interpreting its solution as a minimizer of an adequate optimization problem.

Let us consider the following admissible set

$$\mathcal{A} = \left\{ f \in H^1(\Omega) : 0 < f_{min} \leq f(x) \leq f_{max} \right\}.$$

Here f_{min} and f_{max} are two positive constants which stand respectively for the lower and upper bounds for the unknown f. Now we consider the optimal control problem of finding $\bar{f} \in \mathcal{A}$ the minimizer of the following problem

$$\mathcal{J}(\bar{f}) = \min_{f \in \mathcal{A}} \mathcal{J}(f), \tag{2.1}$$

where

$$\mathcal{J}(f) = \frac{1}{2} \int_0^1 |u(f)(x,T) - g(x)|^2 \, \mathrm{d}x + \frac{\lambda}{2} \int_0^1 |f|^2 \mathrm{d}x, \tag{2.2}$$

u is the solution of the problem (1.1) for the given source term $f \in \mathcal{A}$ and λ is a regularization parameter.

Before considering the inverse problem (1.1)-(1.3), it is essential to provide the well-posedness of the direct problem. It follows form the results established in [7] that there exists a unique weak solution u to the problem (1.1) which belongs to the space $L^2\left(0, T; H_0^1(\Omega)\right)$. Now, we turn to derive an estimate for this solution which we shall need further in the study of the inverse problem.

Lemma 2.1. *Let $u_0, f \in L^2(\Omega)$, then the weak solution u of the problem (1.1) satisfies the following estimate*

$$\|u(t)\|^2_{H^1(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} \mathrm{d}s \leq C(\|f\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}), \quad 0 \leq t \leq T, \tag{2.3}$$

where $C > 0$ is a positive constant independent of any function.

Proof. Multiplying equation (1.1) by u, integrating with respect to x over Ω and applying Green formula, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2 + \|\nabla u(t)\|^2 = \int_0^t f u \mathrm{d}x, \qquad \text{for } t \in (0, T].$$

Applying Young inequality and integrating the above equation with respect to t over $[0,\xi]$ with $\xi \in (0,T]$, we have

$$\|u(t)\|_1^2 + \int_0^{\xi} \|\nabla u(s)\|^2 \mathrm{d}s \leq C\left(\int_0^{\xi} \|f(s)\|^2 \mathrm{d}s + \int_0^{\xi} \|u(s)\|^2 \mathrm{d}s + \|u_0\|^2\right),$$

then, by the use of Gronwall lemma we achieve the proof. □

Lemma 2.2. *For any subsequence $(f_n) \subset \mathcal{A}$, such that $\|f - f_n\|_{L^1(\Omega)} \to 0$ when $n \to \infty$, we have*

$$\lim_{n\to\infty} \int_0^1 \left|u\left(f_n\right)(x,T) - g(x)\right|^2 \mathrm{d}x = \int_0^1 |u(f)(x,T) - g(x)|^2 \mathrm{d}x$$

Proof. The proof is devided into three steps

Step 1 According to Lemma (2.1), there exists a positive constant $C > 0$ such that

$$\|u\left(f_n\right)\|_1^2 + \int_0^T \|\nabla u\left(f_n\right)(s)\|^2 \mathrm{d}s \leq C(\|f_n\|^2 + \|u_0\|^2). \tag{2.4}$$

The inequality (2.4) garantiates that $\{u\left(f_n\right)\}$ is bounded uniformly sequence in $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Hence, there exists a subsequence, again denoted by $\{u\left(f_n\right)\}$, such that

$$u\left(f_n\right) \rightharpoonup u^* \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \tag{2.5}$$

Step 2 Prove $u^*(x,t) = u(f)(x,t)$.

We multiply both sides of the weak formulation

$$\int_0^1 \partial_t u v \mathrm{d}x + \int_0^1 \partial_t \nabla u \cdot \nabla v \mathrm{d}x + \int_0^1 \nabla u \cdot \nabla v \mathrm{d}x = \int_0^1 f v \mathrm{d}x, \quad t \in (0,T], \tag{2.6}$$

by a function $\xi(t) \in C^1[0,T]$ with $\xi(T) = 0$, and taking $f = f_n$, we get

$$\int_0^1 \xi \partial_t u(f_n) v \mathrm{d}x + \int_0^1 \xi \partial_t \nabla u(f_n) \cdot \nabla v \mathrm{d}x + \int_0^1 \xi \nabla u(f_n) \cdot \nabla v \mathrm{d}x = \int_0^1 \xi f_n v \mathrm{d}x,$$

Then by integrating with respect to t over $[0,T]$, we have

$$-\int_0^T \int_0^1 \partial_t \xi u(f_n) v \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^1 \partial_t \xi \nabla u(f_n) \cdot \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^1 \xi \nabla u(f_n) \cdot \nabla v \mathrm{d}x\mathrm{d}t =$$
$$\int_0^T \int_0^1 \xi f_n v \mathrm{d}x\mathrm{d}t - \int_0^1 \xi(0) u_0 v \mathrm{d}x - \int_0^1 \xi(0) \nabla u_0 \nabla v \mathrm{d}x, \tag{2.7}$$

Letting $n \to \infty$ in (2.7) and taking into account (2.5), we obtain

$$-\int_0^T \int_0^1 \partial_t \xi u^* v \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^1 \partial_t \xi \nabla u^* \cdot \nabla v \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^1 \xi \nabla u^* \cdot \nabla v \mathrm{d}x\mathrm{d}t =$$
$$\int_0^T \int_0^1 \xi f v \mathrm{d}x\mathrm{d}t - \int_0^1 \xi(0) u_0 v \mathrm{d}x - \int_0^1 \xi(0) \nabla u_0 \nabla v \mathrm{d}x, \tag{2.8}$$

The identity (2.8) is valid for any $\xi(t) \in C^1[0,T], \xi(T) = 0$. Therefore, we have for all $t \in (0,T]$

$$\int_0^1 \partial_t u^* v \mathrm{d}x + \int_0^1 \partial_t \nabla u^* \cdot \nabla v \mathrm{d}x + \int_0^1 \nabla u^* \cdot \nabla v \mathrm{d}x = \int_0^1 f v \mathrm{d}x,$$

and $u^*(x,0) = u_0$, hence, u^* is a weak solution in the sense (2.6) which yields that $u^* = u(f)$.
Step 3 Prove $\|u(f_n)(.,T) - g\| \to |u(f)(.,T) - g\|$ as $n \to \infty$.
With $f = f_n$, we can rewritten (2.6) in the form

$$\int_0^1 \partial_t \left(u\left(f_n\right) - g\right) v \mathrm{d}x + \int_0^1 \partial_t \nabla \left(u\left(f_n\right) - g\right) \cdot \nabla v \mathrm{d}x + \int_0^1 \nabla \left(u\left(f_n\right) - g\right) \cdot \nabla v \mathrm{d}x = \int_0^1 f_n v \mathrm{d}x - \int_0^1 \nabla g \cdot \nabla v \mathrm{d}x$$

By taking $v = u\left(f_n\right) - g$ in the above formulation, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(f_n) - g\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\left(u\left(f_n\right) - g\right)\|^2 + \|\nabla\left(u\left(f_n\right) - g\right)\|^2 = \int_0^1 f_n \left(u\left(f_n\right) - g\right) \mathrm{d}x - \int_0^1 \nabla g \cdot \nabla \left(u\left(f_n\right) - g\right) \mathrm{d}x. \quad (2.9)$$

We can obtain a similar relation for $u(f)$, namely,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(f) - g\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\left(u(f) - g\right)\|^2 + \|\nabla\left(u(f) - g\right)\|^2 = \int_0^1 f\left(u\left(f\right) - g\right) \mathrm{d}x - \int_0^1 \nabla g \cdot \nabla \left(u\left(f\right) - g\right) \mathrm{d}x. \quad (2.10)$$

From (2.9) and (2.10), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(f_n) - u(f)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(u(f) - g)\|^2 + \int_0^1 \big\{f(u(f) - g) - f_n(u(f_n) - g)\big\}\mathrm{d}x = - \int_0^1 \nabla g \cdot \nabla(u(f) - u(f_n))\mathrm{d}x + \int_0^1 \nabla u(f) \cdot \nabla(u(f) - u(f_n))\mathrm{d}x + \int_0^1 \nabla u(f_n) \cdot \nabla(u(f) - u(f_n))\mathrm{d}x - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big[(u(f) - g)(u(f_n) - u(f))\big]\mathrm{d}x - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big[\nabla(u(f) - g) \cdot \nabla(u(f_n) - u(f))\big]\mathrm{d}x. \quad (2.11)$$

On the other hand, taking $v = u(f_n) - u(f)$ in (2.6) gives

$$\int_0^1 \partial_t u(f)(u(f_n) - u(f))\mathrm{d}x + \int_0^1 \partial_t \nabla u(f) \cdot \nabla(u(f_n) - u(f))\mathrm{d}x = - \int_0^1 \nabla u(f) \cdot \nabla(u(f_n) - u(f))\mathrm{d}x + \int_0^1 f(u(f_n) - u(f))\mathrm{d}x. \quad (2.12)$$

Similarly, by taking $u = u\left(f_n\right) - u(f)$ and $v = u(f) - g$ in (2.6) on can obtain

$$\int_0^1 \partial_t(u(f_n) - u(f))(u(f) - g)\mathrm{d}x + \int_0^1 \partial_t \nabla(u(f_n) - u(f)) \cdot \nabla(u(f) - g)\mathrm{d}x = - \int_0^1 \nabla(u(f_n) - u(f)) \cdot \nabla(u(f) - g)\mathrm{d}x + \int_0^1 (f_n - f)(u(f) - g)\mathrm{d}x. \quad (2.13)$$

From (2.12) and (2.13), we obtain

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left[(u(f)-g)\left(u\left(f_n\right)-u(f)\right)\right]\mathrm{d}x+\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left[\nabla(u(f)-g)\cdot\nabla\left(u\left(f_n\right)-u(f)\right)\right]\mathrm{d}x \\
&=\int_0^1 \nabla g\cdot\nabla\left(u(f_n)-u\left(f\right)\right)\mathrm{d}x-2\int_0^1 \nabla u(f)\cdot\nabla\left(u\left(f_n\right)-u(f)\right)\mathrm{d}x \\
&\qquad +\int_0^1 f\left(u\left(f_n\right)-u(f)\right)\mathrm{d}x+\int_0^1 \left(f_n-f\right)\left(u(f)-g\right)\mathrm{d}x. \quad (2.14)
\end{aligned}$$

Combining (2.11) and (2.14), it follows

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\left(f_n\right)-u(f)\right\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\left(u\left(f\right)-g\right)\right\|^2+\int_0^1\left|\nabla\left(u\left(f_n\right)-u(f)\right)\right|^2\mathrm{d}x \\
=\int_0^1\left(f_n-f\right)\left(u\left(f_n\right)-u(f)\right)\mathrm{d}x,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\left(f_n\right)-u(f)\right\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\left(u\left(f\right)-g\right)\right\|^2 \\
\leq\int_0^1\left(f_n-f\right)\left(u\left(f_n\right)-u(f)\right)\mathrm{d}x,
\end{aligned}$$

By integrating the above inequality with respect to t over $[0,T]$, we obtain

$$\frac{1}{2}\left\|u(f_n)(t)-u(f)(t)\right\|_{H^1(\Omega)}^2\leq\int_0^T\int_0^1\left|f_n-f\right|\cdot\left|u\left(f_n\right)-u(f)\right|\mathrm{d}x\mathrm{d}t. \quad (2.15)$$

Now, from the convergence of $\{f_n\}$ and the weak convergence of $\{u\left(f_n\right)\}$, therefore one can easily obtain

$$\int_0^T\int_0^1\left|f_n-f\right|\cdot\left|u\left(f_n\right)-u(f)\right|\mathrm{d}x\mathrm{d}t\to 0,\qquad \text{as } n\to\infty. \quad (2.16)$$

Combining (2.15) and (2.16), we have

$$\max_{t\in[0,T]}\left\|u\left(f_n;t\right)-u(f;t)\right\|_{L^2(0,1)}\to 0,\qquad \text{as } n\to\infty. \quad (2.17)$$

Let us denote

$$\mathcal{I}=\left|\int_0^1 u\left(f_n\right)(\cdot,T)-g(x)|^2\,dx-\int_0^1\left|u(f)(\cdot,T)-g(x)\right|^2dx\right|,$$

then, using Hölder and Cauchy-Schwartz inequalities, we derive

$$\begin{aligned}
\mathcal{I} &\leq \int_0^1\left|u\left(f_n\right)(\cdot,T)-u(f)(\cdot,T)\right|\cdot\left|u\left(f_n\right)(\cdot,T)+u(f)(\cdot,T)-2g\right|\mathrm{d}x \\
&\leq \left\|u\left(f_n\right)(\cdot,T)-u(f)(\cdot,T)\right\|\cdot\left\|u\left(f_n\right)(\cdot,T)+u(f)(\cdot,T)-2g\right\|.
\end{aligned}$$

Take into account (2.17) we deduce that $\mathcal{I}\to 0$ as $n\to\infty$, which implies in turn

$$\lim_{n\to\infty}\int_0^1\left|u\left(f_n\right)(x,T)-g(x)\right|^2\mathrm{d}x=\int_0^1\left|u(f)(x,T)-g(x)\right|^2\mathrm{d}x.$$

This completes the proof of the desired result. □

Now we pass to prove the existence of a minimizer $\bar{f}\in\mathcal{A}$ to the minimization problem (2.1).

Theorem 2.3. *There exists a minimizer $\bar{f} \in \mathcal{A}$ of $\mathcal{J}(f)$, i.e.*

$$\mathcal{J}(\bar{f}) = \min_{f\in\mathcal{A}} \mathcal{J}(f).$$

Proof. It can be easily seen that $\mathcal{J}$ is non-negative function, and thus it has the greatest lower bound. Let $\{u_n, f_n\}$ be a minimizing sequence, i.e.,

$$\inf_{f\in\mathcal{A}} \mathcal{J}(f) \leq \mathcal{J}(f_n) \leq \inf_{f\in\mathcal{A}} \mathcal{J}(f) + \frac{1}{n}, \quad n \in \mathbb{N}^*.$$

Since $\mathcal{J}(f_n) \leq C$ we deduce from the particular structure of $\mathcal{J}$ that $\{f_n\}$ is uniformly bounded in $H^1(\Omega)$, that is, there exists a positive constant C independent of n such that $\|f_n\|_{H^1(\Omega)} \leq C$. Therefore, there exists a subsequence of $\{f_n\}$, again denoted by $\{f_n\}$, weakly convergent in $H^1(\Omega)$, namely,

$$f_n \rightharpoonup \bar{f} \in H^1(\Omega), \qquad \text{as } n \to \infty. \tag{2.18}$$

By the Sobolev embedding theorem we obtain

$$\left\|f_n - \bar{f}\right\|_{L^1(\Omega)} \to 0, \qquad \text{as } n \to \infty.$$

In particular, since $\{f_n\}$ belongs to $\mathcal{A}$ which is a closed subset, we have

$$f_n \to \bar{f} \in \mathcal{A}, \quad \text{in } L^1(\Omega). \tag{2.19}$$

On the other hand, from (2.18) we obtain

$$\int_0^1 |\bar{f}|^2 \mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n \bar{f} \mathrm{d}x \leq \lim_{n\to\infty} \left(\int_0^1 |f_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^1 |\bar{f}|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.20}$$

By considering (2.19) it follows by applying Lemma (2.2) that

$$\lim_{n\to\infty} \int_0^1 |u(f_n)(x,T) - g(x)|^2 \mathrm{d}x = \int_0^1 |u(\bar{f})(x,T) - g(x)|^2 \mathrm{d}x \tag{2.21}$$

So, by the aid of (2.20) and (2.21), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{J}(f_n) \quad &= \quad \lim_{n\to\infty} \frac{1}{2} \int_0^1 |u(f_n)(x,T) - g(x)|^2 \mathrm{d}x + \lim_{n\to\infty} \frac{\lambda}{2} \int_0^1 |f_n|^2 \mathrm{d}x \\
&\geq \quad \frac{1}{2} \int_0^1 \left|u(\bar{f})(x,T) - g(x)\right|^2 \mathrm{d}x + \frac{\lambda}{2} \int_0^1 |\bar{f}|^2 \mathrm{d}x
\end{aligned}$$

From this observation, we obtain

$$\min_{f\in\mathcal{A}} \mathcal{J}(f) \leq \mathcal{J}(\bar{f}) \leq \liminf_{n\to\infty} \mathcal{J}(f_n) \leq \min_{f\in\mathcal{A}} \mathcal{J}(f). \tag{2.22}$$

Hence

$$\mathcal{J}(\bar{f}) = \min_{f\in\mathcal{A}} \mathcal{J}(f).$$

Thus, $\bar{f}$ is a solution of the optimization problem (2.1). $\square$

3. Necessary condition

Next we turn to state a first order necessary optimality condition which have to be satisfied by each optimal control f.

Theorem 3.1. *Let f be the solution of the optimal control problem* (2.1)-(2.2). *Then for any $h \in \mathcal{A}$, there exists a triple of functions $(u, \psi; f)$ satisfying*

$$\int_0^1 [u(x,T;f) - g(x)]\psi(x,T)dx + \lambda \int_0^1 f(h-f)\mathrm{d}x \geq 0. \tag{3.1}$$

$$\begin{cases} \partial_t u - \partial_t \Delta u - \Delta u = f, \quad (x,t) \in Q_T, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \\ u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq T. \end{cases} \tag{3.2}$$

$$\begin{cases} \partial_t \psi - \partial_t \Delta \psi - \Delta \psi = h - f, \quad (x,t) \in Q_T, \\ \psi(x,0) = 0, \quad 0 \leq x \leq 1, \\ \psi(0,t) = \psi(1,t) = 0, \quad 0 \leq t \leq T. \end{cases} \tag{3.3}$$

Proof. For any $h \in \mathcal{A}$ and $0 \leq \delta \leq 1$, let us put

$$f_\delta \equiv (1-\delta)f + \delta h \in \mathcal{A}.$$

Then

$$\mathcal{J}_\delta \equiv \mathcal{J}(f_\delta) = \frac{1}{2}\int_0^1 |u(x,T;f_\delta) - g(x)|^2 \mathrm{d}x + \frac{\lambda}{2}\int_0^1 |f_\delta|^2 \mathrm{d}x. \tag{3.4}$$

Let u_δ be the solution to the problem (1.1) with given $f = f_\delta$. Since f is an optimal solution, we have

$$\left.\frac{\mathrm{d}\mathcal{J}_\delta}{\mathrm{d}\delta}\right|_{\delta=0} = \int_0^1 [u(x,T;f) - g(x)] \left.\frac{\partial u_\delta}{\partial \delta}\right|_{\delta=0} dx + \lambda \int_0^1 f(h-f)\mathrm{d}x \geq 0. \tag{3.5}$$

Let us take $\tilde{u}_\delta \equiv \frac{\partial u_\delta}{\partial \delta}$, then by direct calculations, we find the following equation:

$$\begin{cases} \partial_t \tilde{u}_\delta - \partial_t \Delta \tilde{u}_\delta - \Delta \tilde{u}_\delta = h - f, \\ \tilde{u}_\delta(x,0) = 0, \\ \tilde{u}_\delta(0,t) = \tilde{u}_\delta(1,t) = 0. \end{cases} \tag{3.6}$$

Let us set $\psi = \tilde{u}_\delta|_{\delta=0}$, then ψ satisfies

$$\begin{cases} \partial_t \psi - \partial_t \Delta \psi - \Delta \psi = h - f, \quad (x,t) \in Q, \\ \psi(x,0) = 0, \quad 0 \leq x \leq 1, \\ \psi(0,t) = \psi(1,t) = 0, \quad 0 \leq t \leq T, \end{cases} \tag{3.7}$$

Regarding to the above finding, the optimality condition becomes

$$\int_0^1 [u(x,T;f) - g(x)]\psi(x,T)dx + \lambda \int_0^1 f(h-f)dx \geq 0, \tag{3.8}$$

and this achieves the proof of Theorem (4.1). □

4. Uniqueness and stability

In this section, we aim to establish a stability result for the inverse problem of identifying the source term f in the state equation (1.1).

Theorem 4.1. *Let f_1 and f_2 be the minimizer of the optimal control problem (2.1)-(2.2) corresponding to the over-specified data g_1 and g_2, respectively. Then, the following estimate holds*

$$\|f_1 - f_2\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2\lambda}} \|g_1 - g_2\|_{L^2(\Omega)}. \tag{4.1}$$

Proof. Let u_1, u_2 and ψ_1, ψ_2 are solutions of the problem (1.1) and (3.3) with source terms f_1, f_2, respectively. By taking $h = f_2$ when $f = f_1$ and taking $h = f_1$ when $f = f_2$ in the necessary optimality condition (3.1), we have

$$-\lambda \int_0^1 f_1 \left(f_2 - f_1\right) \mathrm{d}x \leq \int_0^1 \left[u_1(x,T) - g_1(x)\right] \psi_1(x,T) \mathrm{d}x, \tag{4.2}$$

and

$$-\lambda \int_0^1 f_2 \left(f_1 - f_2\right) \mathrm{d}x \leq \int_0^1 \left[u_2(x,T) - g_2(x)\right] \psi_2(x,T) \mathrm{d}x. \tag{4.3}$$

Setting $U = u_1 - u_2$ and $\Psi = \psi_1 + \psi_2$, then by the superposition principle we deduce that U and Ψ satisfy the following system

$$\begin{cases} \partial_t U_t - \partial_t \Delta U - \Delta U = f_1 - f_2, & (x,t) \in Q_T, \\ U(x,0) = 0, & 0 \leq x \leq 1, \\ U(0,t) = U(1,t) = 0, & 0 \leq t \leq T, \end{cases} \tag{4.4}$$

$$\begin{cases} \partial_t \Psi - \partial_t \Delta \Psi - \Delta \Psi = 0, & (x,t) \in Q_T, \\ \Psi(x,0) = 0, & 0 \leq x \leq 1, \\ \Psi(0,t) = \Psi(1,t) = 0, & 0 \leq t \leq T, \end{cases} \tag{4.5}$$

It is obvious that the problem (4.5), according to the maximum principle, has only zero-solution, therefore

$$\psi_1(x,t) = -\psi_2(x,t), \quad (x,t) \in Q_T. \tag{4.6}$$

In the other hand by noting that ψ_1 is the solution to the following problem

$$\begin{cases} \partial_t \psi_1 - \partial_t \Delta \psi_1 - \Delta \psi_1 = f_2 - f_1, \\ \psi_1(x,0) = 0 \\ \psi_1(0,t) = \psi_1(1,t) = 0. \end{cases}$$

then according the uniqueness property of solutions of systems (3.3) and (4.4) it follows that

$$U(x,t) = -\psi_1(x,t), \quad (x,t) \in Q_T. \tag{4.7}$$

Adding the optimality conditions (4.2) and (4.3) and performing some manipulations on the term by the aid of (4.6) and (4.7) we derive

$$\begin{aligned}
\int_0^1 |f_1 - f_2|^2 \,\mathrm{d}x &\leq \frac{1}{\lambda}\int_0^1 [u_1(x,T) - g_1]\,\psi_1(x,T)\mathrm{d}x + \frac{1}{\lambda}\int_0^1 [u_2(x,T) - g_2]\,\psi_2(x,T)\mathrm{d}x \\
&\leq \frac{1}{\lambda}\int_0^1 U(x,T)\psi_1(x,T)\mathrm{d}x + \frac{1}{\lambda}\int_0^1 (g_2 - g_1)\,\psi_1(x,T)\mathrm{d}x \\
&\leq -\frac{1}{\lambda}\int_0^1 |\psi_1(x,T)|^2 \,\mathrm{d}x + \frac{1}{2\lambda}\int_0^1 |\psi_1(x,T)|^2 \,dx + \frac{1}{2\lambda}\int_0^1 |g_1 - g_2|^2 \,\mathrm{d}x \\
&\leq \frac{1}{2\lambda}\int_0^1 |g_1 - g_2|^2 \,\mathrm{d}x,
\end{aligned}$$

using this last inequality, we arrive at

$$\|f_1 - f_2\|^2_{L^2(\Omega)} \leq \frac{1}{\sqrt{2\lambda}}\|g_1 - g_2\|^2_{L^2(\Omega)}$$

This yields the desired result. □

5. Numerical simulation

After establishing the theoretical results in previous sections, we now aim to developing an efficient numerical algorithm for the numerical reconstruction of the unknown source f in the domain $\Omega = (0,1)$ from the over-specified condition (1.3), and also, provide some computational results.

5.1. **Iterative thresholding algorithm.** For computational purpose, we shall consider in this section a general problem of (1.1)-(1.2)-(3.1), namely, find the pair $\{u, f\}$ solution of the following system

$$\begin{cases}
\partial_t u - \partial_t \Delta u - \Delta u = F(x,t), \\
u(x,0) = u_0(x), \\
u(0,t) = u(1,t) = 0,
\end{cases} \tag{5.1}$$

Let $\mathcal{P}$ be the parameter to data mapping, that is

$$\begin{aligned}
\mathcal{P} :& L^2(\Omega) \to L^2(\Omega) \\
& \mathcal{P}f \mapsto g(x) = u(x,T),
\end{aligned}$$

The nonlinear operator P can be divided into two operators K and H such that

$$\begin{aligned}
\mathcal{P}f &= Kf + Hu_0, \\
&= u_1(x,T) + u_2(x,T),
\end{aligned}$$

where u_1 satisfies the following equation

$$\begin{cases}
\partial_t u_1 - \partial_t \Delta u_1 - \Delta u_1 = f(x), \\
u_1(x,0) = 0, \\
u_1(0,t) = u_1(1,t) = 0,
\end{cases} \tag{5.2}$$

and u_2 is the solution of the following equation

$$\begin{cases}
\partial_t u_2 - \partial_t \Delta u_2 - \Delta u_2 = F(x,t), \\
u_2(x,0) = u_0(x), \\
u_2(0,t) = u_2(1,t) = 0,
\end{cases} \tag{5.3}$$

We know that K is a self-adjoint linear compact operator. Therefore the inverse problem is transformed into

$$\mathcal{P}f = g - Hu_0$$

which can rewritten as

$$f = f - \beta K^* \left(Kf - (g - Hu_0) \right).$$

Then we have the Landweber iteration defined by

$$\begin{aligned} f_{m+1} &= f_m - \beta K^*(Kf_m - (g - Hu_0)) \\ &= f_m - \beta P^*(u_m(x,T) - g), \end{aligned} \tag{5.4}$$

where β is a regularization parameter and u_m is the solution of (1.1) with $f = f_m$. Now, we would like to give the specific forms of the operator K by stating the following lemma.

Lemma 5.1. *For any given $\phi \in L^2(\Omega)$, let $v = K^*\phi$. Then v satisfies the following system*

$$\begin{cases} \partial_t v + \partial_t \Delta v - \Delta v = \phi, \quad (x,t) \in Q_T, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = 0. \end{cases}$$

Proof. Let $\mathcal{L}$ be the differential operator defined as:

$$\mathcal{L}u := \partial_t u - \partial_t \Delta u - \Delta u,$$

and $\mathcal{L}^*$ denotes its adjoint, that is

$$\mathcal{L}^* w := -\partial_t w + \partial_t \Delta w - \Delta w,$$

Suppose that u is the solution of and w satisfies the following system

$$\begin{cases} \mathcal{L}^* w = -\partial_t w + \partial_t \Delta w - \Delta w = h(x), \quad (x,t) \in Q_T, \\ w(0,t) = w(1,t) = 0, \\ w(x,T) = 0. \end{cases} \tag{5.5}$$

then, it follows that

$$\iint_{Q_T} (w(x,t)f(x) - u(x,t)h(x))\, \mathrm{d}x\mathrm{d}t = \iint_{Q_T} (w\mathcal{L}u - u\mathcal{L}^*w)\, \mathrm{d}x\mathrm{d}t,$$

according to definition of operators $\mathcal{L}$ and $\mathcal{L}^*$, we obtain

$$\begin{aligned} \iint_{Q_T} (w\mathcal{L}u - u\mathcal{L}^*w)\, \mathrm{d}x\mathrm{d}t = \iint_{Q_T} (w\partial_t u + u\partial_t w)\, \mathrm{d}x\mathrm{d}t - \iint_{Q_T} (u\partial_t \Delta w + w\partial_t \Delta u)\, \mathrm{d}x\mathrm{d}t + \\ \iint_{Q_T} (u\Delta w - w\Delta u)\, \mathrm{d}x\mathrm{d}t := I_1 + I_2 + I_3, \end{aligned}$$

Next we evaluate the terms I_1, I_2 and I_3 as the following

$$\begin{aligned} I_1 &= \iint_{Q_T} (w\partial_t u + u\partial_t w)\, \mathrm{d}x\mathrm{d}t \\ &= \int_0^1 \left(\int_0^T \partial_t(uw)\mathrm{d}t \right) \mathrm{d}x \\ &= \int_0^1 (u(x,T)w(x,T) - u(x,0)w(x,0))\, \mathrm{d}x = 0, \end{aligned} \tag{5.6}$$

$$
\begin{aligned}
I_2 &= -\iint_{Q_T} (w\partial_t \Delta u + u\partial_t \Delta w)\,\mathrm{d}x\mathrm{d}t \\
&= \iint_{Q_T} (\nabla w \partial_t \nabla u + \nabla u \partial_t \nabla w)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^1 \left(\int_0^T \partial_t (\nabla u \cdot \nabla w)\mathrm{d}t \right) \mathrm{d}x \\
&= \int_0^1 (\nabla u(x,T)\cdot \nabla w(x,T) - \nabla u(x,0)\cdot \nabla w(x,0))\,\mathrm{d}x = 0,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
I_3 &= \iint_{Q_T} (u\Delta w - w\Delta u)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T [u\nabla w - w\nabla u]_{t=0}^{t=T}\,\mathrm{d}x = 0,
\end{aligned} \tag{5.8}
$$

In view of (5.6),(5.7) and (5.8) we have

$$
\int_0^T \int_0^1 u(x,t)h(x)\mathrm{d}x\mathrm{d}t = \int_0^T \int_0^1 w(x,t)f(x)\mathrm{d}x\mathrm{d}t, \tag{5.9}
$$

Making the following variable transformation $s = T - t, v(x,s) = w(x, T-s)$, it is easily to check that v satisfies

$$
\begin{cases}
\partial_s v - \partial_s \Delta v - \Delta v = h(x), \quad (x,s)\in Q_T, \\
v(0,s) = v(1,s) = 0, \\
v(x,0) = 0,
\end{cases} \tag{5.10}
$$

moreover, (5.9) becomes

$$
\int_0^T \int_0^1 u(x,s)h(x)\mathrm{d}x\mathrm{d}s = \int_0^T \int_0^1 v(x,s)f(x)\mathrm{d}x\mathrm{d}s,
$$

Noticing that the above identity holds for any $T > 0$, therefore

$$
\int_0^1 u(x,T)h(x)\mathrm{d}x\mathrm{d}t = \int_0^1 v(x,T)f(x)\mathrm{d}x\mathrm{d}t, \tag{5.11}
$$

From (5.11) we deduce that $\big(Kf, h\big)_{L^2(\Omega)} = \big(f, v\big)_{L^2(\Omega)}$, consequently by the definition of K^* it follows that $v = K^* h$. □

Based on the analysis above, we propose the state the following iterative algorithm for the numerical reconstruction of the unknown source term.

- **Step 1.** Choose $\varepsilon > 0$ a tolerance, $\beta > 0$ regularization parameter, set $k = 0$ and choose an initial value of iteration $f = f^k \in L^2(\Omega)$.
- **Step 2.** Solve the following initial-boundary value problem

$$
\begin{cases}
\partial_t u(x,t) - \partial_t \Delta u(x,t) - \Delta u(x,t) = f(x), \quad (x,t)\in Q_T, \\
u(0,t) = u(1,t) = 0, \quad 0 \le t \le T \\
u(x,0) = u_0(x), \quad x \in \overline{\Omega},
\end{cases} \tag{5.12}
$$

to obtain the solution u^k where $f = f^k$.

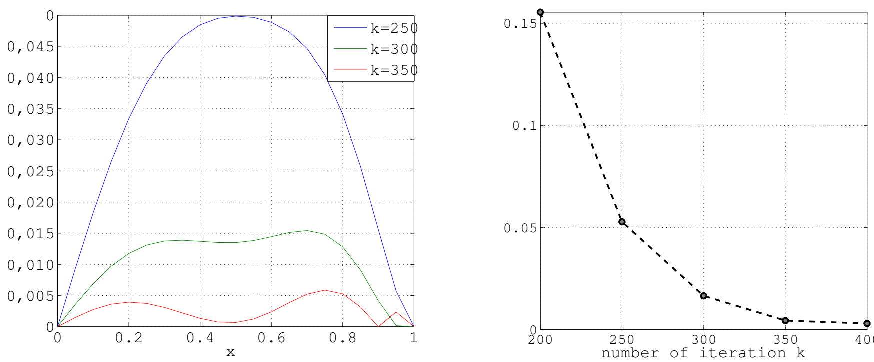


FIGURE 1. Absolute error $|f(x) - f^k(x)|$ of the numerical reconstruction for various iteration numbers k (in the left) and decay rate of the error $\|f - f^k\|_{L^2}$ versus the number of iterations k (in the right).

- **Step 3.** Solve the following initial-boundary value problem

$$\begin{cases} \partial_t v(x,t) - \partial_t \Delta v(x,t) - \Delta v(x,t) = u^k(x,T) - g(x), & (x,t) \in Q_T, \\ v(0,t) = v(1,t) = 0, \quad 0 \le t \le T, \\ v(x,0) = 0, \quad x \in \overline{\Omega}, \end{cases} \tag{5.13}$$

  to obtain the solution v^k.
- **Step 4.** Calculate the error $\|u^k(\cdot,T) - g\|$.
  (i) Set $k = 1$ and Let $f^1 = f^0 - \beta v^0$.
  (ii) If $\|u^1(\cdot,T) - g\| < \varepsilon$ then stop the iteration scheme and take $f = f_1$.
  (iii) Otherwise increase k by one and go to **Step 2**.

By the standard theory of the Landweber iteration (see [10]), we have the following convergence results.

Theorem 5.2. *Let $g \in L^2(\Omega)$ be the input data, and (u^k, f^k) be the k-th approximation in the above iterative procedure. If β satisfies $0 < \beta < \frac{1}{\|\mathcal{P}\|}$, then we have*

$$\lim_{k\to\infty} \|u^k(\cdot,T) - g\|_{L^2(\Omega)} = 0,$$

for every initial function $f^0 \in L^2(\Omega)$.

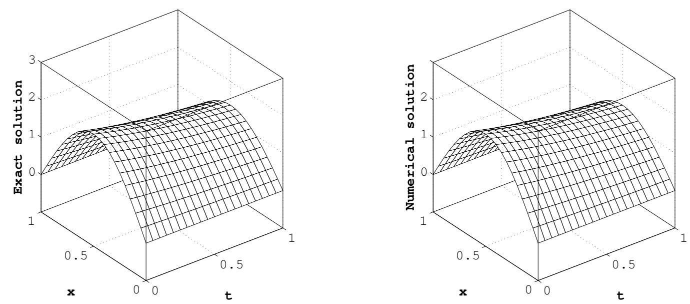
5.2. **Numerical experiments.** In this section we test the effectiveness of the proposed algorithm by performing several numerical experiments. The stopping criterion for the iteration is chosen as

$$\|u(\cdot,T) - g\| < 10^{-4}.$$

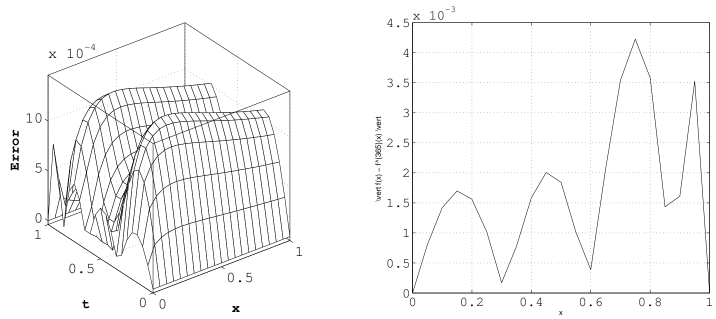
Discretization for problems (5.12) and (5.13) need to be made for numerical implementation. To achieve that, we apply fully-discrete schema, which is based on Legendre-Galerkin spectral method for spatial discretization [5, 6], while for the temporal discretization we employ Cranck-Niclson finite difference schema.

Example. *In our numerical tests, we consider the problem*

$$\begin{cases} \partial_t u(x,t) - \partial_t \Delta u(x,t) - \Delta u(x,t) = \pi^2 e^{-t}\sin(\pi x) + f(x), & (x,t) \in Q_T, \\ u(x,0) = 2\sin(\pi x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T. \end{cases} \tag{5.14}$$



(A) Exact solution u^*. (B) Identified solution u_f^{365}.



(C) $|u(x,t) - u_f^{365}(x,t)|$. (D) $|f(x) - f^{365}(x)|$.

FIGURE 2. Numerical results for the identification of $f(x)$ in (5.14).

For the source term $f(x) = \pi^2 \sin(\pi x)$*, the exact solution of the direct problem is given explicitly as*

$$u^*(x,t) = \sin(\pi x)(1 + e^{-t}), \quad (x,t) \in [0,1] \times [0,T].$$

We begin our test by considering the identification of the source term f for different iteration times (denoted by k) without noise. Figure (1) *(left) displays the profile of the absolute error* $e(f) = |f(x) - f^k(x)|$ *as a function of space variable x for different values of k. In the right-side of Figure* (1) *we plot the line of error rate* $\|f - f^k\|_{L^2(\Omega)}$ *as a function of the number of iteration k.*

Obviously, the main shape can be recovered with 350 *iterations with satisfactory accuracy. Indeed, after* 365 *iterations the identified solutions* (u_f^{365}, f^{365}) *matches* *well with the desired values, which could be illustrated by comparisoning the identified solutions* (u_f^{335}, f^{365}) *with the exact ones* (u^*, f)*. As it can seen from Figure* (2) *the identified solutions and the exact solutions are nearly indistinguishable.*

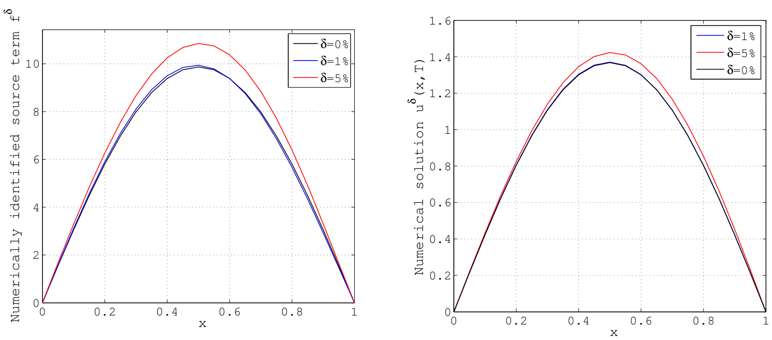


FIGURE 3. The numerical identified source term f^δ (in the left) and the corresponding state $u^\delta(x,T)$ (in the right) with different choices of the noise level ($\delta = 0\%, \delta = 1\%$ and $\delta = 5\%$).

Since the measured data g is usually contain some amount if noise, it is important to consider the numerical identification with noisy data. Let us apply a noisy data generated by adding a random perturbation, namely,

$$g^\delta(x) = g(x)[1 + \delta \times rand(x)], \tag{5.15}$$

where $\delta > 0$ *stands for a relative noise level.*

To observe the performance of our proposed algorithm, the reconstruction of $f(x)$ *from the noisy data* $g^\delta(x)$ *is performed with different choices of the noise level* $\delta = 0\%, \delta = 1\%$ *and* $\delta = 5\%$ *with the same settings from the previous experiment. Figure* (3) *clearly shows that the reconstruction results obtained are also satisfactory.*

6. Concluding remarks and discussion

In this paper, we investigate the inverse problem of identifying the source term in a third-order pseudoparabolic equation. We first established some results concerning the local uniqueness and stability by following a methodology based on optimal control framework. After stating the theoretical results, we have design an easy-to-implement numerical algorithm for the numerical recovering of the unknown parameter. Several experiments are performed to assess the viability and effectiveness of the proposed algorithm.

Based on some observation during the process of numerical computation, it is worth noting that the regularization parameter has a central role in reducing the implementation cost of the proposed algorithm. In fact, for the problem (5.14) we find that for $\beta = 280$ the numerical identified solutions are match well with the exact ones in only 3 iterations. However, for large β the the iterative procedure will diverge whatever the number of iterations.

Another point is connected with the numerical schema used to discretize the problem (5.12) and (5.13) in **Step 2.** and **Step 3.** respectively. Being different form other works, we used a spectral method for the discretization of the problems to be solved, that is in turn reduced the number of iterations for reaching an acceptable numerical results even for a moderate discretization parameters comparing to finite difference schemas, for which, the number of iterations can amount to big numbers such 1 000, 2 000, 5 000 and even 10 000. (see [9, 20]).

The aforementioned points could serve as an interesting course of research, in particularly, studying and adapting spectral methods for the numerical resolution of inverse identification problems, and also extending the proposed algorithm in this paper to solve problems more complicated such as multidimentional inverse problems, and simultaneous inversion problems. These topics are the focus of our future works.

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