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GENERALIZED METRIC SPACES AND LACUNARY STATISTICAL CONVERGENCE

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Abstract. In this study, we examine lacunary statistical convergence on $g-$metric spaces. $g-$metric spaces are metric spaces that have been generalized by defining the distance concept between $n+1$ points. In this sense, we denote the relationships between gS, gS_θ, gN_θ and gC_1 which we have found to be related to each other.

1. Introduction

As it is known, metric spaces are based on the concept of distance and distance function is important in mathematics and many other fields. Today, due to very large and complex data sets, the definition of the distance function needs to be generalized. For this purpose, many studies have already been carried out ([2], [5], [11], [12], [14], [17], [18]). During the sixties, Gahler introduced $2-$metric spaces with $d : X \times X \times X \to \mathbb{R}^+$ where X is a nonempty set and $\mathbb{R}$ is the set of real numbers ([13], [14]) . He claimed that a $2-$metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. Further, there is no easy relationship between results obtained in the two metrics. These considerations led Dhage to define a new generalized metric space called $D-$metric space with $D : X \times X \times X \to \mathbb{R}^+$. In a $D-$metric, two properties of a $2-$metric remained the same, while the other two were replaced by other properties. In addition, one more feature has been added. Afterwards, Dhage tried to develop topological properties in these spaces and these studies formed the basis for those who worked in this field for a long time ([5]). However, several errors for fundamental topological properties in a $D-$metric space were found ([24],[25]). After all these developments, Mustafa and Sims started to study on a more appropriate generalized metric space definition and they defined $G-$metric spaces ([24]). These properties are satisfied when $G(x, y, z)$ is the perimeter of a triangle with vertices at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that $(G5)$ is best possible. $G-$metric function is a distance function that generalizes the concept of distance between 3 points.

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Definition 1.1. ([24]) *Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be function that provides the following five properties. Then, G is called generalized metric or briefly $G-$metric on X. The pair (X, G) is called by a $G-$metric space.*

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$,

$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality)

Example 1.1. *Let $d(x, y, z)$ be the perimeter of the triangle with vertices at $x, y, z \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a $G-$metric space.*

Example 1.2. *Let $X = \{x, y\}$ and let $G(x, x, x) = G(y, y, y) = 0$, $G(x, x, y) = 1$, $G(x, y, y) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a $G-$metric which is not symmetric.*

Example 1.3. *Let (X, d) be a metric space. The function*

$$\psi(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

is a $G-$metric where $\psi : X \times X \times X \to \mathbb{R}^+$.

After Mustafa and Sim's studies, Choi et al. defined $g-$metric functions with degree n ([3]). This means that the distance function is defined between $n + 1$ points. The following definition gives the definition of $g-$metric space with degree l.

Definition 1.2. ([3]) *Let X be a nonempty set. A function $g : X^{l+1} \to \mathbb{R}^+$ that provides the following properties is called $g-$metric with order l on X. The pair (X, g) is called by a $g-$metric space.*

$(g1)$ $g(x_0, x_1, ..., x_l) = 0$ if and only if $x_0 = x_1 = ... = x_l$ for all $x_0, x_1, ..., x_l \in X$

(Positive definiteness)

$(g2)$ $g(x_0, x_1, ..., x_l) = g(x_{\sigma(0)}, x_{\sigma(1)}, ..., x_{\sigma(l)})$ for permutation σ on $\{0, 1, ..., l\}$

(permutation invariancy)

$(g3)$ $g(x_0, x_1, ..., x_l) \leq g(y_0, y_1, ..., y_l)$ for all $(x_0, x_1, ..., x_l), (y_0, y_1, ..., y_l) \in X^{l+1}$

with $\{x_i : i = 0, 1, ..., l\} \subseteq \{y_i : i = 0, 1, ..., l\}$ (monotonicity)

$(g4)$ $g(x_0, x_1, ..., x_s, y_0, y_1, ..., y_t) \leq g(x_0, x_1, ..., x_s, w, w, ..., w) + g(y_0, y_1, ..., y_t, w, w, ..., w)$

for all $x_0, x_1, ..., x_s, y_0, y_1, ..., y_t, w \in X$ with $s + t + 1 = l$ (triangle inequality)

Notice that when $l = 1$ we have ordinary metric space and when $l = 2$ we have $G-$metric space.

Theorem 1.1. ([3]) *Let X be a nonempty set and g be a metric with order l on X. In this case the followings are provided:*

$$i)\ g(\underbrace{x,x,...,x}_{s \text{ times}},\underbrace{y,y,...,y}_{t \text{ times}}) \leq g(\underbrace{x,x,...,x}_{s \text{ times}},w,w,...,w)+g(w,w,...,w,\underbrace{y,y,...,y}_{t \text{ times}}),$$

$$ii)\ g(x,y,...,y) \leq g(x,w,...,w)+g(w,y,...,y),$$

$$iii)\ g(\underbrace{x,x,...,x}_{s \text{ times}},w,w,...,w) \leq (l+1-s)g(w,x,...,x),$$

$$iv)\ g(x_0,x_1,...,x_l) \leq \sum_{i=0}^{n} g(x_i,w,...,w),$$

$$v)\ |g(y,x_1,...,x_l)-g(w,x_1,...,x_l)| \leq \max\{g(y,w,...,w),g(w,y,...,y)\},$$

$$vi)\ \left|g(\underbrace{x,x,...,x}_{s \text{ times}},w,w,...,w)-g(\underbrace{x,x,...,x}_{s' \text{ times}},w,w,...,w)\right| \leq |s-s'|\, g(x,w,w,...,w)$$

$$vii)\ g(x,w,w,...,w) \leq (1+(s-1)(l+1-s)g(\underbrace{x,x,...,x}_{s \text{ times}},w,w,...,w)$$

Statistical convergence, another main concept of our study, was first mentioned by Zygmund in his monograph in 1935 in Warsaw ([31]) and it was formally introduced by Fast ([7]) and Steinhaus ([29]), independently. Later on, Schoenberg gave some basic properties of statistical convergence and studied as a summability method ([27]). After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject. The most well-known of these areas are number theory by Erdös and Tenenbaum ([6]), measure theory by Miller ([23]), trigonometric series by Zygmund ([31]), summability theory by Freedman and Sember ([8]). Fridy has an important study in which he studied the properties of statistical convergence ([9]) and Maio studied statistical convergence in topological spaces ([22]). This concept was also studied with ideals, weak convergence, modulus functions, complex uncertain sequences, $p-$Cesáro convergence with an α number and arithmetic means ([4], [15], [16], [19], [20], [26], [28], [30]). Statistical convergence is based on the definition of natural density of the set $A \subseteq \mathbb{N}$ such as $d(A) = \lim_{n\to\infty} \frac{A_n}{n}$ where $A_n = \{k \in A : k \leq n\}$ and $|A_n|$ gives the cardinality of A_n.

Definition 1.3. ([7]) *A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $d(\{k \leq n : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $st-\lim x_k = L$ and usually the set of statistically convergent sequences is denoted by S.*

Considering the definition of natural density, this definition can also be expressed as for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |k \leq n : |x_k - L| \geq \varepsilon| = 0.$$

Lacunary statistical convergence was defined by Fridy and Orhan in 1993 ([10]). Before giving this definition, let's remind the definition of a lacunary sequence.

Definition 1.4. *A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals $I_r = (k_{r-1}, k_r]$ are determined by θ and the ratio is determined $q_r = \frac{k_r}{k_{r-1}}$.*

Example 1.4. *$\theta = (r^2)$ is a lacunary sequence because $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.*

Example 1.5. *$\theta = (r)$ is not a lacunary sequence because $k_0 = 0$ but $h_r = k_r - k_{r-1} = 1$ for all $r = 0, 1, \ldots$*

Definition 1.5. *([10]) Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is lacunary statistically convergent (or $S_\theta-$convergent) to L if for every $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |k \in I_r : |x_k - L| \geq \varepsilon| = 0.$$

In this case we write $S_\theta - \lim x_k = L$ and usually the set of lacunary statistically convergent sequences is denoted by S_θ.

Another concept closely related to statistical convergence is strong Cesáro summability:

$$|C_1| := \left\{ x : \text{for some } L,\ \lim_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k - L| \right) = 0 \right\}.$$

Similarly, there is a close relationship between strong Cesáro summability and N_θ space:

$$N_\theta := \left\{ x : \text{for some } L,\ \lim_r \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \right) = 0 \right\}.$$

2. Statistical Convergence in $g-$metric spaces

After all this work, Abazari defined statistical convergence in $g-$metric spaces and studied some basic properties ([1]) and Küçük and Gümüş studied about the meaning of lacunary statistical convergence on $G-$metric spaces ([21]).

Definition 2.1. ([1]) *Let $m \in \mathbb{N}$, $A \in \mathbb{N}^m$ and $A(n) = \{i_1, i_2, \ldots, i_m \leq n : (i_1, i_2, \ldots, i_m) \in A\}$ then,*

$$\rho_1(A) := \lim_{n\to\infty} \frac{m!}{n^m} |A(n)|$$

is called $m-$dimensional asymptotic density of the set A.

Definition 2.2. *([1]) Let (x_i) be a sequence in a $g-$metric space (X, g). For every $\varepsilon > 0$, if*

$$\lim_{n\to\infty} \frac{m!}{n^l} |\{(i_1, i_2, \ldots, i_m) \in A : i_1, i_2, \ldots, i_m \leq n,\ g(x, x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \geq \varepsilon\}| = 0$$

then, (x_i) statistically converges to x. This situation is denoted by $gs - \lim x_i = x$ or $x_i \overset{gs}{\to} x$. The set of all statistically convergent sequences in a $g-$metric space is denoted by gS.

The following theorems are some theorems that Abazari proved in his study.

Theorem 2.1. *([1]) In $g-$metric spaces, every convergent sequence is statistically convergent.*

The inverse of this theorem not usually hold.

Example 2.1. *Let* $X = \mathbb{R}$ *and* $g : \mathbb{R}^3 \to \mathbb{R}^+$, $g(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$, *be a* $g-$*metric. Define the set*

$$x_k = \begin{cases} k, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}$$

then, (x_k) *is statistically convergent but it is not convergent.*

Theorem 2.2. *([1]) Statistical limit in a* $g-$*metric space is unique.*

Theorem 2.3. *([1]) In* $g-$*metric spaces, every statistically convergent sequence has a convergent subsequence.*

3. Main Results

In this section the main definitions and results are introduced and discussed. First of all, we consider the definition of lacunary statistical convergence on $g-$metric spaces.

Definition 3.1. *Let* (X, g) *be a* $g-$*metric space,* (x_i) *be a sequence in this space and* θ *be a lacunary sequence. The sequence* (x_i) *is said to be lacunary statistically convergent to* x *provided that for all* $\varepsilon > 0$,

$$\lim_r \frac{m!}{h_r^m} \left|\{i_1, i_2, ..., i_m \in I_r : g(x, x_{i_1}, x_{i_2}, ..., x_{i_m}) \geq \varepsilon\}\right| = 0.$$

This situation is denoted by $gS_\theta - \lim x_i = x$ *or* $x_i \overset{gS_\theta}{\to} x$. *The set of all lacunary statistically convergent sequences is denoted by* gS_θ.

Definition 3.2. *Let* (X, g) *be a* $g-$*metric space and* (x_i) *be a sequence in this space. The sequence* (x_i) *is said to be* gC_1-*statistically summable to* x *provided that*

$$\lim_n \frac{m!}{n^l} \sum_{i_1, i_2, ..., i_m = 1}^{n} g(x, x_{i_1}, x_{i_2}, ..., x_{i_m}) = 0.$$

This situation is denoted by $gC_1 - \lim x_i = x$ *or* $x_i \overset{gC_1}{\to} x$. *The set of all* gC_1-*statistically summable sequences is denoted by* gC_1.

Definition 3.3. *Let* (X, g) *be a* $g-$*metric space,* (x_i) *be a sequence in this space and* θ *be a lacunary sequence. The sequence* (x_i) *is said to be* $gN_\theta-$*statistically summable to* x *provided that*

$$\lim_r \frac{m!}{h_r^m} \sum_{i_1, i_2, ..., i_m \in I_r} g(x, x_{i_1}, x_{i_2}, ..., x_{i_m}) = 0.$$

This situation is denoted by $gN_\theta - \lim x_i = x$ *or* $x_i \overset{gN_\theta}{\to} x$. *The set of all* $gN_\theta-$*statistically summable sequences is denoted by* gN_θ.

The following theorem gives us the relationship between gC_1 and gS

Theorem 3.1. *Let* (X, g) *be a* $g-$*metric space and* (x_i) *be a sequence in this space.*

i) $x_i \overset{gC_1}{\to} x$ implies $x_i \overset{gS}{\to} x$.

ii) If g is a bounded function, $x_i \overset{gS}{\to} x$ implies $x_i \overset{gC_1}{\to} x$.

Proof. $i)$ Suppose that $x_i \overset{gC_1}{\to} x$ and $\varepsilon > 0$ be given. Then,

$$\begin{aligned}\frac{m!}{n^m}\sum_{i_1,i_2,...,i_m=1}^{n} g(x,x_{i_1},x_{i_2},...,x_{i_m}) &\geq \frac{m!}{n^m}\sum_{\substack{i_1,i_2,...,i_m=1\\ g(x,x_{i_1},x_{i_2},...,x_{i_l})\geq\varepsilon}}^{n} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &\geq \varepsilon\frac{m!}{n^m}\left|\{i_1,i_2,...,i_m \leq n : g(x,x_{i_1},x_{i_2},...,x_{i_m}) \geq \varepsilon\}\right|\end{aligned}$$

and considering the limits of both side we have $x_i \overset{gS}{\to} x$ is obtained.

$ii)$ Now, suppose that $x_i \overset{gS}{\to} x$ and $\varepsilon > 0$ be given. From the boundedness of g there is a positive B such that $g(x,x_{i_1},x_{i_2},...,x_{i_m}) \leq B$ for all $x,x_{i_1},x_{i_2},...,x_{i_m} \in X$. Then,

$$\begin{aligned}\frac{m!}{n^m}\sum_{i_1,i_2,...,i_m=1}^{n} g(x,x_{i_1},x_{i_2},...,x_{i_m}) &= \frac{m!}{n^m}\sum_{\substack{i_1,i_2,...,i_m=1\\ g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon}}^{n} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &+ \frac{m!}{n^m}\sum_{\substack{i_1,i_2,...,i_m=1\\ g(x,x_{i_1},x_{i_2},...,x_{i_m})<\varepsilon}}^{n} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &\leq B\frac{m!}{n^m}\left|\{i_1,i_2,...,i_m \leq n : g(x,x_{i_1},x_{i_2},...,x_{i_m}) \geq \varepsilon\}\right| + \varepsilon.\end{aligned}$$

Considering the limits of both side we have $x_i \overset{gC_1}{\to} x$. □

It is possible to prove a similar theorem for gS_θ and gN_θ.

Theorem 3.2. *Let (X,g) be a $g-$metric space, (x_i) be a sequence in this space and θ be a lacunary sequence.*

$i)$ $x_i \overset{gN_\theta}{\to} x$ implies $x_i \overset{gS_\theta}{\to} x$.
$ii)$ If g is a bounded function, $x_i \overset{gS_\theta}{\to} x$ implies $x_i \overset{gN_\theta}{\to} x$.

Proof. $i)$ If $\varepsilon > 0$ and $x_i \overset{gN_\theta}{\to} x$ we have,

$$\lim_r \frac{m!}{h_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) = 0.$$

With this information we can write that

$$\begin{aligned}\frac{m!}{h_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) &\geq \frac{m!}{h_r^m}\sum_{\substack{i_1,i_2,...,i_m\in I_r\\ g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon}} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &\geq \varepsilon\frac{m!}{h_r^m}\left|\{i_1,i_2,...,i_m \in I_r : g(x,x_{i_1},x_{i_2},...,x_{i_m}) \geq \varepsilon\}\right|\end{aligned}$$

which yields the result.

$ii)$ Let $\varepsilon > 0$, g be a bounded function and $x_i \overset{gS_\theta}{\to} x$. From the boundedness of g there is a positive B such that $g(x,x_{i_1},x_{i_2},...,x_{i_m}) \leq B$ for all $x,x_{i_1},x_{i_2},...,x_{i_m} \in$

X. Hence,

$$\begin{aligned}\frac{m!}{h_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) &= \frac{m!}{h_r^m}\sum_{\substack{i_1,i_2,...,i_m\in I_r\\ g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon}} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &+ \frac{m!}{h_r^m}\sum_{\substack{i_1,i_2,...,i_m\in I_r\\ g(x,x_{i_1},x_{i_2},...,x_{i_m})<\varepsilon}} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &\leq B\frac{m!}{h_r^m}\left|\{i_1,i_2,...,i_m\in I_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right| + \varepsilon\end{aligned}$$

Considering that $x_i \overset{gS_\theta}{\to} x$, we have the result. □

In the following theorem, we explain the relationship between gS and gS_θ.

Theorem 3.3. *For any lacunary sequence θ in (X,g) with $\liminf_r q_r > 1$, $gS - \lim x_i = x$ implies $gS_\theta - \lim x_i = x$.*

Proof. Assume that $\liminf_r q_r > 1$. Then, there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r and therefore $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

For every $\varepsilon > 0$, we know from the assumption,

$$\lim_r \frac{m!}{k_r^l}\left|\{i_1,i_2,...,i_m \leq k_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right| = 0.$$

Therefore,

$$\begin{aligned}\frac{m!}{k_r^m}\left|\{i_1,i_2,...,i_m\leq k_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right| &\geq \frac{m!}{k_r^m}\left|\{i_1,i_2,...,i_m\in I_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right| \\ &= \left(\frac{h_r}{k_r}\right)^m \frac{m!}{h_r^m}\left|\{i_1,i_2,...,i_m\in I_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right| \\ &\geq \left(\frac{\delta}{1+\delta}\right)^m \frac{m!}{h_r^m}\left|\{i_1,i_2,...,i_m\in I_r : g(x,x_{i_1},x_{i_2},...,x_{i_m})\geq\varepsilon\}\right|\end{aligned}$$

Considering that $gS - \lim x_i = x$ then, we have $gS_\theta - \lim x_i = x$. □

Finally, in the last theorem, we explain the relationship between gC_1 and gN_θ.

Theorem 3.4. *For any lacunary sequence θ in (X,g) with $\liminf_r q_r > 1$, $gC_1 - \lim x_i = x$ implies $gN_\theta - \lim x_i = x$.*

Proof. Assume that $\liminf_r q_r > 1$. Then, there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$ we have $\frac{k_r}{k_{r-1}} \geq 1+\delta$ for sufficiently large r which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

$$\begin{aligned}\frac{m!}{k_r^m}\sum_{i_1,i_2,...,i_m=1}^{k_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) &\geq \frac{m!}{k_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &= \left(\frac{h_r}{k_r}\right)^m \frac{m!}{h_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m}) \\ &\geq \left(\frac{\delta}{1+\delta}\right)^m \frac{m!}{h_r^m}\sum_{i_1,i_2,...,i_m\in I_r} g(x,x_{i_1},x_{i_2},...,x_{i_m})\end{aligned}$$

If we take the limit of both sides we have the proof. □

4. Conclusions and Future Developments

It is very important to define lacunary statistical convergence, which is one of the most basic concepts of functional analysis, in g-metric spaces and compare the relevant results. g-metric spaces define the distance between n+1 points. These results can be informative for researchers who want to work in this field.

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