# ON THE PERIODIC SOLUTIONS OF NEUTRAL INTEGRODIFFERENTIAL EQUATIONS VIA FIXED POINT METHOD 

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#### Abstract

In the present article, we study existence of periodic solutions (EPSs) of a nonlinear neutral integro- differential equation (NIDE) with multiple variable delays using Krasnoselskiis fixed point theorem. Transforming the considered NIDE to an equivalent integral equation, we prove the EPSs using a fixed point mapping, which is defined as a sum of a contraction and a compact map. The result of this paper has contributions to the topic of the EPSs of NIDEs.


## 1. Introduction

Time-delay often can appear in numerous real-world systems of sciences, engineering, medicine and so on (see, Burton [4], Fridman [10], Hale and Verduyn Lunel [15], Kolmanovskii and Myshkis [17], Kolmanovskii and Nosov [18], Krasovski [19], Kuang [20], Lakshmikantham et al. [21], Rihan [27], Smith [29]). Indeed, delay(s) are also strongly involved in challenging areas of communication and information technologies, and time-delay is a source of instability, in many cases (see, Fridman [10]). Therefore, the qualitative analysis of solutions of functional differential equations (FDEs), i.e., EPSs, stability, boundedness, instability, integrability of solutions of delay differential equations, neutral differential equations (NDEs) and NIDES, etc., has theoretical and practical importance (see, for example, [1-38]).

In particular, we should now summarize a few results on the EPSs for NDEs and NIDEs.

Raffoul [24] used Krasnoselskiis fixed point to investigate that the following NDE with variable delay has a periodic solution:

$$
x^{\prime}(t)=-a(t) x(t)+c(t) x^{\prime}(t-g(t))+q(t, x(t), x(t-g(t))) .
$$

In [24], this equation is transformed to the problem of an integral equation and the author used the contraction mapping principle to demonstrate that this NDE has a unique solution.

[^0]Later, Yankson [38] benefited from a modified Krasnoselskiis fixed point theorem and showed that the following nonlinear NDE with variable delay has a periodic solution:

$$
x^{\prime}(t)=-a(t) h(x(t))+c(t) x^{\prime}(t-g(t))+q(t, x(t), x(t-g(t)))
$$

Chen et al. [7] constructed new criteria for the stability of null solution of the following FDE and NDEs:

$$
\begin{aligned}
x^{\prime}(t) & =-\int_{t-r(t)}^{t} a(t, s) g(s, x(s)) d s \\
x^{\prime}(t)-c(t) x^{\prime}\left(t-r_{1}(t)\right) & =-a(t) x\left(t-r_{2}(t)\right)+\int_{t-r_{3}(t)}^{t} g(t, x(s)) d \mu(t, s)
\end{aligned}
$$

and

$$
x^{\prime}(t)-c(t) x^{\prime}\left(t-r_{1}(t)\right)=-a(t) x\left(t-r_{2}(t)\right)+b(t) g\left(t, x\left(t-r_{3}(t)\right)\right.
$$

As for the technique of the proofs of [7], it depends upon fixed point methods such that here it not needed both boundedness of delays and fixed sign conditions on the coefficient functions of these equations. The work of [7] has contributions to the qualitative theory of FDEs and NDEs.

Motivated by the results above, that can be found in the references of this paper and the relevant literature, we consider the following nonlinear NIDE with multiple variable delays:

$$
\begin{equation*}
x^{\prime}(t)-c(t) x^{\prime}\left(t-r_{1}(t)\right)=-a(t) x\left(t-r_{2}(t)\right)+\int_{t-r_{3}(t)}^{t} g(t, x(s)) d s \tag{1.1}
\end{equation*}
$$

where $r_{j}(t):[0, \infty) \rightarrow[0, \infty), j=1,2,3$, are continuous variable delay functions, $a, c:[0, \infty) \rightarrow R$ and $g:[-r, \infty) \times R \rightarrow R$ are continuous functions, where $r=\max r_{j}(t), j=1,2,3$.

In this paper, we establish suitable mappings and use Krasnoselskiis fixed point theorem to discuss the EPSs of NIDE (1.1). It should be noted that NIDE (1.1) is a new mathematical model and it is different from those studied in the references of this paper. Hence, the results of this paper are new.

Before giving our main result, to make the paper self-contained and to increase its readability, we give Krasnoselskiis fixed point theorem.
Theorem (Krasnoselskiis fixed point theorem, Burton [4, Theorem 1.2.7]). Let M be a closed convex non-empty subset of a Banach space ( $S,\|$.$\| ). Suppose that A$ and $B$ map $M$ into $S$ such that
(i) $A x+B y \in M(\forall x, y \in M)$;
(ii) $A$ is continuous and $A M$ is contained in a compact set;
(iii) $B$ is a large contraction mapping with constant $\alpha<1$. Then there exists $y \in M$ with $A y+B y=y$.

## 2. Existence of periodic solutions

Here, a theorem including sufficient conditions for the EPSs of NIDE (1.1) with multiple variable delays is proved .

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We define $P_{T}=\{(\phi: C(\mathbb{R}, \mathbb{R}), \phi(t+T)=\phi(t))\}$, where $T>0, T \in R$, and $P_{T}$ is the Banach space.

Let

$$
\|x(t)\|=\max _{t \in[0, T]}|x(t)|
$$

Before giving our main theorem related to the EPSs of NIDE (1.1) with variable multiple delays, we suppose the conditions below hold for NIDE (1.1).
(A1) $g(t+T, x)=g(t, x)$,i.e., $g$ is a periodic function. The function $g$ satisfies a Lipschitz condition for $\forall(t, x) \in[0, \infty) \times[-l, l], l>0$, i.e., there is an $L>0, L \in R$, such that

$$
|g(t, x)-g(t, y)| \leq L|x-y| ;
$$

(A2) $r_{3} \in C([0, \infty),[0, \infty))$;

$$
\begin{equation*}
h_{j} \in C\left(\left[r_{0}, \infty\right), \mathbb{R}\right), j=1,2 \tag{A3}
\end{equation*}
$$

(A4) The delay functions satisfies $r_{1} \in C^{2}\left(R^{+},(0, \infty)\right), r_{1}(t) \neq 1$,

$$
r^{\prime}{ }_{1}(t) \neq 1, r_{2} \in C^{1}\left(R^{+},(0, \infty)\right) ;
$$

(A5) $a, c \in C([0, \infty), \mathbb{R}) ; a(t+T)=a(t), c(t+T)=c(t), h_{j}(t+T)=h_{j}(t)$, $j=1,2$, and also $r_{j}(t+T)=r_{j}(t), r_{j}(t) \geq r_{j}^{*} \geq 0, j=1,2,3$;

$$
\begin{equation*}
\int_{0}^{T} h_{i}(s) d s>0, i=1,2 \tag{A6}
\end{equation*}
$$

Lemma 2.1. Suppose that (A1)-(A6) hold and $x(t) \in P_{T}$. Then, $x(t)$ is a solution of NIDE (1.1) with multiple variable delays if and only if

$$
\begin{align*}
x(t)= & \frac{c(t)}{1-r_{1}(t)} x\left(t-r_{1}(t)\right)+\sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t} h_{j}(u) x(u) d s \\
& +\left(1-\exp \left(-\int_{t-T}^{t}\left[\sum_{i=1}^{2} h_{i}(s)\right] d s\right)\right)^{-1} \\
& \times\left\{\int_{t-T}^{t} \exp \left(-\int_{u}^{t}\left[\sum_{i=1}^{2} h_{i}(s)\right] d s\right)\left[h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)-a(u)\right)\right] x\left(u-r_{2}(u)\right) d u\right. \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t}\left[\sum_{i=1}^{2} h_{i}(s)\right] d s\right)\left[h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)-k(u)\right)\right] x\left(u-r_{1}(u)\right) d u \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t}\left[\sum_{i=1}^{2} h_{i}(s)\right] d u\right)\left(\int_{u-r_{3}(u)}^{u} g(u, x(s)) d s\right) d u \\
& \left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t}\left(\left[\sum_{i=1}^{2} h_{i}(s)\right]\right) d s d u\right)\left(\left[\sum_{i=1}^{2} h_{i}(u)\right]\right) \int_{u-r_{j}(u)}^{u} h_{j}(s) x(s) d s d u\right\}, \tag{2.1}
\end{align*}
$$

where

$$
\begin{equation*}
k(u)=\left(\frac{\left.\left(c(u)\left[\sum_{i=1}^{2} h_{i}(u)\right]\right)+c^{\prime}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right)+r_{1}^{\prime \prime}(u) c(u)}{\left(1-r_{1}{ }^{\prime}(u)\right)^{2}}\right) \tag{2.2}
\end{equation*}
$$

Proof. ( $\Rightarrow$ :) We suppose that $x(t) \in P_{T}$ and $x(t)$ is a solution of NIDE (1.1). Firstly, we add the term [ $\left.\sum_{i=1}^{2} h_{i}(t)\right] x(t)$ to the both sides of NIDE (1.1):

$$
\begin{align*}
x^{\prime}(t) & -c(t) x^{\prime}\left(t-r_{1}(t)\right)+\left[h_{1}(t)+h_{2}(t)\right] x(t) \\
& =\left(\sum_{i=1}^{2} h_{i}(t)\right) x(t)-a(t) x\left(t-r_{2}(t)\right)+\int_{t-r_{3}(t)}^{t} g(t, x(s)) d s \tag{2.3}
\end{align*}
$$

Then, the equation (2.3) can be arranged as the following:

$$
\begin{align*}
x^{\prime}(t) & -c(t) x^{\prime}\left(t-r_{1}(t)\right)+\left(\sum_{i=1}^{2} h_{i}(t)\right) x(t) \\
= & \sum_{j=1}^{2} h_{j}\left(t-r_{j}(t)\right)\left(1-r^{\prime}{ }_{j}(t)\right) x\left(t-r_{j}(t)\right) \\
& +\sum_{j=1}^{2} \frac{d}{d t} \int_{t-r_{j}(t)}^{t} h_{j}(u) x(u) d u-a(t) x\left(t-r_{2}(t)\right) \\
& +\int_{t-r_{3}(t)}^{t} g(t, x(s)) d s . \tag{2.4}
\end{align*}
$$

It is obvious that the equations (2.3) and (2.4) are equal. Then, multiplying equation (2.4) by $\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right)$ and then integrating the obtained result from $t-T$ to $t$, we find

$$
\begin{aligned}
& \exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) x^{\prime}(t)-\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) c(t) x^{\prime}\left(t-r_{1}(t)\right) \\
& \quad+\exp \left(\int_{0}^{t}\left[h_{1}(s)+h_{2}(s)\right] d s\right)\left[h_{1}(t)+h_{2}(t)\right] x(t) \\
& =\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) \sum_{j=1}^{2} h_{j}\left(t-r_{j}(t)\right)\left(1-r^{\prime}{ }_{j}(t)\right) x\left(t-r_{j}(t)\right) \\
& \quad+\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) \sum_{j=1}^{2} \frac{d}{d t} \int_{t-r_{j}(t)}^{t} h_{j}(s) x(s) d s \\
& \quad-\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) a(t) x\left(t-r_{2}(t)\right)
\end{aligned}
$$

$$
+\exp \left(\int_{0}^{t}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) \int_{t-r_{3}(t)}^{t} g(t, x(s)) d s
$$

Hence,

$$
\begin{aligned}
\int_{t-T}^{t}[x(u) & \left.\left.\exp \left(\int_{0}^{u}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right)\right)\right]^{\prime}-\int_{t-T}^{t} \exp \left(\int_{0}^{u}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u \\
& =\int_{t-T}^{t} \exp \left(\int_{0}^{u}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) \sum_{j=1}^{2}\left[h_{j}\left(u-r_{j}(u)\right)\left(1-r^{\prime}{ }_{j}(u)\right) x\left(u-r_{j}(u)\right)\right] d u \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t}\left(\sum_{i=}^{2} h_{i}(s)\right) d s\right) \sum_{j=1}^{2} \int_{u-r_{j}(u)}^{u} \frac{d}{d u}\left[h_{j}(s) x(s)\right] d s d u \\
& -\int_{t-T}^{t} \exp \left(\int_{0}^{u}\left(\sum_{i=1}^{2} h_{i}(s)\right) d s\right) a(u) x\left(u-r_{2}(u)\right) d u \\
& +\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right) \int_{u-r_{3}(u)}^{u} g(u, x(s)) d u .
\end{aligned}
$$

Noting these calculations, we find

$$
\begin{aligned}
& x(t) \exp \left(\int_{0}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)-x(t-T) \exp \left(\int_{0}^{t-T} \sum_{i=1}^{2} h_{i}(s) d s\right) \\
& -\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u \\
& =\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right) \sum_{j=1}^{2}\left[h_{j}\left(u-r_{j}(u)\right)\left(1-r_{j}{ }^{\prime}(u)\right) x\left(u-r_{j}(u)\right)\right] d u \\
& \quad+\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right) \sum_{j=1}^{2} \frac{d}{d u} \int_{u-r_{j}(u)}^{u}\left[h_{j}(s) x(s)\right] d s d u \\
& \quad-\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right) a(u) x\left(u-r_{2}(u)\right) d u \\
& \quad+\int_{t-T}^{t} \exp \left(\int_{0}^{u} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(\int_{u-r_{3}(u)}^{u} g(u, x(s)) d s\right) d u
\end{aligned}
$$

As for the next step, consider the equality $x(t)=x(t-T)$ and divide the equation above by

$$
\begin{align*}
x(t)= & \left(1-\exp \left(\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right)^{-1} \times\left\{\int_{t-T}^{t} \exp \left(\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u\right. \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \sum_{j=1}^{2}\left[h_{j}\left(u-r_{j}(u)\right)\left(1-r_{j}{ }^{\prime}(u)\right) x\left(u-r_{j}(u)\right)\right] d u \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \sum_{j=1}^{2} \frac{d}{d u} \int_{u-r_{j}(u)}^{u}\left[h_{j}(s) x(s)\right] d s d u \\
& -\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) a(u) x\left(u-r_{2}(u)\right) d u \\
& \left.+\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(\int_{u-r_{3}(u)}^{u} g(u, x(s)) d s\right) d u\right\} . \tag{2.5}
\end{align*}
$$

Consider the terms

$$
\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u
$$

Multiplying this expression by

$$
\frac{1-r_{1}{ }^{\prime}(u)}{1-r_{1}^{\prime}(u)}
$$

we obtain

$$
\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \frac{c(u) x\left(u-r_{1}(u)\right)\left(1-r^{\prime}{ }_{1}(u)\right)}{\left(1-r^{\prime}{ }_{1}(u)\right)} d u .
$$

Let

$$
\begin{gathered}
U=\frac{c(u)}{1-{r^{\prime}}^{\prime}(u)} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \\
d V=x^{\prime}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right) d u \Rightarrow V=x\left(u-r_{1}(u)\right), \\
d U=\frac{\left[c^{\prime}(u)-c(u)\left(h_{1}(u)+h_{2}(u)\right)\right]\left(1-r^{\prime}{ }_{1}(u)\right)+c(u) r^{\prime \prime}{ }_{1}(u)}{\left(1-{r^{\prime}}^{1}(u)\right)^{2}} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) d u
\end{gathered}
$$

and

$$
\frac{\left[c^{\prime}(u)-c(u)\left(h_{1}(u)+h_{2}(u)\right)\right]\left(1-r^{\prime}{ }_{1}(u)\right)+c(u) r^{\prime \prime}{ }_{1}(u)}{\left(1-r^{\prime}{ }_{1}(u)\right)^{2}}=k(u)
$$

Next, using integration by parts, we have

$$
\begin{aligned}
& \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u \\
= & \left.\frac{c(u)}{1-r^{\prime}{ }_{1}(u)} x\left(u-r_{1}(u)\right) \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right|_{t-T} ^{t} \\
& -\int_{t-T}^{t} x\left(u-r_{1}(u)\right) k(u) \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) d u .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) c(u) x^{\prime}\left(u-r_{1}(u)\right) d u \\
= & \frac{c(t)}{1-r^{\prime}{ }_{1}(t)} x\left(t-r_{1}(t)\right)\left(1-\exp \left(-\int_{t-T}^{t}\left[h_{1}(s)+h_{2}(s)\right] d s\right)\right) \\
& -\int_{t-T}^{t} x\left(u-r_{1}(u)\right) k(u) \exp \left(-\int_{u}^{t}\left[h_{1}(s)+h_{2}(s)\right] d s\right) d u .
\end{aligned}
$$

By the same way, let

$$
\begin{gathered}
U=\exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \\
d V=\sum_{j=1}^{2} \frac{d}{d u} \int_{u-r_{j}(u)}^{u} h_{j}(s) x(s) d s
\end{gathered}
$$

Applying the integration by parts, it is followed that

$$
\begin{array}{r}
\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \sum_{j=1}^{2} \frac{d}{d u} \int_{u-r_{j}(u)}^{u}\left[h_{j}(s) x(s)\right] d s d u \\
=\sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t} h_{j}(u) x(u) d u\left(1-\exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right) \\
\left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(h_{1}(u)+h_{2}(u)\right) \int_{u-r_{j}(u)}^{u} h_{j}(s) x(s) d s\right\} d u
\end{array}
$$

Substituting the expressions above into (2.5), we get that
$x(t)=\frac{c(t)}{1-r_{1}(t)} x\left(t-r_{1}(t)\right)+\sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t} h_{j}(u) x(u) d s$

$$
\begin{align*}
&+\left(1-\exp \left(-\int_{t-T}^{t}\left[h_{1}(s)+h_{2}(s)\right] d s\right)\right)^{-1} \\
& \times\left\{\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)\right)-a(u)\right)\right] x\left(u-r_{2}(u)\right) d u \\
&\left.+\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right)-k(u)\right)\right] x\left(u-r_{1}(u)\right) d u \\
&+\int_{t-T}^{t}\left(\exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right) \int_{u-r_{3}(u)}^{u} g(u, x(s)) d s\right) d u \\
&\left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(h_{1}(u)+h_{2}(u)\right)\left(\int_{u-r_{j}(u)}^{u} h_{j}(s) x(s) d s\right) d u\right\} \tag{2.6}
\end{align*}
$$

This result completes the proof.
$(\Leftarrow:)$ On the contrary, if the derivative of (2.6) is calculated and then substituted in NIDE (1.1), it is seen that the obtained expression satisfies NIDE (1.1). Therefore, $x(t)$ is a solution of NIDE (1.1). Thus, Lemma 2.1 is proved.

Now, to investigate the EPSs of NIDE (1.1), we will benefit from Krasnoselskiis fixed point theorem.

Let $H: P_{T} \rightarrow P_{T}$ be a mapping, where $r_{1}(t) \neq 1$,

$$
\begin{align*}
(H \phi)(t)= & \frac{c(t)}{1-r_{1}(t)} \phi\left(t-r_{1}(t)\right)+\sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t} h_{j}(u) \phi(u) d s \\
& +\left(1-\exp \left(-\int_{t-T}^{t}\left[h_{1}(s)+h_{2}(s)\right] d s\right)\right)^{-1} \\
\times & \left\{\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)\right)-a(u)\right)\right] \phi\left(u-r_{2}(u)\right) d u \\
& \left.+\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right)-k(u)\right)\right] \phi\left(u-r_{1}(u)\right) d u \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d u\right)\left(\int_{u-r_{3}(u)}^{u} g(u, \phi(s)) d s\right) d u \\
& \left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(h_{1}(u)+h_{2}(u)\right)\left(\int_{u-r_{j}(u)}^{u} h_{j}(s) \phi(s) d s\right) d u\right\} \tag{2.7}
\end{align*}
$$

Here, for the application of the fixed point theorem above, it is essential the construction of mappings such as the contraction and the compactness, respectively. In view of these idea, we write equation (2.7) in the form of

$$
(H \phi)(t)=(B \phi)(t)+(A \phi)(t),
$$

where both of $A$ and $B$ are the mappings from $P_{T}$ to $P_{T}$, and these mappings are defined as follows:

$$
\begin{equation*}
(B \phi)(t)=\frac{c(t)}{1-r_{1}(t)} \phi\left(t-r_{1}(t)\right)+\sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t} h_{j}(u) \phi(u) d s \tag{2.8}
\end{equation*}
$$

and

$$
\begin{align*}
(A \phi)(t)= & \left(1-\exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right)^{-1} \\
& \times\left\{\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)\right)-a(u)\right)\right] \phi\left(u-r_{2}(u)\right) d u \\
& \left.+\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right)-k(u)\right)\right] \phi\left(u-r_{1}(u)\right) d u \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(\int_{u-r_{3}(u)}^{u} g(u, \phi(s)) d s\right) d u \\
& \left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left(h_{1}(u)+h_{2}(u)\right) \int_{u-r_{j}(u)}^{u} h_{j}(s) \phi(s) d s d u\right\} . \tag{2.9}
\end{align*}
$$

Lemma 2.2. If the mapping $B$ is defined by (2.8) and

$$
\begin{gather*}
\int_{t-r_{1}(t)}^{t}\left\|h_{1}(t) d t\right\| \leq \zeta_{1}, \int_{t-r_{2}(t)}^{t}\left\|h_{2}(t) d t\right\| \leq \zeta_{2} \\
\left\|\frac{c(t)}{1-r_{1}^{\prime}(t)}\right\| \leq \xi_{3} \text { and } 0<\xi_{1}+\xi_{2}+\xi_{3}<1 \tag{2.10}
\end{gather*}
$$

hold, then the mapping $B$ is a contraction.
Proof. Let $\varphi, \psi \in P_{T}$. Then,

$$
\begin{aligned}
\|(B \phi)(t)-(B \psi)(t)\| & =\max _{t \in[0, T]}|B(\phi)-B(\psi)| \\
& =\max _{t \in[0, T]}\left|\frac{c(t)}{1-r_{1}{ }^{\prime}(t)}\right|\left|\phi\left(t-r_{1}(t)\right)-\psi\left(t-r_{1}(t)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
& \quad+\max _{t \in[0, T]} \sum_{j=1}^{2} \int_{t-r_{j}(t)}^{t}\left|h_{j}(u)\right||\phi(u)-\psi(u)| d u \\
& \leq\left(\xi_{1}+\xi_{2}+\xi_{3}\right)\|\phi-\psi\| .
\end{aligned}
$$

Hence, Lemma 2.2 is proved.
Lemma 2.3. Suppose that (A1)-(A6) hold. Then, the mapping $A: P_{T} \rightarrow P_{T}$ of (2.9) is compact provided that

$$
\begin{equation*}
\max _{t \in[0, T]}\left|\frac{c^{\prime}(t)}{1-r_{1}^{\prime}(t)}+\frac{r_{1}^{\prime \prime}(t) c(t)}{\left(1-r_{1}^{\prime}(t)\right)^{2}}\right| \leq Q \tag{2.11}
\end{equation*}
$$

for $Q>0, Q \in R$.
Proof. Transformation of the variable of (2.9) shows that

$$
(A \phi)(t+T)=(A \phi)(t)
$$

Let $\phi, \psi \in P_{T}$ with $\|\phi\| \leq C$ and $\|\psi\| \leq C$. We prove that the mapping $A$ is continuous. Next, we suppose that

$$
\begin{gathered}
\eta=\max _{t \in[0, T]}\left|\left(1-\exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right)^{-1}\right| \\
\gamma=\max _{u \in[t-T, t]}\left|\exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right|, \\
\sigma=\max _{t \in[0, T]}\left|h_{2}\left(t-r_{2}(t)\right)\left(1-r_{2}{ }^{\prime}(t)-a(t)\right)\right| \\
\beta=\max _{t \in[0, T]}\left|h_{1}\left(t-r_{1}(t)\right)\left(1-r_{1}^{\prime}(t)-k(t)\right)\right|, \\
z \in C\left(R^{+}, \mathbb{R}^{+}\right),\left|\int_{t-r_{3}(s)}^{t} g(t, x(u)) d u\right| \leq z(t) \leq \rho, \rho>0 .
\end{gathered}
$$

For $s \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$,

$$
\begin{equation*}
\left|\left(h_{1}(t)+h_{2}(t)\right) \int_{t-r_{j}(t)}^{t} h_{j}(u) d u\right| \leq s(t) \leq \lambda, \lambda \geq 0 \tag{2.12}
\end{equation*}
$$

provided that $h_{1}, h_{2}$ and $\int_{t-r_{j}(t)}^{t} h_{j}(u) d u$ are bounded.
We use the condition (2.12). Given $\forall \varepsilon>0$, take $\delta=\varepsilon / M$ such that $\|\phi-\psi\| \leq \delta$. Using (2.12) into (2.9), it follows that

$$
\begin{aligned}
\|(A \phi)(t)-(A \psi)(t)\| & \leq \gamma \eta \int_{t-T}^{t}[\sigma\|\phi-\psi\|+\beta\|\phi-\psi\|+\rho\|\phi-\psi\|+\lambda\|\phi-\psi\|] d u \\
& \leq M\|\phi-\psi\|<\varepsilon
\end{aligned}
$$

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where $M=T \gamma \eta(\sigma+\beta+\rho+\lambda)$. Thus, the mapping $A$ is continuous. Next, we will show the mapping $A$ is compact. Let $n \in Z^{+}$and $\phi_{n} \in P_{T}$. Following calculations above, we can obtain that $\left\|A \varphi_{n}\right\| \leq R$.

The calculation of $A\left(\phi_{n}\right)^{\prime}$ gives that

$$
\begin{aligned}
& A\left(\phi_{n}\right)^{\prime}=\left[(-1) \times\left(1-\exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right)^{-2}\right. \\
& \times\left[-\left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)^{\prime} \exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right] \\
& \times\left\{\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)\right)-a(u)\right)\right] \phi\left(u-r_{2}(u)\right) \\
& \left.+\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\left[h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}{ }^{\prime}(u)\right)-k(u)\right)\right] \phi\left(u-r_{1}(u)\right) \\
& +\int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d u\right) \int_{u-r_{3}(u)}^{u} g(u, \phi(s)) d s d u \\
& \left.\left.-\sum_{j=1}^{2} \int_{t-T}^{t} \exp \left(-\int_{u}^{t} \sum_{i=1}^{2} h_{i}(s) d u\right)\left(h_{1}(u)+h_{2}(u)\right) \int_{u-r_{j}(u)}^{u} h_{j}(s) \phi(s) d s d u\right\}\right] \\
& +\left[\left(1-\exp \left(-\int_{t-T}^{t} \sum_{i=1}^{2} h_{i}(s) d s\right)\right)^{-1} \times\left[\left\{\left(h_{2}\left(t-r_{2}(t)\right)\left(1-r_{2}{ }^{\prime}(t)\right)-a(t)\right) \phi\left(t-r_{2}(t)\right)\right.\right.\right. \\
& +\left(h_{1}\left(t-r_{1}(t)\right)\left(1-r_{1}{ }^{\prime}(t)\right)-k(t)\right) \phi\left(t-r_{1}(t)\right) \\
& \left.+\int_{t-r_{3}(t)}^{t}\left\{g(t, \phi(u))-\sum_{i=1}^{2}\left(h_{1}(t)+h_{2}(t)\right) \int_{t-r_{3}(t)}^{t} h_{j}(u) \phi(u) d u\right\} d s\right] \\
& -\left[\left\{\left(h_{2}\left((t-T)-r_{2}(t-T)\right)\left(1-r_{2}{ }^{\prime}(t-T)\right)-a(t)\right) \phi\left((t-T)-r_{2}(t-T)\right)\right.\right. \\
& \left.+h_{1}\left((t-T)-r_{1}(t-T)\right)\left(1-r_{1}^{\prime}(t-T)\right)-k(t)\right) \phi\left((t-T)-r_{1}(t-T)\right) \\
& +\int_{(t-T)-r_{3}(t-T)}^{t-T}\{g(t-T, \phi(u)) \\
& \left.\left.\left.-\sum_{i=1}^{2}\left(h_{1}(t-T)+h_{2}(t-T)\right) \int_{(t-T)-r_{3}(t-T)}^{t-T} h_{j}(u) \phi(u) d u\right\} d s\right] e^{-\int_{t-T}^{t}\left(h_{1}(s)+h_{2}(s)\right) d s}\right] \\
& +\left[\left\{\int_{t-T}^{t}\left(h_{2}\left(u-r_{2}(u)\right)\left(1-r_{2}{ }^{\prime}(u)\right)-a(u)\right) \phi\left(u-r_{2}(u)\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.\left.+h_{1}\left(u-r_{1}(u)\right)\left(1-r_{1}^{\prime}(u)\right)-k(u)\right)\right] \phi\left(u-r_{1}(u)\right)+\int_{t-r_{3}(t)}^{t}\{g(t, \phi(u)) \\
& \left.\left.-\sum_{j=1}^{2}\left(h_{1}(u)+h_{2}(u)\right) \int_{u-r_{j}(u)}^{u} h_{j}(s) \phi(s) d s\right\} d u\left\{\left(-\left(h_{1}(t)+h_{2}(t)\right)\right) e^{-\int_{u}^{t}\left(h_{1}(s)+h_{2}(s)\right) d u}\right\}\right]
\end{aligned}
$$

If we apply the definition of $A\left(\phi_{n}\right)$, then

$$
\begin{aligned}
A\left(\phi_{n}\right)^{\prime}= & -\left[h_{1}(t)+h_{2}(t)\right] \phi_{n}(t) \\
& -\left[\frac{c(t)}{1-r_{1}^{\prime}(t)}+\frac{r_{1}^{\prime \prime}(t) c(t)}{1-r_{1}^{\prime}(t)}\right] \phi_{n}\left(t-r_{1}(t)\right) \\
& +\left[h_{2}\left(t-r_{2}(t)\right)\left(1-r_{2}^{\prime}(t)\right)-a(t)\right] \phi_{n}\left(t-r_{2}(t)\right) \\
& +\left[h_{1}\left(t-r_{1}(t)\right)\left(1-r_{1}^{\prime}(t)\right)-k(t)\right] \phi_{n}\left(t-r_{1}(t)\right) \\
& +\int_{t-r_{3}(t)}^{t} g\left(t, \varphi_{n}(u)\right) d u .
\end{aligned}
$$

Hence, we can conclude that $A\left(\phi_{n}\right)$ is uniformly bounded and equi-continuous. In view of the Arzela- Ascoli theorem, we have a subsequence $A\left(\phi_{n_{k}}\right)$ of $A\left(\varphi_{n}\right)$, which converges uniformly to a continuous $T$-periodic function $\varphi^{*}$. This result verifies that the mapping $A$ is compact.

Theorem 2.4. Let $\gamma, \eta, \beta, \rho, \sigma$ and $\lambda$ be given as in Lemma 2.3. Moreover, suppose conditions (A4)-(A6) hold and the solution $x(t)$ of NIDE (1.1), $x(t) \in P_{T}$, satisfies $|x(t)| \leq G, G>0, G \in R$ and the following inequality

$$
\begin{equation*}
\left\{\left(\zeta_{1}+\zeta_{2}+\zeta_{3}\right)+T \gamma \eta[\sigma+\beta+\rho+\lambda]\right\} G+\eta \gamma T \alpha \leq G \tag{2.13}
\end{equation*}
$$

holds. Then, NIDE (1.1) has a periodic solution.
Proof. Let $M=\left\{\phi \in P_{T}:\|\phi\| \leq G\right\}$. Then, from Lemma 2.3 it follows that that the mapping $A: P_{T} \rightarrow P_{T}$, i.e., $A$ is compact and continuous. Similarly, from Lemma 2.2, it follows that the mapping $B, B: P_{T} \rightarrow P_{T}$, is a contraction. As the next step, if $\phi, \psi \in M$, then $\|A \phi+B \psi\| \leq G$.

Let $\|\phi\|,\|\psi\| \leq G$. Hence, it is obvious that

$$
\begin{aligned}
& \|(A \phi)(t)+(B \psi)(t)\| \\
& \quad \leq\left\{\left(\zeta_{1}+\zeta_{2}+\zeta_{3}\right)+T \gamma \eta[\sigma+\beta+\rho+\lambda]\right\} G+\eta \gamma T \alpha \leq G
\end{aligned}
$$

Therefore, on the set $M$, the conditions of Krasnoselskiis theorem hold. Hence, we conclude that there exists a fixed point $z \operatorname{In} M$ such that $z=A z+B z$. By Lemma 2.2, this fixed point is a solution of NIDE (1.1) and NIDE (1.1) has a $T$-periodic solution.

## 3. Conclusion

In this paper, a nonlinear NIDE with three variable delays is considered. We prove three lemmas and a theorem on the EPSs of the considered NIDE using Krasnoselskiis fixed point theorem. The results of this article have new extensions and contributions to the theory of the EPSs of nonlinear NIDEs.

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