

APPLICATIONS OF GENERALIZED FRACTIONAL OPERATORS IN SUBCLASSES OF UNIFORMLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce new subclasses of the class of uniformly convex functions, using the generalize fractional derivative and integral operators, introduced in 2020 by Issa and Darus. Then, we give coefficient inequalities and we study some analytic properties for this new subclasses.

1. INTRODUCTION

Fractional operators have an effect in exploring the geometric properties of univalent functions. This effect increases as the interest in fractional calculus and differential equations increases.

One of the early fractional operators that many mathematicians used in their research are the fractional integral and differential operators $F^\nu f$ and $G^\nu f$ introduced by Owa [9] as follows.

If we denote by A the class of univalent function and by U the open unit disk, then for a function $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in A we have:

$$F^\nu(g(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\nu)}{\Gamma(n+1-\nu)} a_n z^n,$$

and

$$G^\nu(g(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\nu)}{\Gamma(n+1+\nu)} a_n z^n.$$

These operators can be seen widely in research related to complex geometry, see for example [5, 7] and [8].

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In [4] Issa and Darus, give the following generalization fractional differential and integral operators that generalizes the operators $F^\nu f$ and $G^\nu f$ as follows:

Definition 1.1. Let $f(z)$ be an element belongs to the class A and let $m = 1, 2, 3, \dots$. Then the generalized fractional derivative operator, denoted by $D_z^{\nu,m}$, is

$$D_z^{\nu,m} (f(z)) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n, 0 \leq \nu < 1,$$

where, $m\nu \neq 2, 3, 4, \dots$

Remark. Note that $D_z^{0,m} f = f$ and $D_z^{\nu,1} f = F^\nu f$.

Definition 1.2. Let $f(z)$ be an element of the class A , and let $m = 1, 2, 3, \dots$. Then the generalized fractional Integral operator, denoted by $I_z^{\nu,m}$, is

$$\begin{aligned} I_z^{\nu,m} (f(z)) &= I_z^{\nu,m} \left(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+m\nu)}{\Gamma(n+m\nu+1)} a_n z^n, 0 \leq \nu < 1. \end{aligned} \quad (1.1)$$

Remark. Note that $I_z^{0,m} f = f$ and $I_z^{\nu,1} f = G^\nu f$.

Now, if we denote by T the class of univalent functions that has the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where this class defined by Silverman [6], then applying the previous generalized fractional and integral operators give us the following results.

Corollary 1.3. Let $f(z)$ be a function from the subclass T . Then

$$\begin{aligned} D_z^{\nu,m} (f(z)) &= D_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n, 0 \leq \nu < 1. \end{aligned} \quad (1.2)$$

Also, if we apply the integral operator in Definition 1.2 to the function f , we get,

$$\begin{aligned} I_z^{\nu,m} (f(z)) &= I_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+m\nu)}{\Gamma(n+m\nu+1)} a_n z^n, 0 \leq \nu < 1. \end{aligned} \quad (1.3)$$

In this paper, we give generalization for the class of uniformly convex function by introducing new parameters, where a function $f(z)$ in A is called uniformly convex functions denoted by UCV if the following analytic condition holds:

$$\operatorname{Re}\left\{1 + \frac{zf''}{f'}\right\} \geq \left|\frac{zf''}{f'}\right|, z \in U.$$

We should mention that the concepts of uniformly convex functions were established by Goodman [3].

Many subclasses of the uniformly convex class have been introduced and studied, for example the work by Bharati et al.[1] who obtained coefficient characterization

for some subclass of UCV for functions in the classes A and T . Also, one can refer to Darus et al.[2] who studied subclasses of uniformly convex functions introduced by certain integral operator.

In our work, we introduce new classes of uniformly convex functions using the generalized fractional derivative and integral operators introduced in Definitions 1.1 and 1.2. Also we study the properties of these new classes like distortion theory and coefficients bounds. Moreover, we will see that this new classes generalize the class UCV .

2. PROPERTIES AND GENERALIZATION OF THE SUBCLASS $UCVD_T(\nu, m, \sigma)$

In this section, we will introduce new subclasses of $UCV(\alpha)$ which we call $UCVD_T(\nu, m, \sigma)$ and we will calculate their coefficient criterion.

We will start with the following lemma which is very important in the construction of our work.

Lemma 2.1. *Let $f(z)$ belongs to the class T . Then if $D_z^{\nu, m} f(z) \in T$, we have*

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n \leq 1. \quad (2.1)$$

Proof Assume to the contrary and assume that z is real , then $\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n = 1 + \epsilon, \epsilon > 0$, hence there exist an integer N such that $\sum_{n=2}^N \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n > 1 + \frac{\epsilon}{2}$.

Now, for $\left(\frac{1}{1+\frac{\epsilon}{2}}\right)^{\frac{1}{N-1}} < z < 1$ we have

$$\begin{aligned} \frac{d}{dz} (D_z^{\nu, m} f(z)) &= 1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n z^{n-1} \\ &\leq 1 - \sum_{n=2}^N \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n z^{n-1} \\ &\leq 1 - z^{N-1} \sum_{n=2}^N \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n \\ &< 1 - z^{N-1} \left(1 + \frac{\epsilon}{2}\right) \\ &< 0. \end{aligned}$$

Secondly, since $\frac{d}{dz} (D_z^{\nu, m} f(0)) = 1 > 0$, there exist a real number $z_0, 0 < z_0 < \left(\frac{1}{1+\frac{\epsilon}{2}}\right)^{\frac{1}{N-1}}$ such that $\frac{d}{dz} (D_z^{\nu, m} f(z_0)) = 0$. Hence $D_z^{\nu, m} f$ is not univalent, which contradicts the assumption that $D_z^{\nu, m} f$ belongs to T .

Now, we introduce the new subclass $UCVD_T(\nu, m, \sigma)$.

Definition 2.2. *Let $UCVD_T(\nu, m, \sigma), 0 \leq \nu < 1, m = 1, 2, \dots$ and $0 \leq \sigma \leq 1$ be the class of functions $f \in T$, which satisfies the inequality*

$$\left| \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} \right| \leq \operatorname{Re} \left(1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right), z \in U$$

where $D_z^{\nu,m} f(z)$ is the generalized fractional derivative operator defined in Definition 1.1.

Note that: $UCVD_T(1, 1, 0) = UCV$.

Definition 2.3. Let $UCVI_T(\nu, m, \sigma), 0 \leq \nu < 1, m = 1, 2, \dots$ and $0 \leq \sigma \leq 1$ be the class of functions $f \in T$, which satisfies the inequality

$$\left| \frac{z(I_z^{\nu,m} f(z))''}{(I_z^{\nu,m} f(z))'} \right| \leq \operatorname{Re} \left(1 + \frac{z(I_z^{\nu,m} f(z))''}{(I_z^{\nu,m} f(z))'} - \sigma \right), z \in U$$

where $I_z^{\nu,m} f(z)$ is the generalized fractional integral operator defined in Definition 1.2.

Theorem 2.4. Let the function f be a function in T that has the form given in equation (1.2). Then $f \in UCVD_T(\nu, m, \sigma)$, and for $m = 1, 2, 3, \dots$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (2n-1-\sigma) na_n \leq 1-\sigma. \quad (2.2)$$

Proof Suppose that the inequality (2.2) holds, so

$$\left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| - \operatorname{Re} \left(\frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right) \leq 2 \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right|.$$

Which implies,

$$\begin{aligned} \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| - \operatorname{Re} \left(\frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right) &\leq 2 \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| \\ &= 2 \left| \frac{z \left(0 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-2} \right)}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \right| \\ &\leq \frac{2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n| |z|^{n-1}} \\ &\leq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} 2n(n-1) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n|}. \end{aligned}$$

In addition, by Lemma 2.1 we have $1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n > 0$, and by equation (2.2) the expression $\frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} 2n(n-1) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n|}$ is bounded by $1 - \sigma$, and hence the result. In the another direction, let $f \in UCVD_T(\nu, m, \sigma)$, and suppose

that z is real. Then by Definition 2.2 we have,

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} - \sigma \right) &\geq \left| \frac{z \left(-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1)a_n z^{n-2} \right)}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \right| \\ 1 + \frac{-\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} - \sigma &\geq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \\ 1 - \sigma &\geq 2 \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}}. \end{aligned}$$

Now, let $z \rightarrow 1-$ along the real axis, so our inequality become

$$\begin{aligned} 1 - \sigma &\geq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} 2n(n-1)a_n}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n} \\ (1 - \sigma) \left(1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n \right) &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} 2n(n-1)a_n \\ (1 - \sigma) - (1 - \sigma) \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} 2n(n-1)a_n \\ (1 - \sigma) &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} [2n(n-1) + (1 - \sigma)n] a_n \\ &= \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n [2n-1-\sigma] a_n, \end{aligned}$$

that completes the proof.

Remark. The result in Theorem 2.4 is sharp for all functions of the form:

$$F_n(z) = D_z^{\nu, m} f_n(z) = z - \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{1-\sigma}{n(2n-1-\sigma)} z^n, n \geq 2,$$

this function belongs to the class $UCVD_T(\nu, m, \sigma)$.

Corollary 2.5. If $f \in UCVD_T(\nu, m, \sigma)$, Then

$$a_n \leq \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{1-\sigma}{n(2n-1-\sigma)}, n \geq 2.$$

Proof Since $f \in UCVD_T(\nu, m, \sigma)$, then by Theorem 2.4 we have:

$$\begin{aligned} &\frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(2n-1-\sigma) a_n \\ &\leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(2n-1-\sigma) a_n \\ &\leq 1 - \sigma, \end{aligned}$$

consequently,

$$a_n \leq \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{1-\sigma}{n(2n-1-\sigma)}, n \geq 2.$$

Corollary 2.6. Let $f \in UCVD_T(\nu, m, \sigma)$ and let $|z| = r < 1$. Then:

- a. $|D_z^{\nu, m} f(z)| \leq r + \frac{1-\sigma}{2(3-\sigma)} r^2$.
- b. $|D_z^{\nu, m} f(z)| \geq r - \frac{1-\sigma}{2(3-\sigma)} r^2$.

Proof From Theorem 2.4, for functions f in $UCVD_T(\nu, m, \sigma)$:

$$\begin{aligned} & \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} 2(3-\sigma) \sum_{n=2}^{\infty} a_n \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(2n-1-\sigma) a_n \\ & \leq 1-\sigma. \end{aligned}$$

So, for a function f in $UCVD_T(\nu, m, \sigma)$, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\sigma}{2(3-\sigma)}.$$

To show (a) we will start by considering equation (1.2), so

$$\begin{aligned} |D_z^{\nu, m}(f(z))| &= \left| D_z^{\nu, m} \left(z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\ &= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, \quad 0 \leq \nu < 1 \\ &\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\ &\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\sigma}{2(3-\sigma)} \\ &= r + \frac{1-\sigma}{2(3-\sigma)} r^2. \end{aligned}$$

Secondly, to show (b) :

$$\begin{aligned}
|D_z^{\nu,m}(f(z))| &= \left| D_z^{\nu,m} \left(z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\
&= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, 0 \leq \nu < 1 \\
&\geq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\
&\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\
&\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\sigma}{2(3-\sigma)} \\
&= r - \frac{1-\sigma}{2(3-\sigma)} r^2.
\end{aligned}$$

Remark. (a) and (b) in the previous remark are sharp. to see this, take the function $F_2(z) = D_z^{\nu,m} f_2(z) = z - \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\sigma}{2(3-\sigma)} z^2$, at $z = \pm ir, \pm r$ which belongs to $UCVD_T(\nu, m, \sigma)$.

3. THE SUBCLASS $UCVD_T(\nu, m, \sigma, \gamma)$ AND ITS ANALYTICAL PROPERTIES

In the following definition we generalize the class $UCVD_T(\nu, m, \sigma)$ to the class $UCVD_T(\nu, m, \sigma, \gamma)$ by considering the new parameter γ .

Definition 3.1. Let $UCVD_T(\nu, m, \sigma, \gamma), 0 \leq \nu < 1, m = 1, 2, \dots, 0 \leq \sigma \leq 1$ and $0 \leq \gamma \leq 1$ be the class of functions $f \in T$, satisfying the condition:

$$\operatorname{Re} \left(1 + \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right) \geq \sigma \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| + \gamma, z \in U.$$

Note that $:UCVD_T(\nu, m, 1, \gamma) = UCVD_T(\nu, m, \gamma)$, and $UCVD_T(1, 1, 1, 0) = UCV$.

Theorem 3.2. Let the function f be a function in T . Then $f \in UCVD_T(\nu, m, \sigma, \gamma)$ for some $0 \leq \nu < 1, m = 1, 2, \dots, 0 \leq \sigma \leq 1$ and $0 \leq \gamma \leq 1$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n [n(1+\sigma) - (\sigma+\gamma)] a_n \leq 1 - \gamma. \quad (3.1)$$

Proof From Definition 3.1 it is enough to show that

$$\sigma \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| \leq \operatorname{Re} \left(\frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} + 1 \right) - \gamma,$$

now

$$\begin{aligned}
\sigma \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| - \operatorname{Re} \left(\frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right) &\leq (\sigma + 1) \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right| \\
&= (\sigma + 1) \left| \frac{z \left(- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-2} \right)}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \right| \\
&\leq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (\sigma + 1) n(n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n| |z|^{n-1}} \\
&\leq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (\sigma + 1) n(n-1) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n|},
\end{aligned}$$

by Lemma 2.1 we have $1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n > 0$, so the expression $\frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (\sigma + 1) n(n-1) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n|}$ is bounded by $1 - \gamma$ if and only if equation (3.1) holds and hence the result.

Conversely, let $f \in UCVDT(\nu, m, \sigma, \gamma)$, and suppose that z is real; then by Definition 3.1 we have,

$$\begin{aligned}
\left| 1 + \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} - \gamma \right| &\geq \operatorname{Re} \left(1 + \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right) - \gamma \\
&\geq \sigma \left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} \right|,
\end{aligned}$$

which gives,

$$\begin{aligned}
1 + \frac{- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} &\geq \sigma \left| \frac{- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \right| + \gamma \\
1 + \frac{- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} - \gamma &\geq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \\
1 - \gamma &\geq (1 + \sigma) \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}}.
\end{aligned}$$

Now, let $z \rightarrow 1-$ along the real axis. Then we get

$$\begin{aligned}
1 - \gamma &\geq (1 + \sigma) \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n} \\
(1 - \gamma) \left(1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n \right) &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (1 + \sigma) n(n-1) a_n \\
(1 - \gamma) - (1 - \gamma) \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} (1 + \sigma) n(n-1) a_n \\
(1 - \gamma) &\geq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n [n(1 + \sigma) - (\sigma + \gamma)] a_n,
\end{aligned}$$

that completes the proof.

Remark. *The result in Theorem 3.2 is sharp for all functions of the form*

$$F_n(z) = D_z^{\nu, m} f_n(z) = z - \frac{\Gamma(n - m\nu + 1)}{\Gamma(n + 1)\Gamma(2 - m\nu)} \frac{1 - \gamma}{n[n(1 + \sigma) - (\sigma + \gamma)]} z^n, n \geq 2,$$

where this function belongs to the class $UCVD_T(\nu, m, \sigma, \gamma)$.

Corollary 3.3. *Suppose that $f \in UCVD_T(\nu, m, \sigma, \gamma)$. Then*

$$a_n \leq \frac{\Gamma(n - m\nu + 1)}{\Gamma(n + 1)\Gamma(2 - m\nu)} \frac{1 - \gamma}{n[n(1 + \sigma) - (\sigma + \gamma)]}, n \geq 2.$$

Proof Since $f \in UCVD_T(\nu, m, \sigma, \gamma)$, then by Theorem 3.2 we have the following inequalities which give us the needed result:

$$\begin{aligned} & \frac{\Gamma(n + 1)\Gamma(2 - m\nu)}{\Gamma(n - m\nu + 1)} n[n(1 + \sigma) - (\sigma + \gamma)] a_n \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)\Gamma(2 - m\nu)}{\Gamma(n - m\nu + 1)} n[n(1 + \sigma) - (\sigma + \gamma)] a_n \\ & \leq 1 - \gamma, \end{aligned}$$

consequently,

$$a_n \leq \frac{\Gamma(n - m\nu + 1)}{\Gamma(n + 1)\Gamma(2 - m\nu)} \frac{1 - \gamma}{n[n(1 + \sigma) - (\sigma + \gamma)]}, n \geq 2.$$

Corollary 3.4. *Let $f \in UCVD_T(\nu, m, \sigma, \gamma)$ and let $|z| = r < 1$. Then:*

- a. $|D_z^{\nu, m} f(z)| \leq r + \frac{1-\gamma}{2(2+\sigma-\gamma)} r^2$.
- b. $|D_z^{\nu, m} f(z)| \geq r - \frac{1-\gamma}{2(2+\sigma-\gamma)} r^2$.

Proof From Theorem 3.2, for functions f in $UCVD_T(\nu, m, \sigma, \gamma)$:

$$\begin{aligned} & \frac{\Gamma(3)\Gamma(2 - m\nu)}{\Gamma(3 - m\nu)} 2(2(1 - \sigma)(\gamma + \sigma)) \sum_{n=2}^{\infty} a_n \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)\Gamma(2 - m\nu)}{\Gamma(n - m\nu + 1)} n(n - \gamma + \sigma) a_n \\ & \leq 1 - \gamma. \end{aligned}$$

So, for a function f in $UCVD_T(\nu, m, \sigma, \gamma)$, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\Gamma(3 - m\nu)}{\Gamma(3)\Gamma(2 - m\nu)} \frac{1 - \gamma}{2(2 + \sigma - \gamma)}.$$

Now, from (1.2) , we have the following inequalities that proves (a)

$$\begin{aligned}
|D_z^{\nu,m}(f(z))| &= \left| D_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\
&= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, 0 \leq \nu < 1 \\
&\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\
&\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\
&\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\gamma}{2(2+\sigma-\gamma)} \\
&= r + \frac{1-\gamma}{2(2+\sigma-\gamma)} r^2.
\end{aligned}$$

Secondly, to show (b) , from the inequality:

$$\begin{aligned}
|D_z^{\nu,m}(f(z))| &= \left| D_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\
&= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, 0 \leq \nu < 1 \\
&\geq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\
&\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\
&\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{1-\gamma}{2(2+\sigma-\gamma)} \\
&= r - \frac{1-\gamma}{2(2+\sigma-\gamma)} r^2,
\end{aligned}$$

hence we had shown both bounds.

Remark. The result in Corollary 3.4 is sharp for all functions of the form

$$F_2(z) = D_z^{\nu,m} f_2(z) = z - \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{1-\gamma}{2(2+\sigma-\gamma)} z^2, \text{ at } z = \pm ir, \pm r$$

where, this function belongs to the class $UCVD_T(\nu, m, \sigma, \gamma)$.

4. THE CLASS $CD_T(\nu, m, \sigma)$

Definition 4.1. Let $CD_T(\nu, m, \sigma)$, $0 \leq \nu < 1$, $m = 1, 2, \dots$, and $0 \leq \sigma \leq 1$ be the subclass of T that satisfies the following inequality:

$$\left| \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} + 1 - \sigma \right| \leq \operatorname{Re} \left\{ \frac{z(D_z^{\nu,m} f(z))''}{(D_z^{\nu,m} f(z))'} + 1 + \sigma \right\}, z \in U.$$

Theorem 4.2. Let the function f be a function in T . Then $f \in CD_T(\nu, m, \sigma)$, for some $0 \leq \nu < 1, m = 1, 2, \dots, 0 \leq \sigma \leq 1$ and $0 \leq \gamma \leq 1$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[n-1+\sigma] a_n \leq \sigma. \quad (4.1)$$

Proof Let $f \in T$ and assume that $\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[n-1+\sigma] a_n \leq \sigma$, we need to show that $f \in CD_T(\nu, m, \sigma)$, so we need to show the inequality $\left| \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} + 1 - \sigma \right| \leq \operatorname{Re} \left\{ 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} + 1 + \sigma \right\}, z \in U$, which is equivalent to showing that

$$\left| 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right| - \operatorname{Re} \left\{ 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right\} \leq 2\sigma.$$

Now

$$\begin{aligned} & \left| 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right| - \operatorname{Re} \left\{ 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right\} \\ & \leq 2 \left| 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - \sigma \right| \\ & = 2 \left| 1 + \frac{z \left(- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-2} \right)}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} - \sigma \right| \\ & \leq 2 \left\{ \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n| |z|^{n-1}} + 1 - \sigma \right\} \\ & \leq 2 \left\{ \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n |a_n|} + 1 - \sigma \right\}, \end{aligned}$$

by Lemma 2.1 we have $1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n > 0$, so the last expression is bounded by 2σ if and only if (4.1) holds and hence the result. Conversely, let $f \in CD_T(\nu, m, \sigma)$, and suppose that z is real; then by Definition 4.1 we have,

$$\begin{aligned} & 1 + \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} \geq 1 - \frac{z(D_z^{\nu, m} f(z))''}{(D_z^{\nu, m} f(z))'} - 2\sigma \\ & 1 + \frac{- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \geq 1 - \frac{- \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n n z^{n-1}} - 2\sigma \\ & 1 - \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n a_n z^{n-1}} \geq 1 + \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n n z^{n-1}} - 2\sigma, \end{aligned}$$

now, let $z \rightarrow 1-$ along the real axis. Then we get

$$\begin{aligned} & 2 \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n} \leq 2\sigma \\ & \frac{\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n} \leq \sigma \\ & \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n(n-1) a_n \leq \sigma \left(1 - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} na_n \right) \\ & \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} [n(n-1) + n\sigma] a_n \leq \sigma \\ & \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[(n-1) + \sigma] a_n \leq \sigma. \end{aligned}$$

Hence the result.

Remark. The result in Theorem 4.2 is sharp for all functions of the form

$$F_n(z) = D_z^{\nu,m} f_n(z) = z - \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{\sigma}{n[n-1+\sigma]} z^n, \quad n \geq 2,$$

this function belongs to the class $CD_T(\nu, m, \sigma)$.

Corollary 4.3. Suppose that $f \in CD_T(\nu, m, \sigma)$. Then

$$a_n \leq \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{\sigma}{n[n-1+\sigma]}, \quad n \geq 2.$$

Proof Since $f \in CD_T(\nu, m, \sigma)$, then by Theorem 4.2 we have the following inequalities which give us the needed result:

$$\begin{aligned} & \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[n-1+\sigma] a_n \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[n-1+\sigma] a_n \\ & \leq \sigma, \end{aligned}$$

consequently,

$$a_n \leq \frac{\Gamma(n-m\nu+1)}{\Gamma(n+1)\Gamma(2-m\nu)} \frac{\sigma}{n[n-1+\sigma]}, \quad n \geq 2.$$

Corollary 4.4. Let $f \in CD_T(\nu, m, \sigma)$ and let $|z| = r < 1$. Then:

- (1) $|D_z^{\nu,m} f(z)| \leq r + \frac{\sigma}{2(1+\sigma)} r^2$.
- (2) $|D_z^{\nu,m} f(z)| \geq r - \frac{\sigma}{2(1+\sigma)} r^2$.

Proof: From Theorem 4.2, for functions f in $CD_T(\nu, m, \sigma)$:

$$\begin{aligned} & \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} 2(1+\sigma) \sum_{n=2}^{\infty} a_n \\ & \leq \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} n[n-1+\sigma] a_n \\ & \leq \sigma. \end{aligned}$$

So, for a function f in $CD_T(\nu, m, \sigma)$, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{\sigma}{2(1+\sigma)}.$$

Now, from (1.2), we have the following inequalities that proves (a)

$$\begin{aligned} |D_z^{\nu,m}(f(z))| &= \left| D_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\ &= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, \quad 0 \leq \nu < 1 \\ &\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\ &\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{\sigma}{2(1+\sigma)} \\ &= r + \frac{\sigma}{2(1+\sigma)} r^2. \end{aligned}$$

Secondly, to show the second inequality:

$$\begin{aligned} |D_z^{\nu,m}(f(z))| &= \left| D_z^{\nu,m} \left(f(z) = z - \sum_{n=2}^{\infty} a_n z^n \right) \right| \\ &= \left| z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n z^n \right|, \quad 0 \leq \nu < 1 \\ &\geq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-m\nu)}{\Gamma(n-m\nu+1)} a_n |z|^n \\ &\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\Gamma(3)\Gamma(2-m\nu)}{\Gamma(3-m\nu)} |z|^2 \frac{\Gamma(3-m\nu)}{\Gamma(3)\Gamma(2-m\nu)} \frac{\sigma}{2(1+\sigma)} \\ &= r - \frac{\sigma}{2(1+\sigma)} r^2, \end{aligned}$$

hence we had shown both bounds.

Remark. The result in Corollary 4.4 is sharp for all functions of the form

$$F_2(z) = D_z^{\nu, m} f_2(z) = z - \frac{\Gamma(3 - m\nu)}{\Gamma(3)\Gamma(2 - m\nu)} \frac{\sigma}{2(1 + \sigma)} z^2, \text{ at } z = \pm ir, \pm r.$$

This function belongs to the class $CD_T(\nu, m, \sigma)$.

Remark. By taking $\nu = -\nu$ in the previous results, gives us new subclasses and results of uniformly convex functions. are also obtained.

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