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FIXED AND BEST PROXIMITY POINTS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we extend the fixed point (respectively, best proximity point) theorems for Reich type and Hardy-Rogers type contraction mappings in partial metric space and validate our results with non-trivial examples. In addition, we apply our results to Fredholm integral equation.

1. Introduction

In 1994, Matthews [15] introduced the notion of partial metric as a generalization of metric and extended the Banach contraction principle [6] in partial metric space. Afterward, partial metrics became of great interest to many researchers (for more details, one can check in [1], [4], [8], [9], [22], [24], etc., and references therein).

Ran and Reurings [20] gave an analogue of Banach fixed point theorem in partially ordered sets and applications to linear and nonlinear matrix equations were discussed. Nieto and López ([16], [17]) extended the results of Ran and Reurings[20] by weakening the continuity condition and applied to solve first-order ordinary differential equations with periodic boundary conditions. Recently there has been a trend of discussing metric spaces equipped with partial order([5], [10], [16], [17], [18], [20], [26], [27], etc.). Aydi et al. [5] gave some fixed point results using an ICS mapping and involving Boyd-Wong type contractions in partially ordered metric spaces. Choudhury et al. [10] established some coincidence point results for generalized weak contractions with discontinuous control functions in metric spaces with a partial order. Shatanawi and Postolache [26] obtained common fixed point results for mappings satisfying nonlinear contractive conditions of a cyclic form based on the notion of an altering distance function in ordered metric space.

On the other hand, the study of the best proximity points in the context of fixed point theory is also interesting and some works on the best proximity point problem can be found in [7], [19], [23], [28], etc.

The main purpose of our work is to obtain fixed point and the best proximity point theorems for Reich type and Hardy-Rogers type contraction mappings in the

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setting of partial metric space. Our works also extend some results of Altun et al. [2] and other similar results in the existing literature.

2. Preliminaries

In this section, we recall the following definitions and results which are directly or indirectly related to our work. We denote $\mathbb{R}^+$ the set of positive real numbers and $\mathbb{N}$ the set of natural numbers.

Let (X,d) be a metric space and let $T : X \to X$ be a self mapping. In 1971, Reich [21] generalized Banach's [6] and Kannan's ([12], [13]) theorems by using the following new type of contraction mapping:

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty), \ \forall \ x,y \in X$$

where a,b,c are nonnegative and $a+b+c<1$.

Example 2.1. *Let us consider $X=[0,1]$ with the usual metric d, where $d(x,y) = |x-y|$. Define a mapping T on X as $Tx = \frac{3x}{10}$ for all $x \in X$. Then, we have $d(x,Tx) = \frac{7x}{10}$, $d(y,Ty) = \frac{7y}{10}$, $d(Tx,Ty) = \frac{3}{10}|x-y|$. For all $x,y \in X$, we have*

$$\begin{aligned} d(Tx,Ty) =& \frac{3}{10}|x-y| \\ \leq& \frac{3}{10}\Big(|x|+|y|\Big) = \frac{3}{7}\Big(\frac{7}{10}|x| + \frac{7}{10}|y|\Big) \\ \leq& \frac{3}{7}|x - \frac{3x}{10}| + \frac{43}{100}|y - \frac{3y}{10}|. \end{aligned}$$

Setting $b=\frac{3}{7}, c=\frac{43}{100}$ and $0 \leq a < \frac{99}{700}$, then $a+b+c<1$. Thus, we have

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty), \text{ for all } x,y \in X$$

and hence T is a Reich type contraction mapping.

Example 2.2. *Let (X,d) be a metric space where $X=[0,1]$ and $d(x,y) = |x-y|$. Define T on X as*

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0,1) \\ \frac{1}{3}, & x = 1 \end{cases}$$

Two cases arise:
Case (i): When $x,y \in [0,1)$ or $x,y = 1$

$$d(Tx,Ty) = 0 \leq ad(x,y) + bd(x,Tx) + cd(y,Ty)$$

for nonnegative a,b,c such that $a+b+c<1$.
Case (ii): When $x \in [0,1)$ and $y=1$
$d(Tx,Ty) = \frac{1}{6}$ and for $a=\frac{1}{5}, b=\frac{1}{4}, c=\frac{1}{2}$ with $a+b+c<1$

$$\begin{aligned} ad(x,y) + bd(x,Tx) + cd(y,Ty) &= a|x-1| + b\Big|x - \frac{1}{2}\Big| + c\Big|1 - \frac{1}{3}\Big| \\ &\geq d(Tx,Ty). \end{aligned}$$

From the above two cases, we conclude that T is Reich type contraction mapping but T is not continuous at $x=1$.

In 1973, Hardy and Rogers [11] gave a generalization of fixed point theorem of Reich [21] using the following contraction mapping:

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty) + ed(x,Ty) + fd(y,Tx),$$

$\forall\ x, y \in X$, where a, b, c, f, e are nonnegative and $a + b + c + e + f < 1$.

Further, in 2010 Altun et al. [2] proved the following Reich's [21] and Hardy-Rogers's [11] theorems in partial metric space.

Theorem 2.3. [2] *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping such that*

$$p(Tx,Ty) \leq ap(x,y) + bp(x,Tx) + cp(y,Ty),$$

$\forall\ x, y \in X$, where $a, b, c \geq 0$ and $a + b + c < 1$. Then T has a unique fixed point.

Theorem 2.4. [2] *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping such that*

$$p(Tx,Ty) \leq ap(x,y) + bp(x,Tx) + cp(y,Ty) + dp(x,Ty) + ep(y,Tx),$$

$\forall\ x, y \in X$, where $a, b, c, d, e \geq 0$ and if $d \geq e$, then $a + b + c + d + e < 1$, if $d < e$, then $a + b + c + d + 2e < 1$. Then T has a unique fixed point.

Definition 2.5. [15] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that $\forall\ x, y, z \in X$:*
(p_1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$*(equality);*
(p_2) $p(x,x) \leq p(x,y)$*(small self-distance);*
(p_3) $p(x,y) = p(y,x)(symmetry)$;
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)(triangularity)$.

The pair (X, p) is called a partial metric space. For partial metric, self distance need not be zero and if self distance is zero, a partial metric reduces to a metric. But $p(x, y) = 0$ implies $x = y$ by (p_1) and (p_2).

Example 2.6. [15] *Let $X = \mathbb{R}^+$ and define $p : X \times X \to \mathbb{R}^+$ as $p(x,y) = \max\{x,y\}$. Then p is a partial metric.*

Example 2.7. [15] *Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then p is a partial metric.*

Example 2.8. [14] *Let d be a metric and p a partial metric on a nonempty set X respectively. Define $p_i : X \times X \to \mathbb{R}^+, i \in \{1,2,3\}$ as*

$$p_1(x,y) = d(x,y) + p(x,y);$$
$$p_2(x,y) = d(x,y) + \max\{\omega(x), \omega(y)\};$$
$$p_3(x,y) = d(x,y) + a;$$

where $\omega : X \to \mathbb{R}^+$ is an arbitrary function and $a \geq 0$. Then $p_i, i \in \{1,2,3\}$ is a partial metric on X.

Example 2.9. *Let (X, p) be a partial metric space on a nonempty set X and k be a non-zero positive real number. Then $p_k : X \times X \to \mathbb{R}^+$ defined as $p_k(x,y) = kp(x,y)$ is also a partial metric on X.*

Example 2.10. *Let $X = \mathbb{R}$ and define $p : X \times X \to \mathbb{R}^+$ as*

$$p(x, y) = |x - y| + |x| + |y|.$$

We show that p is a partial metric. Clearly, (p_1) and (p_3) of Definition 2.5 hold. To show (p_2) and (p_4), we have

$$p(x, x) = |x| + |x| = |x - y + y| + |x| \leq |x - y| + |y| + |x| = p(x, y).$$

This shows that (p_2) is satisfied. Also we have

$$\begin{aligned} p(x, y) &= |x - y| + |x| + |y| \\ &= |x - z + z - y| + |x| + |y| \\ &\leq |x - z| + |z - y| + |x| + |y| \\ &= |x - z| + |x| + |z| + |z - y| + |z| + |y| - 2|z| \\ &= p(x, z) + p(z, y) - p(z, z). \end{aligned}$$

Thus (p_4) is satisfied.

Note that every partial metric p on a non-empty set X generates a topology $\tau(p)$ on X, whose base is a family of open balls, $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ $\forall$ $x \in X$ (for more details, we refer to [9] and [15]).

Definition 2.11. [15] *(i) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to some $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.*
(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence in X converges with respect to $\tau(p)$ to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(iv) Let (X, p) be a partial metric space and $T : X \to X$ be a mapping on X. T is said to be continuous at $x \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$T(B_p(x, \delta)) \subseteq B_p(Tx, \varepsilon).$$

Let (X, p) be a partial metric space. Define $d_p : X \times X \to \mathbb{R}^+$ as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \ \forall \ x, y \in X.$$

Then (X, d_p) is a metric space. This shows that every partial metric on a nonempty set induces a metric [15].

Proposition 2.12. [15] *Let (X, p) be a partial metric space and (X, d_p) be the corresponding induced metric space. Also let $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges in (X, d_p) with respect to $\tau(d_p) \Longrightarrow \{x_n\}$ converges with respect to $\tau(p)$.
(ii) $\{x_n\}$ is Cauchy in (X, p) if and only if $\{x_n\}$ is Cauchy in (X, d_p).
(iii) (X, p) is complete if and only if (X, d_p) is complete.
Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \iff p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 2.13. [3] *A sequence* $\{x_n\}$ *in a partial metric space* (X,p) *converges to* $c \in X$ *if for each* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$p(x_n, c) - p(c,c) < \varepsilon \text{ and}$$
$$p(x_n, c) - p(x_n, x_n) < \varepsilon \ \forall\ n \geq N.$$

Let A and B be two nonempty subsets of a partial metric space (X,p). Then, the distance between A and B is given as

$$p(A,B) = \inf\{p(a,b) : a \in A \text{ and } b \in B\}.$$

Definition 2.14. *(*[7], [28]*) For a mapping* $T : A \to B$ *an element* $x \in A$ *is called a best proximity point of* T *if* $p(x,Tx) = p(A,B)$.

Example 2.15. *Let* $X = \mathbb{R}^+$ *be a partial metric space with partial metric* $p(x,y) = \max\{x,y\},\ \forall\ x,y \in X$ *and define* $T : A = [1,2] \to B = [0,1]$ *by* $Tx = \frac{x}{3}$. *Then* 1 *is a best proximity point of* T *as* $p(1,T1) = p(1, \frac{1}{3}) = p(A,B)$.

Lemma 2.16. [28] *Let* (X,p) *be a partial metric space with partial metric* p. *If* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *such that* $x_n \to x \in X$ *and* $y_n \to y \in X$, *then* $p(x_n,y_n) \to (x,y)$ *as* $n \to \infty$.

3. Fixed point theorems in partially ordered partial metric space

In this section, inspired by Ran and Reurings [20] (respectively, Nieto and López [16, 17]), we extend Theorem 2.3 and Theorem 2.4 of Altun et al. [2] in partially ordered partial metric space.

Let $(X,\preceq)$ be a partially ordered set and p be a partial metric on X. Then $(X,\preceq,p)$ is called a partially ordered partial metric space. Two elements x and y in X are said to be comparable if either $x \preceq y$ or $y \preceq x$. A mapping $T : X \to X$ is said to monotone nondecreasing if $x \preceq y$ implies $Tx \preceq Ty$, $x,y \in X$.

Theorem 3.1. *Let* $(X,\preceq,p)$ *be a complete partially ordered partial metric space. Let* T *be a continuous and monotone nondecreasing self mapping on* X *such that*

$$p(Tx,Ty) \leq ap(x,y) + bp(x,Tx) + cp(y,Ty) + dp(x,Ty) + ep(y,Tx), \tag{3.1}$$

$\forall\, x,y \in X$ *with* $x \preceq y$, *where* $a,b,c,d,e \geq 0$ *and if* $d \geq e$, *then* $a+b+c+d+e < 1$, *if* $d < e$, *then* $a+b+c+d+2e < 1$. *Further, if there exists* $x_0 \in X$ *with* $x_0 \preceq Tx_0$, *then* T *has a fixed point.*

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. Since T is monotone nondecreasing, then we have

$$Tx_0 \preceq T^2x_0$$
$$T^2x_0 \preceq T^3x_0$$
$$\dots$$
$$T^nx_0 \preceq T^{n+1}x_0.$$

Inductively, we obtain

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \dots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \dots$$

Thus, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \geq 0$. If $x_{n_0+1} = x_{n_0}$ i.e., $Tx_{n_0} = x_{n_0}$, for some $n_0 \geq 0$, then x_{n_0} is a fixed point of T. Assume that $x_{n+1} \neq x_n$, for all $n \geq 0$. Since $x_n \preceq x_{n+1}$, for all $n \geq 0$, then using (3.1), we have

$$\begin{aligned} p(x_{n+1}, x_n) =& p(Tx_n, Tx_{n-1}) \\ \leq& ap(x_n, x_{n-1}) + bp(x_n, Tx_n) + cp(x_{n-1}, Tx_{n-1}) \\ &+ dp(x_n, Tx_{n-1}) + ep(x_{n-1}, Tx_n) \\ =& ap(x_n, x_{n-1}) + bp(x_n, x_{n+1}) + cp(x_{n-1}, x_n) \\ &+ dp(x_n, x_n) + ep(x_{n-1}, x_{n+1}) \\ \leq& (a+c)p(x_n, x_{n-1}) + bp(x_n, x_{n+1}) \\ &+ dp(x_n, x_n) + e[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)] \\ =& (a+c+e)p(x_n, x_{n-1}) + (b+e)p(x_n, x_{n+1}) \\ &+ (d-e)p(x_n, x_n). \end{aligned} \tag{3.2}$$

Since, by (p_2), we have

$$p(x_n, x_n) \leq p(x_n, x_{n-1}) \text{ and } p(x_n, x_n) \leq p(x_n, x_{n+1}). \tag{3.3}$$

If $d \geq e$, then using (3.2), and (3.3), we have

$$p(x_{n+1}, x_n) \leq \max\left\{\frac{a+c+d}{1-b-e}, \frac{a+c+e}{1-b-d}\right\} p(x_n, x_{n-1}) \tag{3.4}$$

for all $n \geq 1$. Similarly, for all $n \geq 1$ if $d < e$, then by omitting the $-ve$ term from (3.2) and using (3.3), we have

$$p(x_{n+1}, x_n) \leq \max\left\{\frac{a+c+d+e}{1-b-e}, \frac{a+c+e}{1-b-d-e}\right\} p(x_n, x_{n-1}). \tag{3.5}$$

Setting

$$k = \begin{cases} \max\left\{\frac{a+c+d}{1-b-e}, \frac{a+c+e}{1-b-d}\right\}, & \text{if } d \geq e; \\ \max\left\{\frac{a+c+d+e}{1-b-e}, \frac{a+c+e}{1-b-d-e}\right\}, & \text{if } d < e. \end{cases}$$

Clearly, $k \in [0, 1)$ and hence from (3.4) and (3.5), we obtain

$$p(x_{n+1}, x_n) \leq kp(x_n, x_{n-1}) \leq k^n p(x_1, x_0). \tag{3.6}$$

Also, we have

$$\begin{aligned} d_p(x_{n+1}, x_n) =& 2p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) - p(x_n, x_n) \\ \leq& 2p(x_{n+1}, x_n) \\ \leq& 2k^n p(x_1, x_0). \end{aligned}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, d_p). Let $m, n \in \mathbb{N}, m > n$.

$$\begin{aligned} d_p(x_m, x_n) \leq & d_p(x_m, x_{m-1}) + d_p(x_{m-1}, x_n) \\ \leq & d_p(x_m, x_{m-1}) + d_p(x_{m-1}, x_{m-2}) + d_p(x_{m-2}, x_n) \\ \leq & d_p(x_m, x_{m-1}) + d_p(x_{m-1}, x_{m-2}) + d_p(x_{m-2}, x_{m-3}) \\ & + ... + d_p(x_{n+1}, x_n) \\ \leq & 2(k^{m-1} + k^{m-2} + k^{m-3} + ... + k^n)p(x_1, x_0) \\ \leq & 2\frac{k^n(1 - k^{m-n})}{1 - k}p(x_1, x_0) \\ \leq & 2\frac{k^n}{1 - k}p(x_1, x_0). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p), so we have $\lim_{m,n\to\infty} d_p(x_m, x_n) = 0$. Hence the sequence $\{x_n\}$ is also a Cauchy sequence in (X, p) i.e., $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite.

Since (X, p) is complete, so the sequence $\{x_n\}$ converges in the metric space (X, d_p) i.e., $\lim_{n\to\infty} d_p(x_n, \zeta) = 0$ and hence $p(\zeta, \zeta) = \lim_{n\to\infty} p(x_n, \zeta)$.

Also, from Proposition 2.12 (iii), we have

$$\lim_{n\to\infty} d_p(x_n, \zeta) = 0 \iff p(\zeta, \zeta) = \lim_{n\to\infty} p(x_n, \zeta) = \lim_{m,n\to\infty} p(x_m, x_n). \tag{3.7}$$

From (3.3) and (3.6) and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} p(x_n, x_n) = 0 = \lim_{n\to\infty} p(x_{n+1}, x_n). \tag{3.8}$$

On the other hand, we have

$$d_p(x_m, x_n) = 2p(x_m, x_n) - p(x_m, x_m) - p(x_n, x_n).$$

Letting $m, n \to \infty$ in the above and using (3.8), we obtain

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0.$$

Thus, from (3.7), we obtain

$$p(\zeta, \zeta) = \lim_{n\to\infty} p(x_n, \zeta) = \lim_{m,n\to\infty} p(x_m, x_n) = 0.$$

Finally, we show ζ is a fixed point of T. By continuity of T at ζ, given $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(\zeta, \delta)) \subseteq B_p(T\zeta, \varepsilon)$. Since $p(\zeta, \zeta) = 0 = \lim_{n\to\infty} p(\zeta, x_n)$, there exists $K \in \mathbb{N}$ such that

$$p(x_n, \zeta) < p(\zeta, \zeta) + \delta, \ \forall\ n \geq K.$$

This implies $x_n \in B_p(\zeta, \delta), \ \forall\ n \geq K$. Thus, we obtain

$$\begin{aligned} & Tx_n \in T(B_p(\zeta, \delta)) \subseteq B_p(T\zeta, \varepsilon) \\ & \implies p(Tx_n, T\zeta) < p(T\zeta, T\zeta) + \varepsilon, \ \forall\ n \geq K. \end{aligned}$$

Therefore, $\lim_{n\to\infty} p(Tx_n, T\zeta) = p(T\zeta, T\zeta)$. Now, we have

$$\begin{aligned} p(\zeta, T\zeta) & \leq p(\zeta, Tx_n) + p(Tx_n, T\zeta) - p(Tx_n, Tx_n) \\ & \leq p(\zeta, Tx_n) + p(Tx_n, T\zeta). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} p(\zeta, T\zeta) \leq & p(T\zeta, T\zeta) \\ & \leq ap(\zeta,\zeta) + bp(\zeta, T\zeta) + cp(\zeta, T\zeta) + dp(\zeta, T\zeta) \\ & \quad + ep(\zeta, T\zeta) \end{aligned}$$

$$\implies (1 - b - c - d - e)p(\zeta, T\zeta) \leq 0.$$

This implies $T\zeta = \zeta$ i.e. ζ is a fixed point of T. ☐

Theorem 3.2. *In the above Theorem 3.1, omitting the continuity of T and enclose the following condition:*
For an increasing sequence $\{x_n\}$ with $x_n \to \zeta$ in X, then $x_n \preceq \zeta$, $\forall\ n$.
Then T has a fixed point.

Proof. Follow the same steps as the proof of Theorem 3.1 up to getting the Cauchy sequence $\{x_n\}$ converging to ζ with

$$p(\zeta,\zeta) = \lim_{n\to\infty} p(x_n,\zeta) = \lim_{n,m\to\infty} p(x_m, x_n) = 0.$$

From (3.1), we have

$$\begin{aligned} p(\zeta, T\zeta) \leq & p(\zeta, Tx_n) + p(Tx_n, T\zeta) - p(Tx_n, Tx_n) \\ & \leq p(\zeta, x_{n+1}) + p(Tx_n, T\zeta) \\ & \leq p(\zeta, x_{n+1}) + ap(x_n, \zeta) + bp(x_n, Tx_n) + cp(\zeta, T\zeta) \\ & \quad + dp(x_n, T\zeta) + ep(\zeta, Tx_n) \\ & \leq p(\zeta, x_{n+1}) + ap(x_n, \zeta) + bp(x_n, x_{n+1}) + cp(\zeta, T\zeta) \\ & \quad + dp(x_n, \zeta) + dp(\zeta, T\zeta) + ep(\zeta, x_{n+1}). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} p(\zeta, T\zeta) & \leq (c + d)p(\zeta, T\zeta) \\ \implies (1 - c - d)p(\zeta, T\zeta) & \leq 0. \end{aligned}$$

This shows $T\zeta = \zeta$ i.e. ζ is a fixed point of T. ☐

Theorem 3.3. *To the hypothesis of Theorem 3.1 and 3.2, subsume the following hypothesis:*
(H): Every pair of fixed points of T are comparable.
Then T has a unique fixed point.

Proof. Let T has two fixed points ζ and η. By hypothesis (H), ζ and η are comparable i.e. $\zeta \preceq \eta$ or $\eta \preceq \zeta$.
Now, from (3.1), we have

$$\begin{aligned} p(\zeta, \eta) = & p(T\zeta, T\eta) \\ & \leq ap(\zeta, \eta) + bp(\zeta, T\zeta) + cp(\eta, T\eta) + dp(\zeta, T\eta) + ep(\zeta, T\eta) \\ & \leq ap(\zeta, \eta) + bp(\zeta, \zeta) + cp(\eta, \eta) + dp(\zeta, \eta) + ep(\zeta, \eta) \\ & \leq ap(\zeta, \eta) + dp(\zeta, \eta) + ep(\zeta, \eta). \end{aligned}$$

This implies that $(1 - a - d - e)p(\zeta, \eta) \leq 0$ and hence $p(\zeta, \eta) = 0$. Therefore, $\zeta = \eta$. ☐

We have the following Reich type corollaries from Theorem 3.1 (respectively, 3.2 and 3.3).

Corollary 3.4. *Let $(X, \preceq, p)$ be a complete partially ordered partial metric space. Let T be a continuous and nondecreasing self mapping on X such that there exists nonnegative numbers a, b, c with $a + b + c < 1$ satisfying*

$$p(Tx, Ty) \leq ap(x, y) + bp(x, Tx) + cp(y, Ty),\ \forall\ x, y \in X \text{ with } x \preceq y.$$

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Corollary 3.5. *In the above Corollary 3.4, exclude the continuity of T and enclose the following condition:*
For an increasing sequence $\{x_n\}$ with $x_n \to \zeta$ in X, then $x_n \preceq \zeta$, $\forall\ n$.
Then T has a fixed point.

Corollary 3.6. *To the hypothesis of Corollary 3.4 and 3.5 subsume the following hypothesis:*
(H): Every pair of fixed points of T are comparable.
Then T has a unique fixed point.

Example 3.7. *Let $X = [0, \infty)$, $\preceq$ the natural ordering of real numbers and $p(x, y) = \max\{x, y\}$. Then $(X, \preceq, p)$ is a complete partially ordered partial metric space. Define $T : X \to X$ as*

$$Tx = \begin{cases} 0, & 0 \leq x < 2, \\ \frac{x+2}{5}, & x \geq 2. \end{cases}$$

T is a nondecreasing mapping. Consider the following cases:
Case I: If $x, y \in [0, 2)$ and $x \leq y$, then

$$p(Tx, Ty) = 0 \leq ap(x, y) + bp(x, Tx) + cp(y, Ty)$$

for any nonnegative numbers a, b, c with $a + b + c < 1$.
Case II: If $x, y \geq 2$ and $x \leq y$, then

$$\begin{aligned} p(Tx, Ty) &= \max\left\{\frac{x+2}{5}, \frac{y+2}{5}\right\} = \frac{y+2}{5} \\ &\leq \frac{y+y}{5} \leq \frac{2}{5}y \\ &\leq ap(x, y) + bp(x, Tx) + cp(y, Ty), \end{aligned}$$

where $a = \frac{2}{5}, b = c = 0$ and $p(x, y) = y$.
Case III: If $x \in [0, 2)andy \geq 2$, then

$$\begin{aligned} p(Tx, Ty) &= \max\left\{0, \frac{y+2}{5}\right\} \\ &= \frac{y+2}{5} \\ &\leq \frac{y+y}{5} \\ &\leq ap(x, y) + bp(x, Tx) + cp(y, Ty), \end{aligned}$$

where $a = \frac{2}{5}, b = c = 0, p(x, y) = y$ and $p(x, Tx) = x$. Thus, T satisfies all the condition of Corollary 3.6 and 0 is the unique fixed point of T.

4. Best proximity point theorems

Let (X,p) be a partial metric space and A,B be nonempty subsets of X. Let A_0 and B_0 denote the following sets:

$$A_0 = \{x \in A : p(x,y) = p(A,B), \text{ for some } y \in B\};$$
$$B_0 = \{y \in B : p(x,y) = p(A,B), \text{ for some } x \in A\}.$$

Definition 4.1. *([25], [28]) Let (X,p) be a partial metric space and A,B be two nonempty subsets of X with $A_0 \neq \emptyset$. The pair (A,B) is said to have the P-property if and only if*

$$\left.\begin{matrix} p(x_1,y_1) = p(A,B) \\ p(x_2,y_2) = p(A,B) \end{matrix}\right\} \implies p(x_1,x_2) = p(y_1,y_2),$$

$x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

Lemma 4.2. *([8], [28]) B_0 is closed with respect to (X,d_p).*

Lemma 4.3. *([8], [28]) $T(\overline{A}_0) \subseteq B_0$ for a mapping $T : A \to B$.*

Definition 4.4. *Let (X,p) be a partial metric space and A,B be nonempty subsets of X. A mapping $T : A \to B$ is said to be Reich type contraction mapping if it satisfies:*

$$p(Tx,Ty) \leq ap(x,y) + b[p(x,Tx) - p(A,B)] + c[p(y,Ty) - p(A,B)], \tag{4.1}$$

$\forall\ x,y \in X$, where $a,b,c \geq 0$ and $a+b+c < 1$.

Theorem 4.5. *Let A and B be nonempty closed subsets of a complete partial metric space (X,p) with $A_0 \neq \emptyset$. Let $T : A \to B$ be a continuous Reich type contraction mapping satisfying the following conditions:*
(i) $T(A_0) \subseteq B_0$;
(ii) the pair (A,B) has the P-property.
Then T has a unique best proximity point.

Proof. By Lemma 4.2, B_0 is closed with respect to (X,d_p). Also, by Lemma 4.3 $T(\overline{A}_0) \subseteq B_0$.

Now we consider an operator $P_{A_0} : T(\overline{A}_0) \to A_0$ defined by

$$P_{A_0}y = \{x \in A_0 : p(x,y) = p(A,B)\}.$$

Let $x_1,x_2 \in \overline{A}_0$. As the pair (A,B) has the P-property, we get

$$\begin{aligned} P_{A_0}Tx_1 &= \{u \in A_0 : p(u,Tx_1) = p(A,B)\} \\ P_{A_0}Tx_2 &= \{v \in A_0 : p(v,Tx_2) = p(A,B)\} \\ \implies p(Tx_1,Tx_2) &= p(u,v). \end{aligned}$$

Now, we have

$$\begin{aligned} p(P_{A_0}Tx_1, P_{A_0}Tx_2) = p(Tx_1,Tx_2) \\ &\leq ap(x_1,x_2) + b[p(x_1,Tx_1) - p(A,B)] + c[p(x_2,Tx_2) - p(A,B)] \\ &\leq ap(x_1,x_2) + b[p(x_1,P_{A_0}Tx_1) + p(P_{A_0}Tx_1,Tx_1) - p(A,B)] + \\ &\quad c[p(x_2,P_{A_0}Tx_2) + p(P_{A_0}Tx_2,Tx_2) - p(A,B)]. \end{aligned}$$

This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a continuous Reich type contraction mapping on a complete partial metric subspace $\overline{A}_0$. By Theorem 2.3 , we conclude that $P_{A_0}T$ has a unique fixed point $x \in A_0$ i.e. $P_{A_0}Tx = x$. This shows that $p(x, Tx) = p(A, B)$ i.e., x is a unique best proximity point of T. □

Example 4.6. *Let $X = (0, \infty)$ and $A = [2, \infty)$, $B = [1, \infty)$ be subsets of X with partial metric $p(x, y) = \max\{x, y\}$. Define $T : A \to B$ as $T(x) = \frac{x}{2}$. Then $p(A, B) = 2, A_0 = \{2\}$ and $B_0 = [1, 2]$. Also $T(A_0) \subseteq B_0$. Now, for $x, y \in A$*

$$T(x) = [1, \infty) = B,\ p(Tx, Ty) = 1,\ p(x, Tx) = p(y, Ty) = 2 \text{ and } p(x, y) = 2.$$

We have

$$p(Tx, Ty) \leq ap(x, y) + b[p(x, Tx) - p(A, B)] + c[p(y, Ty) - p(A, B],$$

for nonnegative $a = \frac{3}{5}, b = 0, c = \frac{1}{5}$ with $a + b + c < 1$. Thus, T satisfies the conditions of Theorem 4.5 and 2 is the unique best proximity point of T.

Definition 4.7. *Let $A, B \neq \emptyset$ be subsets of a partial metric space (X, p). A mapping $T : A \to B$ is said to be Hardy-Rogers type contraction mapping if T satisfies:*

$$\begin{aligned} p(Tx, Ty) \leq & ap(x, y) + b[p(x, Tx) - p(A, B)] + c[p(y, Ty) - p(A, B)] + \\ & d[p(x, Ty) - p(A, B)] + e[p(y, Tx) - p(A, B)],\ \forall\ x, y \in A. \end{aligned}$$

where $a, b, c, d, e \geq 0$ and if $d \geq e$, then $a + b + c + d + e < 1$ and if $d < e$, then $a + b + c + d + 2e < 1$.

Theorem 4.8. *Let (X, p) be a complete partial metric space and $A, B \subseteq X$, closed and non-empty with $A_0 \neq \emptyset$. Assume a continuous Hardy-Rogers type contraction mapping $T : A \to B$ satisfying the following conditions:*
(i) $TA_0 \subseteq B_0$;
(ii) the pair (A, B) has the P-property.
Then T has a unique best proximity point.

Proof. By Lemma 4.2, B_0 is closed with respect to (X, d_p). Also, by Lemma 4.3 $T(\overline{A}_0) \subseteq B_0$. Now we consider an operator $P_{A_0} : T(\overline{A}_0) \to A_0$ defined by

$$P_{A_0}y = \{x \in A_0 : p(x, y) = p(A, B)\}.$$

Let $x_1, x_2 \in \overline{A}_0$. As the pair (A, B) has the P-property, we get

$$\begin{aligned} & P_{A_0}Tx_1 = \{u \in A_0 : p(u, Tx_1) = p(A, B)\} \\ & P_{A_0}Tx_2 = \{v \in A_0 : p(v, Tx_2) = p(A, B)\} \\ \Longrightarrow & p(Tx_1, Tx_2) = p(u, v). \end{aligned}$$

Now, we have

$$\begin{aligned} p(P_{A_0}Tx_1, P_{A_0}Tx_2) = & p(Tx_1, Tx_2) \\ \leq & ap(x_1, x_2) + b[p(x_1, Tx_1) - p(A, B)] + c[p(x_2, Tx_2) - \\ & p(A, B)] + d[p(x_1, Tx_2) - p(A, B)] + e[p(x_2, Tx_1) - p(A, B)] \\ \leq & ap(x_1, x_2) + b[p(x_1, P_{A_0}Tx_1) + p(P_{A_0}Tx_1, Tx_1) - p(A, B)] + \\ & c[p(x_2, P_{A_0}Tx_2) + p(P_{A_0}Tx_2, Tx_2) - p(A, B)] + \\ & d[p(x_1, P_{A_0}Tx_2) + p(P_{A_0}Tx_2, Tx_2) - p(A, B)] + \\ & e[p(x_2, P_{A_0}Tx_1) + p(P_{A_0}Tx_1, Tx_1) - p(A, B)] \\ \leq & ap(x_1, x_2) + bp(x_1, P_{A_0}Tx_1) + cp(x_2, P_{A_0}Tx_2) + \\ & dp(x_1, P_{A_0}Tx_2) + ep(x_2, P_{A_0}Tx_1). \end{aligned}$$

This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a Hardy-Rogers type contraction mapping on a complete partial metric subspace $\overline{A}_0$. By Theorem 2.4, we conclude that $P_{A_0}T$ has a unique fixed point $x \in A_0$ i.e. $P_{A_0}Tx = x$. This shows that $p(x, Tx) = p(A, B)$ i.e., x is a unique best proximity point of T. □

Example 4.9. *Let (X, p) be a partial metric space where $X = (0, \infty), p(x, y) = \max\{x, y\}$ and $A = [3, \infty), B = [1, 2]$ two subsets of X. Then $p(A, B) = 3$, $A_0 = \{3\}$ and $B_0 = [1, 2]$. Define $T : A \to B$ by*

$$Tx = \frac{x+1}{3}.$$

Now $T(A_0) \subseteq B_0$ and for $x, y \in A$

$$Tx = \left[\frac{4}{3}, \infty\right), p(Tx, Ty) = \frac{4}{3},$$

$$p(x, Tx) = 3 = p(y, Ty) = p(x, Ty) = p(y, Tx) = p(x, y).$$

We have

$$\begin{aligned} p(Tx, Ty) \leq & ap(x, y) + b[p(x, Tx) - p(A, B)] + c[p(y, Ty) - p(A, B)] + \\ & d[p(x, Ty) - p(A, B)] + e[p(y, Tx) - p(A, B)] \end{aligned}$$

for $a = \frac{5}{9}, b = c = d = \frac{1}{9}$, $e = 0$ and $a + b + c + d + e = \frac{8}{9} < 1$. Thus, T satisfies all conditions of Theorem 4.8 and hence 3 is the unique bet proximity point of T.

Corollary 4.10. *If $A = B$ in Theorem 4.5 and 4.8 that is T is a self-mapping, then T has a unique fixed point. For this case, continuity of T is not necessary.*

5. Application

Here, we establish the existence of solution for a Fredholm integral equation. Consider a Fredholm integral equation

$$x(t) = q(t) + \int_{\alpha}^{\beta} K(t, s, x(s))ds, \tag{5.1}$$

where $t \in I = [\alpha, \beta]$, $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $q : I \to \mathbb{R}$ are continuous functions.

Let $X = C(I, \mathbb{R})$ be the space of real continuous functions defined on I. Define p on X by

$$p(x, y) = \max_{t \in [\alpha, \beta]} \Big[|x(t) - y(t)| + \gamma\Big] \text{ where } \gamma \in \mathbb{R}^+.$$

Then, (X, p) ia a complete partial metric space. Suppose $T : X \to X$ is a self-mapping defined by

$$Tx(t) = q(t) + \int_{\alpha}^{\beta} K(t, s, x(s))ds, \forall\ x \in X \text{ and } \forall\ t \in I.$$

Obviously, $x(t)$ is a solution of (5.1) if and only if it is a fixed point of T.

Theorem 5.1. *Besides above conditions, consider the following hypothesis hold:*
(h_1) There exists a continuous function $f : I \times I \to \mathbb{R}^+$ such that

$$\begin{aligned} |K(t, s, x(s)) - K(t, s, y(s))| \leq f(t, s)\Big[&|x(s) - y(s)| + \frac{b}{a}|x(s) - Tx(s)| + \frac{c}{a}|y(s) - \\ &Ty(s)| + \frac{\gamma}{a}(a + b + c - 1)\Big] \end{aligned}$$

$\forall\ x, y \in X$ and $\forall\ t, s \in I$;
(h_2) $\max_{t\in[\alpha,\beta]} \int_\alpha^\beta f(t,s)ds \leq \frac{a}{\beta-\alpha}$, where $0 < a < 1, 0 \leq b, c < 1$ and $a+b+c < 1$.
Then, the integral equation (5.1) has a unique solution in X.

Proof. We know that (X, p) is a complete partial metric space.
Now, we show T is a Reich-type contractive mapping.

$$\begin{aligned}
p(Tx, Ty) &= \max_{t\in[\alpha,\beta]} \Big[|Tx(t) - Ty(t)| + \gamma \Big] \\
&= \max_{t\in[\alpha,\beta]} \Big[\Big| \int_\alpha^\beta K(t, s, x(s)) - \int_\alpha^\beta K(t, s, y(s)) \Big| + \gamma \Big] \\
&\leq \max_{t\in[\alpha,\beta]} \Big[\int_\alpha^\beta |K(t, s, x(s)) - K(t, s, y(s))| + \gamma \Big] \\
&\leq \max_{t\in[\alpha,\beta]} \Big[\int_\alpha^\beta f(t,s)ds \big[|x(t) - y(t)| + \frac{b}{a}|x(t) - Tx(t)| + \frac{c}{a}|y(t) - Ty(t)| \\
&\quad + \frac{\gamma}{a}(a + b + c - 1) \big] + \gamma \Big] \\
&\leq a[|x - y| + \gamma] + b[|x - Tx| + \gamma] + c[|y - Ty| + \gamma] \\
&\leq ap(x, y) + bp(x, Tx) + cp(y, Ty)
\end{aligned}$$

where $0 < a < 1$ and b, c are nonnegative with $a+b+c < 1$. This shows that T is a Reich-type contractive mapping on X and hence by Corollary 4.10, T has a unique solution. □

Example 5.2. *Let (X, p) be the complete partial metric space as in Theorem 5.1. Define $T : X \to X$ as*

$$Tx(t) = q(t) + \int_0^1 K(t, s, x(s))ds, \tag{5.2}$$

where $K(t, s, x(s)) = \frac{t(1+s)}{4}x(s)$ and $q(t) = \frac{19}{48}t$. Setting $f(t, s) = \frac{t(1+s)}{4}$. Then

$$\int_0^1 f(t,s)ds = \int_0^1 \frac{t(1+s)}{4}ds = \frac{3}{8}t$$

which implies that $\max_{t\in[0,1]} \int_0^1 f(t,s)ds \leq \frac{3}{8}$. Then we can easily show T satisfies all condition of Theorem 5.1 and $x(t) = \frac{t}{2}$ is a unique solution of the integral equation (5.2).

CONCLUSION

We discuss the fixed point and the best proximity point theorems of Reich type and Hardy-Rogers type contraction mappings in partial metric space by generalizing some results of Altun et al.[2]. We also give non-trivial examples and an application to certify the achieved results. It is interesting to highlight that our result can be further generalized using control functions.

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