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GENERALIZED WEIGHTED HARDY OPERATORS AND THEIR COMMUTATORS IN THE LOCAL "COMPLEMENTARY" GENERALIZED VARIABLE EXPONENT WEIGHTED MORREY SPACES

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Abstract. In this paper we study the boundedness of weighted Hardy operators H_u^α and $\mathcal{H}_u^\alpha$ in local "complementary" generalized variable exponent weighted Morrey spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$, characterized by variable exponent $p(x)$, a general function $\omega(r)$ and a weight φ. We also study the boundedness of the commutators of H_u^α and $\mathcal{H}_u^\alpha$ in the spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$.

1. Introduction

Let u denote a weight function on $(0,\infty)$, i.e. a positive measurable function on $(0,\infty)$. We consider the following generalized weighted Hardy operators

$$H_u^\alpha f(x) = |x|^{\alpha-n} u(|x|) \int_{|y|\le|x|} \frac{f(y)}{u(|y|)} dy,$$

$$\mathcal{H}_u^\alpha f(x) = |x|^{\alpha} u(|x|) \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n u(|y|)} dy,$$

where $\alpha \ge 0$.

Given a measurable function b, the commutators $[b, H_u^\alpha]$ and $[b, \mathcal{H}_u^\alpha]$ are defined by

$$[b, H_u^\alpha] f(x) = |x|^{\alpha-n} u(|x|) \int_{|y|\le|x|} [b(x)-b(y)] \frac{f(y)}{u(|y|)} dy,$$

$$[b, \mathcal{H}_u^\alpha] f(x) = |x|^{\alpha} u(|x|) \int\limits_{|y|>|x|} [b(x)-b(y)] \frac{f(y)}{|y|^n u(|y|)} dy$$

respectively.

We will study the boundedness of generalized weighted Hardy operators H_u^α, $\mathcal{H}_u^\alpha$ and their commutators in local "complementary" generalized variable exponent

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weighted Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{0\}}$, where $x_0 \in \mathbb{R}^n$, $1 \le p_- \le p(x) \le p_+ < \infty$, for which

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{x_0\}}} = \sup_{r>0} \frac{\|\varphi\|_{L^{p'(\cdot)}(B(x_0,r))}}{\omega(r)} \|f\|_{L^{p(\cdot)}_{\varphi}(\Omega\setminus B(x_0,r))} < \infty.$$

In 1938, C. Morrey [24] considered the integral growth condition on derivatives over balls, in order to study the existence and regularity for partial differential equations. A family of functions with the integral growth condition is then called a Morrey space after his name. Until recently, a rapid growth has been seen in the study of Morrey type spaces because of its applications in major fields of engineering and sciences. Function spaces with non-standard growth has seen a major focus in recent times because of its wide range of applications in the area of image processing, the study of thermorheological fluids and modeling of electrorheological fluids. It would be next to impossible to give a complete account of the literature which is available to this subject. Let us quote at least the long series of papers (see [7, 10, 11, 12, 13, 14, 16, 19, 22, 23, 20, 21, 25, 26, 27]). Nowadays there is an evident increase of investigations, related to variable exponent Morrey spaces. Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ were introduced and studied in [1] in the Euclidean setting. In [1] the boundedness of the maximal operator was proved in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$, and for potential operators a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$- theorem was proved under the same log-condition in the case of bounded set $\Omega \subset \mathbb{R}^n$.

The generalized variable exponent Morrey spaces were introduced and studied in [10] in the case of bounded sets. In [10] the boundedness of the maximal operator, potential operators and singular integral operators in variable exponent Morrey spaces under the certain conditions were proved. In [10] the authors studied the boundedness of the classical integral operators in the generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ over an open bounded set $\Omega \subset \mathbb{R}^n$. Generalized Morrey spaces of such a kind in the case of constant p, with the norm

$$\|f\|_{\mathcal{M}^{p,\omega}(\mathbb{R}^n)} := \sup_{x,\, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)} \|f\|_{L^p(B(x,r))}, \tag{1.1}$$

under some assumptions on ω were studied in [8], [19], [22].

In [9] there were introduced and studied local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$ with constant p, the space of all functions $f \in L^p(\mathbb{R}^n\setminus B(x_0,r))$, $r > 0$ with finite norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\mathbb{R}^n\setminus B(x_0,r))}.$$

We denote by [9] the local "complementary" Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ with constant p, the space of all functions $f \in L^p(\mathbb{R}^n\setminus B(x_0,r))$, $r > 0$ with the finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(\mathbb{R}^n\setminus B(x_0,r))} < \infty, \quad x_0 \in \mathbb{R}^n,$$

where $1 \le p < \infty$ and $0 \le \lambda < n$. Note that ${}^{\complement}\mathcal{L}^{p,0}_{\{x_0\}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Boundedness of maximal operators, singular integral operators and their commutators in the local "complementary" generalized variable exponent Morrey spaces

${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$ on unbounded sets was proved in [2] and boundedness of potential operators and their commutators in the spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$ on unbounded sets was proved in [3].

In this paper we consider local "complementary" generalized variable exponent weighted Morrey spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$ characterized by variable exponent $p(x)$, a general function $\omega(r)$ and a weight φ, see Definition 3.2.

We find the conditions on the functions (ω_1, ω_2) for the boundedness of the weighted Hardy operators H_u^α, $\mathcal{H}_u^\alpha$ and their commutators in local "complementary" generalized variable exponent weighted Morrey spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we give the definition and basic properties of in local "complementary" generalized variable exponent weighted Morrey spaces ${}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega,\varphi}$. In Section 4 we deal with the weighted Hardy operators and their commutators.

The main results are given in Theorems 4.1, 4.2, 4.3, 4.4. We emphasize that the results we obtain for generalized Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\omega(r)$.

We use the following notation: Everywhere in the sequel the functions $\omega(r)$, $\omega_1(r)$ and $\omega_2(r)$ used in the body of the paper, are non-negative measurable function on $(0,\infty)$. $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\})$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c,c_1, c_2, C etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries

In this section we give the definitions and some basic facts of variable exponent Lebesgue and Morrey spaces. We also give the definition of BMO spaces. We refer to the sources [5, 18, 28] for variable exponent Lebesgue spaces.

Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with values in $[1,\infty)$. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1, \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$

By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the space of all measurable functions $f(x)$ on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \mathbb{R}^n$, we denote the conjugate exponent.

$\mathcal{P}(\mathbb{R}^n)$ is the set of bounded measurable functions $p : \mathbb{R}^n \to [1,\infty)$;

$\mathcal{P}^{log}(\mathbb{R}^n)$ is the set of exponents $p \in \mathcal{P}(\mathbb{R}^n)$ satisfying the local log-condition

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \quad x, y \in \mathbb{R}^n, \tag{2.2}$$

where $A = A(p) > 0$ does not depend on x, y.

We will use also the following decay conditions:

$$|p(x) - p(0)| \le \frac{A_0}{|\ln|x||}, \quad |x| \le \frac{1}{2}, \tag{2.3}$$

$$|p(x) - p(\infty)| \le \frac{A_\infty}{|\ln|x||}, \quad |x| \ge 2, \tag{2.4}$$

where $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$.

$\mathcal{A}^{log}(\mathbb{R}^n)$ is the set of bounded exponents $p : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition (2.2); $\mathbb{P}^{log}(\mathbb{R}^n)$ is the set of exponents $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $1 < p_- \le p(x) \le p_+ < \infty$; for $\mathbb{R}^n$ which may be unbounded, by $\mathcal{P}_\infty(\mathbb{R}^n)$, $\mathcal{P}^{log}_\infty(\mathbb{R}^n)$, $\mathbb{P}^{log}_\infty(\mathbb{R}^n)$, $\mathcal{A}^{log}_\infty(\mathbb{R}^n)$ we denote the subsets of the above sets of exponents satisfying the decay condition (2.4).

For brevity, by $\mathbb{P}^{log}_{0,\infty}(\mathbb{R}^n)$ we denote the set of bounded measurable functions (not necessarily with values in $[1,\infty)$), which satisfy the decay conditions (2.3) and (2.4).

We will also make use of the estimate provided by the following lemma (see [5], Corollary 4.5.9).

$$\|\chi_{B(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\theta_p(x,r)}, \qquad x \in \mathbb{R}^n, \ p \in \mathbb{P}^{log}_\infty(\mathbb{R}^n), \tag{2.5}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r > 1 \end{cases}$.

By φ we always denote a weight, i.e. a positive, locally integrable function with $\mathbb{R}^n$. The weighted Lebesgue space $L^{p(\cdot)}_\varphi(\mathbb{R}^n)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_\varphi(\mathbb{R}^n)} = \|f\varphi\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Let us define the class $A_{p(\cdot)}(\mathbb{R}^n)$ (see [6], [17]) to consist of those weights φ for which

$$[\varphi]_{A_{p(\cdot)}} \equiv \sup_B |B|^{-1} \|\varphi\|_{L^{p(\cdot)}(B(x,r))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(x,r))} < \infty.$$

A weight function φ belongs to the class $A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if

$$[\varphi]_{A_{p(\cdot),q(\cdot)}} \equiv \sup_B r^{\theta_p(x,r)-\theta_q(x,r)-n} \|\varphi\|_{L^{q(\cdot)}(B(x,r))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(x,r))} < \infty.$$

Let $\lambda(x)$ be a measurable function on $\mathbb{R}^n$ with values in $[0,n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is defined as the set of integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda(x)\theta_p(x,r)}{n}} \|f\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

Definition 2.1. *We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that*

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where $f_{B(x,t)}(x) = |B(x,t)|^{-1} \int_{B(x,t)} f(y)dy$.

Definition 2.2. *We define the* $BMO_{p(\cdot),\varphi}(\mathbb{R}^n)$ *space as the set of all locally integrable functions* f *such that*

$$\|f\|_{BMO_{p(\cdot),\varphi}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n)}} < \infty.$$

Theorem 2.3. [15] *Let* $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ *and* φ *be a Lebesgue measurable function. If* $\varphi \in A_{p(\cdot)}(\mathbb{R}^n)$*, then the norms* $\|\cdot\|_{BMO_{p(\cdot),\varphi}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

Definition 2.4. *We define the* $BMO_{0,p(\cdot),\varphi}(\mathbb{R}^n)$ *space as the set of all local integrable functions* f *such that*

$$\|f\|_{BMO_{0,p(\cdot),\varphi}} = \sup_{r>0,\, \sigma\in S^{n-1}} \frac{\|(f(\cdot) - f(r\sigma))\chi_{B(0,r)}\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n)}}{\|\varphi\chi_{B(0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} < \infty.$$

3. Variable exponent local "complementary" generalized Morrey spaces

The local generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n)$ and local generalized weighted Morrey spaces $\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ with variable exponent are defined (see [10]) by the norms

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{-\theta_p(x_0,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(B(x_0,r))}$$

and

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{x_0\}}} = \sup_{r>0} \frac{1}{\omega(r)\|\varphi\|_{L^{p(\cdot)}(B(x_0,r))}} \|f\|_{L^{p(\cdot)}_{\varphi}(B(x_0,r))},$$

where $x_0 \in \mathbb{R}^n$ and $1 \le p_- \le p(x) \le p_+ < \infty$ for all $x \in \mathbb{R}^n$.

We find it convenient to introduce the variable exponent version of the local "complementary" space as follows.

Definition 3.1. *Let* $x_0 \in \mathbb{R}^n$, $1 \le p_- \le p(x) \le p_+ < \infty$. *Variable exponent local "complementary" Morrey spaces* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}_{\{x_0\}}(\mathbb{R}^n)$ *are defined by the norm*

$$\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{\lambda(x)\theta_{p'}(x_0,t)}{n}} \|f\chi_{\mathbb{R}^n\setminus B(x_0,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Definition 3.2. *Let* $x_0 \in \mathbb{R}^n$, $1 \le p_- \le p(x) \le p_+ < \infty$. *Local "complementary" generalized variable exponent Morrey spaces* ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n)$ *and local "complementary" generalized variable exponent weighted Morrey spaces* ${}^{\complement}\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ *are defined by the norms*

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus B(x_0,r))},$$

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{x_0\}}} = \sup_{r>0} \frac{\|\varphi\|_{L^{p'(\cdot)}(B(x_0,r))}}{\omega(r)} \|f\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n\setminus B(x_0,r))}$$

respectively.

Everywhere in the sequel we assume that

$$\sup_{r>0} \frac{\|\varphi\|_{L^{p'(\cdot)}(B(x_0,r))}}{\omega(r)} < \infty, \tag{3.1}$$

which makes the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n)$ non-trivial, since it contains $L^{p(\cdot)}(\mathbb{R}^n)$ in this case.

If also $\inf\limits_{r>0} \frac{1}{\omega(r)} > 0$, then ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, to guarantee that the "complementary" space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n)$ is strictly larger than $L^p(\mathbb{R}^n)$, one should be interested in the cases where

$$\lim_{r\to 0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} = 0. \tag{3.2}$$

Clearly, the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\mathbb{R}^n)$ may contain functions with a non-integrable singularity at the point x_0, if no additional assumptions are introduced.

Remark. *The introduction of global "complementary" Morrey-type spaces has no big sense, neither in case of constant exponents, nor in the case of variable exponents. In the case of constant exponents this was noted in* [4]*, pp 19-20; in this case the global space defined by the norm*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\mathbb{R}^n\setminus B(x,r))}$$

reduces to $L^p(\mathbb{R}^n)$ *under the assumption* (3.1)*. In the case of variable exponents there happens the same. In general, to make it clear, note that for instance under the assumption* (3.2) *if we admit that* $\sup\limits_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} \|f\|_{L^p(\mathbb{R}^n\setminus B(x,r))}$ *for two different points* $x = x_0$ *and* $x = x_1$, $x_0 \neq x_1$*, this would immediately imply that* $f \in L^{p(\cdot)}$ *in a neighbourhood of both the points* x_0 *and* x_1*.*

4. Weighted Hardy operator in the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega,\varphi}_{\{0\}}(\mathbb{R}^n)$

The proof of the main result of this section presented in Theorem 4.1 is based on the estimate given in the following preliminary theorem.

Theorem 4.1. *Let* $p \in \mathbb{P}^{log}_{0,\infty}(\mathbb{R}^n)$, $\varphi \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*. Suppose also that function* $u(r)r^\gamma$, $\gamma > 0$ *almost increasing and the functions* (ω_1, ω_2) *satisfy the conditions*

$$\int_0^t \frac{s^{\alpha-\theta_p(0,s)+\theta_q(0,s)}\omega_1(s)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\omega_2(t)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,s))}} \tag{4.1}$$

$$\int_0^t \frac{\omega_1(s)}{u(s)\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\omega_1(t)}{u(t)\|\varphi\|_{L^{p'(\cdot)}(B(0,t))}}, \tag{4.2}$$

where $t > 0$*.*

Then the weighted Hardy operator H^α_u *is bounded from the space* ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)$ *to the space* ${}^{\complement}\mathcal{M}^{q(\cdot),\omega_2,\varphi}_{\{0\}}(\mathbb{R}^n)$*.*

Proof. Let $f \in {}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)$, $\beta > 0$. We have

$$\begin{aligned}\int\limits_{|z|<r} \frac{|f(z)|}{u(|z|)}dz &= \int\limits_{B(0,r)} \frac{|f(z)|}{u(|z|)}dz \\ &= \beta \int_{B(0,r)} \frac{|z|^{-\beta}|f(z)|}{u(|z|)} \left(\int_0^{|z|} s^{\beta-1}ds\right) dz \\ &= \beta \int_0^r s^{\beta-1} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} \frac{|z|^{-\beta}|f(z)|}{u(|z|)}dz\right) ds.\end{aligned}$$

Since $\frac{1}{r^\beta u(r)} \le \frac{C}{t^\beta u(t)}$, $0 < t < r$, and applying Hölder inequality, we get

$$\int\limits_{|z|<r} \frac{|f(z)|}{u(|z|)}dz \le C \int_0^r \frac{1}{u(s)} \|f\|_{L^{p(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,s))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,r))} \frac{ds}{s}.$$

Then we get

$$\int\limits_{|z|<r} \frac{|f(z)|}{u(|z|)}dz \le C \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,r))} \int_0^r \frac{1}{u(s)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus B(0,s))} \frac{ds}{s}. \tag{4.3}$$

Therefore by (4.1), (4.2) and (4.3) we have

$$\begin{aligned}&\|H_u^\alpha f\|_{L^{q(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,t))} \\ &\le C \left\| \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,|\cdot|))} |\cdot|^{\alpha-n} u(|\cdot|) \int_0^{|\cdot|} \frac{1}{u(s)} \|f\|_{L^{p(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,s))} \frac{ds}{s} \right\|_{L^{q(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,t))} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)} u(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}} \int_0^{|\cdot|} \frac{\omega_1(s)}{u(s)\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \right\|_{L^{q(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,t))} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)} u(|\cdot|)\omega_1(|\cdot|)}{u(|\cdot|)\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}} \right\|_{L^{q(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,t))} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)} \omega_1(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \right\|_{L^{q(\cdot),\varphi}(\mathbb{R}^n\setminus B(0,t))} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \int_0^t \frac{s^{\alpha-\theta_p(0,s)+\theta_q(0,s)}\omega_1(s)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \\ &\le C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \frac{\omega_2(t)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,s))}}.\end{aligned}$$

□

Theorem 4.2. *Let $p \in \mathbb{P}^{log}_{0,\infty}(\mathbb{R}^n)$, $\varphi \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $b \in BMO_{0,p'(\cdot),\varphi^{-1}}(\mathbb{R}^n)$. Suppose also that function $u(r)r^\gamma$, $\gamma > 0$ almost increasing and the functions (ω_1, ω_2) satisfy the conditions (4.1) and (4.2).*

Then the commutator of weighted Hardy operator H_u^α, $[b, H_u^\alpha]$, is bounded from the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)$ to the space ${}^{\complement}\mathcal{M}^{q(\cdot),\omega_2,\varphi}_{\{0\}}(\mathbb{R}^n)$.

Proof. Let $f \in {}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{0\}}(\mathbb{R}^n)$, $\beta > 0$. We have

$$\begin{aligned}
&\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{u(|z|)}dz \\
&= \int\limits_{B(0,r)} \frac{|b(z)-b(x)||f(z)|}{u(|z|)}dz \\
&= \beta \int_{B(0,r)} \frac{|z|^{-\beta}|b(z)-b(x)||f(z)|}{u(|z|)} \left(\int_0^{|z|} s^{\beta-1}ds\right) dz \\
&= \beta \int_0^r s^{\beta-1} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} \frac{|z|^{-\beta}|b(z)-b(x)||f(z)|}{u(|z|)}dz\right) ds.
\end{aligned}$$

Since $\frac{r^{-\beta}}{u(r)} \le \frac{Ct^{-\beta}}{u(t)}$, $0 < t < r$ and applying Hölder inequality, we get

$$\begin{aligned}
&\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{u(|z|)}dz \\
&\le C\int_0^r \frac{1}{u(s)} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} |b(z)-b(x)||f(z)|dz\right) \frac{ds}{s} \\
&\le C\int_0^r \frac{1}{u(s)} \|f\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n\setminus B(0,s))} \|b(\cdot)-b(x)\|_{L^{p'(\cdot)}_{\varphi^{-1}}(B(0,r))} \frac{ds}{s}.
\end{aligned}$$

Then, we obtain

$$\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{u(|z|)}dz$$

$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,r))} \int_0^r \frac{1}{u(t)} \|f\|_{L^{p(\cdot)}_{\varphi}(\mathbb{R}^n\setminus B(0,t))} \frac{dt}{t}. \tag{4.4}$$

Therefore by (4.4), we get

$$\|[b,H_u^\alpha]f\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,\rho))}\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}$$
$$\times\left\|\,|\cdot|^{\alpha-n}u(|\cdot|)\|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,|\cdot|))}\int_0^{|\cdot|}\frac{1}{u(t)}\,\|f\|_{L_\varphi^{p(\cdot)}(\mathbb{R}^n\setminus B(0,t))}\frac{dt}{t}\right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\|f\|_{\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)}$$
$$\times\left\|\frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)}u(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}}\int_0^{|\cdot|}\frac{\omega_1(t)}{u(t)\|\varphi\|_{L^{p'(\cdot)}(B(0,t))}}\frac{dt}{t}\right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\|f\|_{\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)}$$
$$\times\left\|\frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)}u(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}}\frac{\omega_1(|\cdot|)}{u(|\cdot|)\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}}\right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\|f\|_{\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)}\left\|\frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)}\omega_1(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}}\right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\|f\|_{\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)}\int_0^\rho\frac{t^{\alpha-\theta_p(0,t)+\theta_q(0,t)}\omega_1(t)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,t))}}\frac{dt}{t}$$
$$\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\|f\|_{\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)}\frac{\omega_2(\rho)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,\rho))}}.$$

□

Theorem 4.3. *Let $p\in\mathbb{P}_{0,\infty}^{log}(\mathbb{R}^n)$, $\varphi\in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ Suppose also that function u almost decreasing and the functions (ω_1,ω_2) satisfy the conditions*

$$\int_0^t\frac{s^{\alpha-\theta_p(0,s)+\theta_q(0,s)}\omega_1(s)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}}\frac{ds}{s}\le C\frac{\omega_2(t)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,t))}},\tag{4.5}$$

$$\int_t^\infty\frac{s^{-\theta_p(0,s)+\theta_q(0,s)-1}}{u(s)\|\varphi\|_{L^{q(\cdot)}(B(0,s))}}ds\le C\frac{t^{-\theta_p(0,t)+\theta_q(0,t)}}{u(t)\|\varphi\|_{L^{q(\cdot)}(B(0,t))}}\tag{4.6}$$

where $t>0$.

Then the weighted Hardy operator $\mathcal{H}_u^\alpha$ is bounded from the space $\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)$ to the space $\complement\mathcal{M}_{\{0\}}^{q(\cdot),\omega_2,\varphi}(\mathbb{R}^n)$.

Proof. Let $f\in\complement\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1}(\mathbb{R}^n)$. We have

$$\int\limits_{|z|>r}\frac{|f(z)|}{|z|^nu(|z|)}dz=\int\limits_{\mathbb{R}^n\setminus B(0,r)}\frac{|f(z)|}{|z|^nu(|z|)}dz$$
$$\le C\int_{\mathbb{R}^n\setminus B(0,r)}\frac{|f(z)|}{u(|z|)}\left(\int_{|z|}^\infty s^{-n-1}ds\right)dz$$
$$\le C\int_r^\infty s^{-n-1}\left(\int_{\{z\in\mathbb{R}^n:r<|z|<s\}}\frac{|f(z)|}{u(|z|)}dz\right)ds.$$

Hence applying Hölder inequality we get

$$\int\limits_{|z|>r} \frac{|f(z)|}{|z|^n u(|z|)} dz \le C \int_r^\infty \frac{s^{-n-1}}{u(s)} \|f\|_{L^{p(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,r))} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,s))} ds. \quad (4.7)$$

Therefore by (4.7), we obtain

$$\begin{aligned}
&\|\mathcal{H}^\alpha_u f\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C \left\| |\cdot|^\alpha u(|\cdot|)\|f\|_{L^{p(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,|\cdot|))} \int_{|\cdot|}^\infty \frac{s^{-n-1}}{u(s)} \|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,s))} ds \right\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C \left\| |\cdot|^\alpha u(|\cdot|)\|f\|_{L^{p(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,|\cdot|))} \int_{|\cdot|}^\infty \frac{s^{-\theta_p(0,s)+\theta_q(0,s)-1}}{u(s)\|\varphi\|_{L^{q(\cdot)}(B(0,s))}} ds \right\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|f\|_{\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^\alpha u(|\cdot|)\omega_1(|\cdot|)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \int_{|\cdot|}^\infty \frac{s^{-\theta_p(0,s)+\theta_q(0,s)-1}}{u(s)\|\varphi\|_{L^{q(\cdot)}(B(0,s))}} ds \right\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|f\|_{\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)} u(|\cdot|)\omega_1(|\cdot|)}{u(|\cdot|)\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \right\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|f\|_{\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)} \omega_1(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \right\|_{L^{q(\cdot)}_\varphi(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|f\|_{\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \int_0^t \frac{s^{\alpha-\theta_p(0,s)+\theta_q(0,s)}\omega_1(s)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \\
&\le C\|f\|_{\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)} \frac{\omega_2(t)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,t))}}.
\end{aligned}$$

□

Theorem 4.4. *Let $p \in \mathbb{P}^{log}_{0,\infty}(\mathbb{R}^n)$, $\varphi \in A_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Suppose also that function u almost decreasing, $b \in BMO_{0,p'(\cdot),\varphi^{-1}}(\mathbb{R}^n)$ and the functions (ω_1, ω_2) satisfy the conditions* (4.5) *and* (4.6).

Then the commutator of weighted Hardy operator $\mathcal{H}^\alpha_u$, $[b, \mathcal{H}^\alpha_u]$, is bounded from the space $\complement\mathcal{M}^{p(\cdot),\omega_1,\varphi}_{\{0\}}(\mathbb{R}^n)$ to the space $\complement\mathcal{M}^{q(\cdot),\omega_2,\varphi}_{\{0\}}(\mathbb{R}^n)$.

Proof. Let $f \in {}^{\complement}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\int\limits_{|z|>r} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^n u(|z|)}dz \\
&= \int\limits_{\mathbb{R}^n\setminus B(0,r)} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^n u(|z|)}dz \\
&\le C\int_{\mathbb{R}^n\setminus B(0,r)} \frac{|b(z)-b(x)|\,|f(z)|}{u(|z|)}\left(\int_{|z|}^{\infty} s^{-n-1}ds\right)dz \\
&\le C\int_r^{\infty} s^{-n-1}\left(\int_{\{z\in\mathbb{R}^n: r<|z|<s\}} \frac{|b(z)-b(x)|\,|f(z)|}{u(|z|)}dz\right)ds \\
&\le C\int_r^{\infty} \frac{s^{-n-1}}{u(s)}\left(\int_{\{z\in\mathbb{R}^n: r<|z|<s\}} |b(z)-b(x)|\,|f(z)|dz\right)ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\{z\in\mathbb{R}^n: r<|z|<s\}} |b(z)-b(x)|\,|f(z)|dz \\
&\le \int_{\{z\in\mathbb{R}^n: r<|z|<s\}} \|b(\cdot)-b(x)\|_{L^{p'(\cdot),\varphi^{-1}}(B(0,s))}\,\|f\|_{L_{\varphi}^{p(\cdot)}(\mathbb{R}^n\setminus B(0,r))} \\
&\le \|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,s))}\|f\|_{L_{\varphi}^{p(\cdot)}(\mathbb{R}^n\setminus B(0,r))},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int\limits_{|z|>r} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^n u(|z|)}dz \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{L_{\varphi}^{p(\cdot)}(\mathbb{R}^n\setminus B(0,r))}\int_r^{\infty} \frac{s^{-n}\|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,s))}}{u(s)}\frac{ds}{s}. \qquad (4.8)
\end{aligned}$$

Therefore by (4.8), we get

$$\begin{aligned}
\|[b,\mathcal{H}_u^\alpha]f\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} &\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}} \\
&\times \left\| |\cdot|^\alpha u(|\cdot|)\|f\|_{L_\varphi^{p(\cdot)}(\mathbb{R}^n\setminus B(0,|\cdot|))} \int_{|\cdot|}^\infty \frac{s^{-n-1}\,\|\varphi^{-1}\|_{L^{p'(\cdot)}(B(0,s))}}{u(s)}ds \right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}} \\
&\times \left\| |\cdot|^\alpha u(|\cdot|)\|f\|_{L_\varphi^{p(\cdot)}(\mathbb{R}^n\setminus B(0,|\cdot|))} \int_{|\cdot|}^\infty \frac{s^{-\theta_p(0,s)+\theta_q(0,s)-1}}{u(s)\|\varphi\|_{L^{q(\cdot)}(B(0,s))}}ds \right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{\mathtt{C}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)} \\
&\times \left\| \frac{|\cdot|^\alpha u(|\cdot|)\omega_1(|\cdot|)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \int_{|\cdot|}^\infty \frac{s^{-\theta_p(0,s)+\theta_q(0,s)-1}}{u(s)\|\varphi\|_{L^{q(\cdot)}(B(0,s))}}ds \right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{\mathtt{C}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)} \\
&\times \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)}u(|\cdot|)\omega_1(|\cdot|)}{u(|\cdot|)\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{\mathtt{C}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)} \\
&\times \left\| \frac{|\cdot|^{\alpha-\theta_p(0,|\cdot|)+\theta_q(0,|\cdot|)}\omega_1(|\cdot|)}{\|\varphi\|_{L^{q(\cdot)}(B(0,|\cdot|))}\|\varphi\|_{L^{p'(\cdot)}(B(0,|\cdot|))}} \right\|_{L_\varphi^{q(\cdot)}(\mathbb{R}^n\setminus B(0,t))} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{\mathtt{C}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)} \int_0^t \frac{s^{\alpha-\theta_p(0,s)+\theta_q(0,s)}\omega_1(s)}{\|\varphi\|_{L^{p'(\cdot)}(B(0,s))}}\frac{ds}{s} \\
&\le C\|b\|_{BMO_{0,p'(\cdot),\varphi^{-1}}}\,\|f\|_{\mathtt{C}\mathcal{M}_{\{0\}}^{p(\cdot),\omega_1,\varphi}(\mathbb{R}^n)} \frac{\omega_2(t)}{\|\varphi\|_{L^{q'(\cdot)}(B(0,t))}}.
\end{aligned}$$

□

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