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ISOMETRIES ON SOME GENERAL FAMILY FUNCTION SPACES AMONG COMPOSITION OPERATORS

M. A. BAKHIT

Abstract. In this paper, we discuss the isometries of composition operators on the holomorphic general family function spaces $F(p, q, s)$. First, we classify the isometric composition operators acting on a general Banach spaces. For $1 < p < 2$, we display that an isometry of C_ϕ is caused only by a rotation of the disk. We scrutinize the previous work on the case for $p \geq 2$. Also, we characterize many of the foregoing results about all α-Besov-type spaces $F(p, \alpha p - 2, s)$, $\alpha > 0$. We exhibit that in every classes $F(p, \alpha p - 2, s)$ except for the Dirichlet space $\mathcal{D} = F(2, 0, 0)$, rotations are the only that produce isometries.

1. Introduction

Initially, the notion has to be set up and the main concept has also to be clarified, in addition to some other fundamental facts that are requested for what is to follow. Throughout this study, the symbol $\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disk of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ denotes its boundary. The symbol $\mathcal{H}(\mathbb{D})$ denotes the holomorphic functions group on $\mathbb{D}$. The symbol A denotes two-dimensional Lebesgue measure on $\mathbb{D}$, so that $A(\mathbb{D}) \equiv 1$.

For $a \in \mathbb{D}$, let $\varphi_a(z)$ be the Möbius transformations defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \text{ for } z \in \mathbb{D},$$

where $\varphi_a(0) = a, \varphi_a(a) = 0, \varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$.
The Green's function for $\mathbb{D}$ with logarithmic singularity at $a \in \mathbb{D}$ is defined by $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$. For all $0 < p, s < \infty$ and $-2 < q < \infty$, the general family function spaces $F(p, q, s)$ is defined as the set of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty.$$

The family of spaces $F(p, q, s)$ was introduced by Zhao in [23] and Rättyä in [19]. For that $p \geq 1$, the spaces $F(p, q, s)$ are Banach spaces under the norm

$$\|f\|_* = |f(0)| + \|f\|_{F(p,q,s)}.$$

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The spaces $F(p,q,s)$ are $\frac{q+2}{p}-$Möbius invariant spaces see [23]. Moreover, as special cases, many classical holomorphic function spaces are included in the spaces $F(p,q,s)$, like the weighted Bergman spaces A^p_q with $0<p<\infty, -2<q<\infty$, and the Besov spaces B_p. Similarly, usual Dirichlet type space $D_\alpha, 0\le\alpha<\infty$, in addition to $BMOA$ and Q_s spaces (see [2]).
For suitability, we let $F(p,\alpha p-2,s)$ be the α-Besove type spaces, where $\alpha>0$. We already have that, for $F(p,\alpha p-2,s)$ spaces if $\alpha p+s\le 1$, then it contains constant functions only (see [23]). Therefore, we will presume that $\alpha p+s>1$, which warrants that $F(p,\alpha p-2,s)$ are nontrivial. Amongst these spaces $F(p,\alpha p-2,s)$, the case may be $\alpha=1$ is interesting in particular, since the classes $F(p,p-2,s)$ are Möbius invariant. It means that $\|f\circ\varphi_a\|_{F(p,p-2,s)}=\|f\|_{F(p,p-2,s)}$ for any $f\in F(p,p-2,s)$ and any $a\in\mathbb{D}$. It is known from [19] and [23], as special cases of $F(p,\alpha p-2,s)$ that

(1) $F(p,\alpha p-2,0)=B_{p,\alpha}$ for $1<p<\infty$.

(2) $F(2,0,s)=Q_s$ and $F(2,0,1)=BMOA$.

(3) $F(2,1,0)=H^2$ and $F(2,\alpha,0)=D_\alpha$.

Let $f\in\mathcal{H}(\mathbb{D})$, for all $0<p,s<\infty$ and $-1<q<\infty$. Then, the function f has been said to belong to $\mathcal{N}(p,q,s)$ if (see [13])

$$\|f\|^p_{\mathcal{N}(p,q,s)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)<\infty.$$

We note that by the operator $f\mapsto f'$, the spaces $F(p,q,s)$ are closely related to $\mathcal{N}(p,q,s)$ and $F(p,\alpha p-2,s)$, are related to $\mathcal{N}(p,\alpha p-2,s)$.
For a holomorphic map $\phi:\mathbb{D}\to\mathbb{D}$, the composition operator C_ϕ is defined by $C_\phi=f\circ\phi$, for any $f\in\mathcal{H}(\mathbb{D})$. If $\psi\in\mathcal{H}(\mathbb{D})$ are given in addition, then the weighted composition operator $W_{\psi,\phi}$ is defined by $W_{\psi,\phi}f=\psi(z)(f\circ\phi)$ for any $f\in\mathcal{H}(\mathbb{D})$. It goes without saying that $W_{\psi,\phi}f$ is a generalization of $C_\phi f=f\circ\phi$ and multiplication operator $M_\psi f=\psi\cdot f$. The demeanor of these operators is extensively studied on numerous spaces of holomorphic functions (see for example [1, 3, 4, 6, 14, 15, 22] and others).

Let X be a normed linear space, and let an operator $T:X\to X$, if $\|Tf\|_X=\|f\|_X$ then, T is an isometry on X, for all $f\in X$. Several authors have classified the isometries of C_ϕ operators on many holomorphic function spaces. Martin and Vukotić in [17] characterized the isometries of C_ϕ operators on the full range of H^p and the Bergman spaces A^p_α for $1\le p<\infty$. Allen et. al. in [1], they showed that the isometries among C_ϕ operators on the Besov spaces B_p for $1\le p<\infty$ are induced by rotations. Shabazz and Tjani extended this work to all Besov type spaces $B_{p,\alpha}$ for $p>1,\alpha>-1$ see [22]. Carswell and Hammond in [5] obtained that the C_ϕ operators is an isometry on the weighted Bergman spaces A^p_α if and only if a holomorphic self-map ϕ is a rotation; this truth varies relatively from the analogs results that are famed for other Hilbert spaces. For the isometries among C_ϕ operators on the Bloch space $\mathcal{B}$ and the Bloch type spaces $\mathcal{B}^\alpha,\alpha>0,\alpha\ne 1$ see [6], [16] and [24]. For the classical Dirichlet spaces $\mathcal{D}$ and the weighted Dirichlet-type space see [10, 18]. Moreover, for the isometries among C_ϕ operators on $BMOA$ space see [14].

It is noticeable that isometries among weighted composition operators were studied at the beginning of this century on several spaces too, (see for example [4, 15] and [25]). Forelli in [9] demonstrated that isometries are in the form of the weighted composition operators, for the H^p spaces, where $1 \leq p < \infty$ with $p \neq 2$, Rudin extended this work to several variables in [21]. In order to have a comprehensive list in this regard, see the monographs [8] and [11].

For our special goals, in Section 2 we classify the isometric composition operators acting on general Banach spaces and we extend some results due to Kolaski in [12] to weighted Bergman type spaces $\mathcal{N}(p,q,s)$. In Section 3, we give integrated proof of the descriptions of all possible composition operators that are isometries of the general family function spaces $F(p,q,s)$, in terms of their counting function. During Section 4, we are expanding some of the previous results to include all α-Besov type spaces $F(p, \alpha p-2, s)$, also we show that C_ϕ is isometries on $F(p, \alpha p-2, s)$ if and only if ϕ is a rotation, for each $p > 1, \alpha > 1$ except when $p = 2$ and $\alpha p = 2$. Moreover, acting as if C_ϕ is an isometry on $F(2, 2\alpha-2, s), \alpha \neq 1$, then our assumption is only if and only if ϕ is a rotation.

2. Isometries of general Banach space

As well obvious that, if the composition operator C_ϕ is isometries on different spaces that include the Hardy space, the weighted Bergman spaces in addition to the Bloch type spaces together with the Besov and Besov-type spaces and $BMOA$, then $\phi(0) = 0$, see [6, 16, 17] and [24]. Shabazz and Tjani in [22], they observed that a Banach space X under some common conditions, the isometries among C_ϕ and $W_{\psi,\phi}$ fix the origin. In particular, the following theorem unified all such known results (see [22], Theorem 2.1, 2.2 and Corollary 2.1).

Theorem 2.1. *Let X be a Banach space of functions $f \in \mathcal{H}(\mathbb{D})$ containing $Aut(\mathbb{D})$, whose norm $\|f\|_*^p = |f(0)|^p + \|f\|_X^p$ for some fixed natural number p. Moreover, suppose that for all $f \in X$ and also for any constant m, $\|f + m\|_X = \|f\|_X$. So, we have the following:*

1. *For all $f \in X$, then C_ϕ is an isometric operator on X, if and only if $\phi(0) = 0$ and $\|f \circ \phi\|_X = \|f\|_X$.*
2. *For $\phi \in Aut(\mathbb{D})$, then C_ϕ is an isometric operator on X, if and only if ϕ is a disk rotation.*
3. *For all $z \in \mathbb{D}$, then $W_{\psi,\phi}$ is an isometric operator on X, if and only if the symbols ϕ and ψ such that $\psi(z) = \psi(0), |\psi(0)| = 1$ and $\phi(0) = 0$.*

In the next theorem, we extend the results of Theorem 3.1 in [22]. We derive a necessary and sufficient condition that warranties the isometric of a composition operator C_ϕ on any Banach space X is exactly the weighted composition operators $W_{\psi,\phi}$ with symbols ϕ and ϕ' on else Banach space Y.

Theorem 2.2. *Assume that X and Y are Banach spaces of functions $f \in \mathcal{H}(\mathbb{D})$ inclоding $Aut(\mathbb{D})$ and such that $f \in X \Leftrightarrow f' \in Y$. Then, C_ϕ is an isometric operator on X, if and only if the symbol ϕ, such that $\phi(0) = 0$ and $W_{\phi',\phi}$ is an isometric operator on Y.*

Proof. First, let C_ϕ be an isometric operator on X. Then, for each $f \in X$ by Theorem 2.1, we have $\phi(0) = 0$ and $\|C_\phi f\|_X = \|f\|_X$.
Now let $g \in Y$ and choose $f \in X$ with $f' = g$, then we have

$$\|(W_{\phi',\phi})g\|_Y = \|f\|_X = \|g\|_Y.$$

Thus, $W_{\phi',\phi}$ is an isometric operator on Y.
Next, suppose that $\phi(0) = 0$ and that $W_{\phi',\phi}$ is an isometry on Y. For all $g \in Y$, we have $\|(W_{\phi',\phi})g\|_Y = \|g\|_Y$.
Therefore, since $f \in X$ if and only if $f' \in Y$, we conclude that $\|C_\phi f\|_X = \|f\|_X$ is valid for all $f \in X$ and C_ϕ is an isometry on X. □

Kolaski in ([12], Theorem 2.1), clarifies that the isometries on A^p_α, with $p \neq 2$, are weighted composition operators of the form $W_{\psi,\phi}$, whereas $\phi(\mathbb{D})$ is dense and subset of $\mathbb{D}$. In addition, in the proof the theorem, it is clear that ϕ is a full map of $\mathbb{D}$.

The main theorem on [12] can be extended to some weighted Bergman type spaces $\mathcal{N}(p, q, s)$ without much difficulty.

Theorem 2.3. *For all $1 < p < \infty$ with $p \neq 2$, $-1 < q < \infty$ and $0 < s < \infty$. If an operator T is a linear isometry on $\mathcal{N}(p, q, s)$ and such that $T1 = \psi$, then there exist $\phi \in \mathcal{H}(\mathbb{D})$ which maps $\mathbb{D}$ onto $\phi(\mathbb{D})$, where $\phi(\mathbb{D})$ is a dense subset of $\mathbb{D}$ and such that $Tf = W_{\psi,\phi}f$ for all $f \in \mathcal{N}(p, q, s)$. Furthermore, for each bounded Borel function u on $\mathbb{D}$,*

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} u(\phi(z))|\psi(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) \\ = \ & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} u(z)(1 - |z|^2)^q g^s(z, a)dA(z). \qquad (2.1) \end{aligned}$$

Proof. First, assume that T is a linear isometry on $\mathcal{N}(p, q, s)$ and set $T1 = \psi$. Define the measure

$$d\mu_{p,q,s} = |u|^p(1 - |z|^2)^q g^s(z, a)dA(z).$$

For $f \in \mathcal{H}(\mathbb{D})$ define $Mf = Tf/\psi$. Since $\psi \in \mathcal{N}(p, q, s), \psi \neq 0$, almost everywhere two-dimensional Lebesgue measure $[A]$. Thus, Mf is well defined. It follows that M is a multiplicative linear map of $\mathcal{N}(p, q, s)$ into $L^p(d\mu_{p,q,s})$ which preserves sup norms (see [20]). Moreover, if $f \in L^\infty(d\mu_{p,q,s})$, then

$$\psi(Mf)^k = \psi M(f^k) = T(f^k), \text{ for all } k = 1, 2, \ldots.$$

As [20], this implies that $Mf \in L^\infty(d\mu_{p,q,s})$. Thus M is a multiplicative linear sup norm isometry on $L^\infty(d\mu_{p,q,s})$, with $M1 = 1$.

Now let $u_i(z) = z_i$, where $z_i = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, and define $\phi : \mathbb{D} \longrightarrow \mathbb{C}^n$ with $\phi = Mu_i$. It follows (as in [12]) that $\mu_{p,q,s}(\phi^{-1}B) = A(B)$ for every Borel set $B \subset \mathbb{C}^n$. This gives (2.1). The proof of $\phi(\mathbb{D})$ is a dense subset of $\mathbb{D}$, can be found in [12]. □

We close this section with an interesting natural result. This result is true in various spaces, and several proofs have been presented regarding it. Here, we rely on a proof by a special flavor that is due to Allen et.al. in [1], so we omitted the details. One of the tactics is used for using a fact that adjoint of a surjective isometry is also an isometry with the fact that the point estimates are really bounded linear functions.

Corollary 2.4. *If C_ϕ is a surjective isometry on a given Banach space X of holomorphic functions (contains univalent functions), then ϕ is univalent.*

3. Isometries on $F(p,q,s)$ spaces

The first result is a direct consequence of Theorem 2.1.

Lemma 3.1. *For all $1 < p < \infty, -2 < q < \infty, 0 < s < \infty$ and $f \in F(p,q,s)$. If C_ϕ is an isometry on $F(p,q,s)$, then $\phi(0) = 0$ and $\|C_\phi f\|_{F(p,q,s)} = \|f\|_{F(p,q,s)}$.*

Also, the next result is a direct consequence of Corollary 2.4.

Lemma 3.2. *For all $1 < p < \infty, -2 < q < \infty, 0 < s < \infty$. If C_ϕ is a surjective isometry on $F(p,q,s)$, then ϕ is univalent in $\mathbb{D}$.*

The next lemma relates isometries of C_ϕ on $F(p,q,s)$ to isometries of $W_{\phi',\phi}$ on $\mathcal{N}(p,q,s)$.

Lemma 3.3. *Let $1 < p < \infty, -1 < q < \infty$ and $0 < s < \infty$. If C_ϕ is an isometry on $F(p,q,s)$, then $W_{\phi',\phi}$ is an isometry on $\mathcal{N}(p,q,s)$ and $A\big[\mathbb{D}\backslash\phi(\mathbb{D})\big]= 0$.*

Proof. First, let $f \in \mathcal{N}(p,q,s)$ and choose $g \in F(p,q,s)$ with $g' = f$. By using Lemma 3.1, we have

$$\begin{aligned}
\|W_{\phi',\phi} f\|^p_{\mathcal{N}(p,q,s)} &= \|\phi'(f\circ\phi)\|^p_{\mathcal{N}(p,q,s)} = \|(g\circ\phi)'\|^p_{\mathcal{N}(p,q,s)} \\
&= \|g\circ\phi\|^p_{F(p,q,s)} = \|g\|^p_{F(p,q,s)} \\
&= \|f\|^p_{\mathcal{N}(p,q,s)},
\end{aligned}$$

which verifies that $W_{\phi',\phi}$ is an isometry on $\mathcal{N}(p,q,s)$.

The second inference follows from [12], there the author displays that ϕ is a full map of $\mathbb{D}$, in fact, tenor that $A\big[\mathbb{D}\backslash\phi(\mathbb{D})\big]= 0$ (see also [16]). For ϕ a holomorphic self-map of $\mathbb{D}$ and $w \in \mathbb{D}$, we let $\{z_j(w)\}$ denote the sequence of zeros of the function $\phi(z) - w$ and let $n_\phi(w)$ denote the cardinality of the set $\phi^{-1}(\{w\})$. This function is lower semi-continuous for holomorphic maps by Rouche's Theorem, and is therefore measurable with respect to A. Also, we let $E = \{w \in \mathbb{D} : n_\phi(w) > 1\}$ be a measurable set; since n_ϕ is a measurable function and that $n_\phi(w) = 1$ a.e. on $\mathbb{D} \setminus E$. □

Proposition 3.4. *Let $1 < p < 2, -2 < q < \infty, 0 < s < \infty$. If C_ϕ is an isometry on $F(p,q,s)$, then ϕ is a univalent map in $\mathbb{D}$.*

Proof. First, let $f(z) = z$, by the change of variable $w = \phi(z)$, the Schwarz-Pick Lemma (with $p < 2$) and Theorem 2.1, we have

$$\begin{aligned}
\|f\|^p_{F(p,q,s)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |\phi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |\phi'(z)|^2(1-|\phi(z)|^2)^q g^s(\phi(z),a)dA(z)\\
&= \sup_{a\in\mathbb{D}}\int_{\phi(\mathbb{D})} \sum_{j\geq 1}\{(1-|\phi(z_j(w))|^2)^q g^s(\phi(z_j(w)),a)\}dA(w)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \sum_{j\geq 1}\{(1-|w|^2)^q g^s(z,a)\}dA(w)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}\setminus E} (1-|w|^2)^q g^s(z,a)dA(w) + \sup_{a\in\mathbb{D}}\int_{E} \sum_{j\geq 1}\{(1-|w|^2)^q g^s(z,a)\}dA(w)\\
&\geq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}\setminus E} (1-|w|^2)^q g^s(z,a)dA(w) + 2\sup_{a\in\mathbb{D}}\int_{E} (1-|w|^2)^q g^s(z,a)dA(w)\\
&= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} (1-|w|^2)^q g^s(z,a)dA(w) + \sup_{a\in\mathbb{D}}\int_{E} (1-|w|^2)^q g^s(z,a)dA(w)\\
&= \|f\|^p_{F(p,q,s)} + \sup_{a\in\mathbb{D}}\int_{E} (1-|w|^2)^q g^s(z,a)dA(w).
\end{aligned}$$

Thus, it is clear that $A(E) = 0$ and thus $n_\phi(w) = 1$ a.e. on $\mathbb{D}$.
For the univalent of ϕ, in the fact an univalent functions are open maps. A proof can be found in the proof of Theorem A in [16]. □

The following result follows immediately from Theorem 2.1 and Lemma 3.2.

Proposition 3.5. *Let $p > 2, -2 < q < \infty, 0 < s < \infty$. Then C_ϕ is a surjective isometry on $F(p,q,s)$ if and only if ϕ is a rotation of $\mathbb{D}$.*

Corollary 3.6. *Let $p > 2, -2 < q < \infty, 0 < s < \infty$ and suppose that C_ϕ is an isometry on $F(p,q,s)$. If the range of C_ϕ contains a univalent function, then C_ϕ is a rotation of $\mathbb{D}$.*

Theorem 3.7. *Let $p > 2, -2 < q < \infty, 0 < s < \infty$ and suppose that $n_\phi(w) = 1$ a.e. in some neighborhood of the origin. Then C_ϕ is an isometry on $F(p,q,s)$, if and only if ϕ is a rotation of $\mathbb{D}$.*

Proof. When considering ϕ a rotation of unit disk $\mathbb{D}$, so this clarifies that C_ϕ is an isometric operator on $F(p,q,s)$.
Now, suppose that $n_\phi(w) = 1$ almost in the neighborhood of the origin, it follows from that there exists a neighborhood U of the symbol $\phi(0) = 0$ such that $n_\phi(w) = 1$ almost everywhere on U. We know ϕ is continuous and $\phi^{-1}(U)$ is an open set. Accordingly, there exists an $a \in \mathbb{D}$ and $r \in (0,1)$ are achieved that $\phi(\mathbb{D}(a,r)) \subseteq U$, for $\mathbb{D}(a,r)$ is the pseudo-hyperbolic disk defined by $\mathbb{D}(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$.

If we let $B = \phi(\mathbb{D}(a,r))$, then ϕ^{-1} becomes well defined on B, which is a single-valued inverse mapping and $\phi^{-1}(B) = \mathbb{D}_r$. By Lemma 3.3 and (2.1) we have

$$\begin{aligned}
& \sup_{a\in B}\int_B (1-|z|^2)^q g^s(z,a)dA(z) \\
= & \sup_{a\in \mathbb{D}(a,r)}\int_{\mathbb{D}(a,r)} |\phi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z) \\
\le & \sup_{a\in \mathbb{D}(a,r)}\int_{\mathbb{D}(a,r)} |\phi'(z)|^2(1-|\phi(z)|^2)^q g^s(\phi(z),a)dA(z) \\
= & \sup_{a\in B}\int_B (1-|w|^2)^q g^s(z,a)dA(w).
\end{aligned}$$

This computation implies that equality must be held throughout. For sure, ϕ must be a rotation of $\mathbb{D}$. □

4. Isometries on $F(p,\alpha p-2,s)$ spaces

The isometries on α-Besov type spaces $F(p,\alpha p-2,s)$ among composition operators C_ϕ are going to be classified in this section. In order for us to get a full description of when C_ϕ is an isometric operator on $F(p,\alpha p-2,s)$ we will have to use Nevanlinna type counting function (see [3]) as follows

$$N_{p,\alpha,s}(\phi,w) = \sum_{\phi(z)=w}\{|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}g^s(z,a)\}, \tag{4.1}$$

for all $w\in\phi(\mathbb{D}), 1\le p<\infty, 0<s<\infty$ and $1<\alpha<\infty$. Note that the counting function $N_{p,\alpha,s}(\phi,w)$ appear at the change of variables formula in the particular spaces. By means of making a non-univalent change of variables, for all $f\in F(p,\alpha p-2,s)$, as done in [3], we can see that

$$\|C_\phi f\|^p_{F(p,\alpha p-2,s)} = \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(w)|^p N_{p,\alpha,s}(\phi,w)dA(w). \tag{4.2}$$

Let $H^2(N_{2,\alpha,s}dA(w))$ denote the Hilbert space of $f\in\mathcal{H}(\mathbb{D})$ with

$$\|f\|^2_{N_{2,\alpha,s}} := \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f(w)|^2 N_{2,\alpha,s}(\phi,w)dA(w) < \infty.$$

Hereafter, we assort isometric composition operator on $F(2,2\alpha-2,s)$ spaces expressing with it's a counting function.

Theorem 4.1. *Let $\alpha>\frac{1}{2}$ and $0<s<\infty$. Then C_ϕ is an isometry on $F(2,2\alpha-2,s)$, if and only if $\phi(0)=0$ and for almost every $w,a\in\mathbb{D}$,*

$$N_{2,\alpha,s}(\phi,w) = \sum_{\phi(z)=w}\{(1-|z|^2)^{2\alpha-2}g^s(z,a)\} = (1-|w|^2)^{2\alpha-2}g^s(z,a). \tag{4.3}$$

Proof. First, suppose that C_ϕ is an isometric operator on $F(2,2\alpha-2,s)$. Then, by Lemma 3.3, this is synonymous to $\phi(0)=0$ and $W_{\phi',\phi}$ being an isometric operator on : $\mathcal{N}(2,2\alpha-2,s)$. Then $\|W_{\phi',\phi}u\|_{\mathcal{N}(2,2\alpha-2,s)} = \|u\|_{\mathcal{N}(2,2\alpha-2,s)}$, for all

$u \in \mathcal{N}(2, 2\alpha - 2, s)$. By making a non-univalent change of variables, we obtain

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u(w)|^2 N_{2,\alpha,s}(\phi, w)dA(w) \\ = & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |u(z)|^2 (1 - |z|^2)^{2\alpha-2} g^s(z, a)dA(z). \end{aligned} \tag{4.4}$$

By using (4.4), we have $u \in \mathcal{N}(2, 2\alpha - 2, s)$ if and only if $u \in H^2(N_{2,\alpha,s}dA(w))$, in fact $\|u\|_{\mathcal{N}(2,2\alpha-2,s)} = \|h\|_{N_{2,\alpha,s}}$. Using the polarization identities (see [7], Lemma 3.3) in $\mathcal{N}(2, 2\alpha - 2, s)$ and in $H^2(N_{2,\alpha,s}dA(w))$ and for functions $f, h \in \mathcal{N}(2, 2\alpha - 2, s)$, we get

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} f(w)\overline{h(w)} N_{2,\alpha,s}(\phi, w)dA(w) \\ = & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} f(w)\overline{h(w)}(1 - |z|^2)^{2\alpha-2} g^s(z, a)dA(z); \end{aligned}$$

using $f(z) = z^m$ and $h(z) = z^n$ for $m, n = 0, 1, 2, ...$
Now, for all polynomials $P(z, \overline{z})$, in the two variables z and $\overline{z}$, we see that

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} P(z, \overline{z}) N_{2,\alpha,s}(\phi, w)dA(w) \\ = & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} P(z, \overline{z})(1 - |z|^2)^{2\alpha-2} g^s(z, a)dA(z). \end{aligned}$$

Secondly, by Theorem 2.40 in [7] (the Stone-Weierstrass Theorem), for all $u \in C(\overline{\mathbb{D}})$, we have

$$\begin{aligned} & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} u(z) N_{2,\alpha,s}(\phi, w)dA(w) \\ = & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} u(z)(1 - |z|^2)^{2\alpha-2} g^s(z, a)dA(z). \end{aligned}$$

In the end, by Theorem 2.14 in [21] (the Riesz Representation Theorem), we conclude that (4.3) holds almost for all $w, a \in \mathbb{D}$. The sufficiency condition is clear and the proof is completed. □

For $\alpha > 1$, if C_ϕ is an isometry on $F(2, 2\alpha - 2, s)$, then by Theorem 4.1 and almost for all $w, a \in \mathbb{D}$ and $0 < (1 - |w|^2)^{2\alpha-2} g^s(z, a) \leq \eta_\phi(w)$ (the cardinality of the set $\phi^{-1}(w)$). So, we get the following result.

Corollary 4.2. *For all $\alpha > 1$ and $0 < s < \infty$. If we let C_ϕ be an isometric operator on $F(2, 2\alpha - 2, s)$, then ϕ is a full self-map of $\mathbb{D}$ and $\phi(0) = 0$.*

Proposition 4.3. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a non-constant map of $\mathbb{D}$. Then, C_ϕ is an isometric operator on $F(2, 2\alpha - 2, s)$, for all $\frac{1}{2} < \alpha < 1$ and $0 < s < \infty$, if and only if the map ϕ is a rotation of $\mathbb{D}$.*

Proof. First, suppose that C_ϕ is an isometric operator on $F(2, 2\alpha - 2, s)$. If $z \in \mathbb{D}$, then we have by Schwarz's lemma, $(1-|\phi(z)|^2)^{2\alpha-2} \leq (1-|z|^2)^{2\alpha-2}$ and $g(\phi(z), a) \leq g(z, a)$.
Thus, by using Theorem 4.1 and nearly for every $w, a \in \mathbb{D}$,

$$(1 - |w|^2)^{2\alpha-2} g^s(z, a) = N_{2,\alpha,s}(\phi, w) \geq \eta_\phi(w)(1 - |w|^2)^{2\alpha-2} g^s(z, a),$$

and $\eta_\phi(w) \le 1$. Pick $w \in \mathbb{D}$ with $\eta_\phi(w) = 1$ and (4.3) holds. In Schwarz's lemma we get equality and ϕ must be a rotation map of $\mathbb{D}$. Disk rotations are clear to be isometries. $\square$

Below that, the result in Theorem 4.1 is extended to encompass the rest of the symbol indices $p > 1$.

Theorem 4.4. *Let $p > 1$ with $p \neq 2, 1 < \alpha < \infty$ and $0 < s < \infty$. Then C_ϕ is an isometry on $F(p, \alpha p - 2, s)$, if and only if $\phi(0) = 0$ and for almost every $w, a \in \mathbb{D}$,*

$$N_{p,\alpha,s}(\phi, w) = (1 - |w|^2)^{2\alpha-2} g^s(z, a).$$

Proof. First, we suppose that C_ϕ is an isometric operator on $F(p, \alpha p - 2, s)$ classes. From Lemma 3.3 this will be equivalent to $W_{\phi',\phi}$ being an isometric operator on $\mathcal{N}(p, \alpha p - 2, s)$ and $\phi(0) = 0$. Next, for all bounded Borel functions h and using equation (2.1), we obtain

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} h(\phi(z))|\phi'(z)|^p (1 - |z|^2)^{\alpha p-2} g^s(z, a) dA(z) \\ = \ &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} h(z)(1 - |z|^2)^{\alpha p-2} g^s(z, a) dA(z); \end{aligned}$$

by making a non-univalent change of variables, we get

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} h(w) N_{p,\alpha,s}(\phi, w) dA(w) = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} h(z)(1 - |z|^2)^{\alpha p-2} g^s(z, a) dA(z). \tag{4.5}$$

Now, let $\mathcal{C}(\mathbb{D})$ be the group of all continuous functions with compact support on the unit disk $\mathbb{D}$. If we let a function $h \in \mathcal{C}(\mathbb{D})$, so it is a bounded Borel function. Thus, (4.5) holds for every function $h \in \mathcal{C}(\mathbb{D})$. Moreover, by Theorem 2.14 in [21] (the Riesz Representation Theorem), for almost every $w, a \in \mathbb{D}$ it is concluded that

$$N_{p,\alpha,s}(\phi, w) = (1 - |w|^2)^{2\alpha-2} g^s(z, a).$$

By (4.2), the above condition's sufficiency clear. $\square$

Proposition 4.5. *For all $p > 1$ with $p \neq 2$, $\alpha > \frac{1}{2}$ with $\alpha p \neq 2$ and $0 < s < \infty$, let ϕ be a non-constant with $\phi'(0) \neq 0$. Then, C_ϕ is an isometric operator on $F(p, \alpha p - 2, s)$ classes, if and only if the self-map ϕ is a rotation map of $\mathbb{D}$.*

Proof. Suppose that C_ϕ is an isometric operator on $F(p, \alpha p - 2, s)$ classes. Since, ϕ is univalent in small disk D in $\mathbb{D}$ centered at 0 and such that $\phi'(0) \neq 0$. Then

$$\sup_{a\in\mathbb{D}} \int_{\phi(D)} (1 - |w|^2)^{\alpha p-2} g^s(z, a) dA(w) = \sup_{a\in\mathbb{D}} \int_{\phi(D)} N_{p,\alpha,s}(\phi, w) dA(w). \tag{4.6}$$

By making a non-univalent change of variables in both integrals of (4.6), we have

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{D} |\phi'(z)|^2 (1 - |\phi(z)|^2)^{\alpha p-2} g^s(\phi(z), a) dA(z) \\ = \ &\sup_{a\in\mathbb{D}} \int_{\phi(D)} |\phi'(z)|^p (1 - |z|^2)^{\alpha p-2} g^s(z, a) dA(z), \end{aligned}$$

equivalently

$$\sup_{a\in\mathbb{D}}\int_D |\phi'(z)|^2\Big\{(1-|\phi(z)|^2)^{\alpha p-2}g^s(\phi(z),a)$$
$$-|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}g^s(z,a)\Big\}dA(z)=0.$$

Using Theorem 4.4, for almost every $w,a\in\mathbb{D}$ and for

$$N_{p,\alpha,s}(\phi,w)=(1-|w|^2)^{\alpha p-2}g^s(z,a).$$

It is concluded that the above integrand as non-negative, thus for almost every $z\in D$, we get

$$|\phi'(z)|^{p-2}(1-|w|^2)^{\alpha p-2}g^s(z,a)=(1-|\phi(z)|^2)^{\alpha p-2}g^s(\phi(z),a). \tag{4.7}$$

Now, with the picked sequence $(z_n)\in D$ such that z_n converges to 0, as well (4.7) holds for each z_n. Since, $\phi(0)=0$ by Corollary 4.2, it is concluded that $|\phi'(0)|=1$. We could finalize the proof by using Schwarz's lemma, from which we can deduce that ϕ is a rotation of $\mathbb{D}$. Since, the converse is clear, the proof has been completed. □

Theorem 4.6. *Let $\alpha>\frac{1}{2}$ with $\alpha\neq 1$ and $0<s<\infty$. Then, C_ϕ is an isometry on $F(2,2\alpha-2,s)$ if and only if the map ϕ is a rotation of $\mathbb{D}$.*

Proof. First, for all $\alpha>\frac{1}{2}$ suppose that C_ϕ is an isometric operator on $F(2,2\alpha-2,s)$ classes. Then, by Theorem 4.1, $\phi(0)=0$ and for almost all $w,a\in\mathbb{D}$,

$$\sum_{\phi(z)=w}\{(1-|z|^2)^{2\alpha-2}g^s(z,a)\}=(1-|w|^2)^{2\alpha-2}g^s(z,a). \tag{4.8}$$

Since, the self-map ϕ is non-constant, then there is $m\in\mathbb{N}$, for $j=0,1,\dots,m-1$ such that $\phi^{(j)}=0$ however $\phi^{(m)}\neq 0$. Furthermore, there is a univalent function $h\in\mathcal{H}(D)$, where D centred at zero and such that $h(0)=0$ also $\phi(z)=h(z)^m$ for all $z\in D$, in addition to $h(D)$ contains a disk Δ with radius R and centred at zero. Set $w\in\Delta$ where $0<|w|<R^m$, and let $w_1,w_2,\dots,w_m$ denote the mth roots of w which is $w_k^m=w$ for $k=1,2,\dots,m$. Then $h^{-1}(w_k)\in\phi^{-1}(w), k=1,2,\dots,m$ and by (4.8)

$$\sum_{k=1}^m\Big[1-|h^{-1}(w_k)|^2\Big]^{2\alpha-2}g^s(h^{-1}(w_k),a)\le(1-|w|^2)^{2\alpha-2}g^s(z,a). \tag{4.9}$$

Note that, for w close enough zero,

$$h^{-1}(w_k)=\big[(h^{-1})'(0)+o(1)\big]w=\Big[(h'(0))^{-1}+o(1)\Big]w.$$

Now, determine $b\in\mathbb{C}^m$ such that $h^{-1}(w_k)=e^{i\theta}\varphi_b(w_k)$ for $k=1,2,\dots,m$.
Then, it is easy to check that $\varphi_a\big(h^{-1}(w_k)\big)=e^{i\gamma}\varphi_{\widetilde{a}}(w_k)$, where $e^{i\gamma}=\frac{a\bar{b}-e^{i\theta}}{1-\overline{a}be^{i\theta}}$ and $\widetilde{a}=\varphi_b(ae^{i\theta})$. So, $g(h^{-1}(w_k),a)=g(w,\widetilde{a})$.
By (4.9), we conclude that

$$\begin{aligned}(1-|w|^2)^{2\alpha-2} &\ge \sum_{k=1}^m\Big[1-\Big|(h'(0))^{-1}+o(1)\Big|^2|w_k|^2\Big]^{2\alpha-2}\\ &= m\Big[1-\Big|(h'(0))^{-1}+o(1)\Big|^2|w|^{\frac{2}{m}}\Big]^{2\alpha-2}g^s(w,\widetilde{a})\end{aligned}$$

or equivalently that

$$m^{\frac{1}{2\alpha-2}}\left|(h'(0))^{-1}+o(1)\right|^2|w|^{\frac{2}{m}}-|w|^2\geq m^{\frac{1}{2\alpha-2}}-1$$

for w near 0. Hence

$$\lim_{w\to 0} m^{\frac{1}{2\alpha-2}}\left|(h'(0))^{-1}+o(1)\right|^2|w|^{\frac{2}{m}}-|w|^2=0,$$

to conclude, $m=1$ and ϕ is univalent close zero and $\phi'(0)\neq 0$. Apply Proposition 4.5, the self-map ϕ is evidently a disk rotation. If $\frac{1}{2}<\alpha<1$, so Proposition 4.3 is applicable. It is evident that disk rotations are isometries. □

However, the norm in Besov-type and the Dirichlet spaces are slightly different than the one in $F(2,2\alpha-2,0)$, by Theorem 2.1 and the main results in [16] and [22], we conclude that they have the same isometries.

Corollary 4.7. *C_ϕ is an isometric operator on $F(2,0,0)$, if and only if the self-map ϕ is a univalent full map fixing the origin.*

Remark. *Note that $H^2=F(2,1,0)$ but the isometries in H^2 among C_ϕ are all the inner functions that are fixing the origin, whereas the isometries in the $F(2,1,0)$ norm are only the disk rotations.*

Lemma 4.8. (see Lemma 3.1 in [22])
Assume that ϕ is an analytic full map of $\mathbb{D}$ which such that $|\phi'(z)|\leq|z|$ for all $z\in\mathbb{D}$ and $\phi(0)=0$. Then the map ϕ is locally univalent at zero, that means $\phi'(0)\neq 0$.

Theorem 4.9. *Let $\alpha>1$ and $p>1$ with $p\neq 2$, then C_ϕ is an isometric operator on $F(p,\alpha p-2,s)$ classes, if and only if the self-map ϕ is a rotation of $\mathbb{D}$.*

Proof. First, we suppose that C_ϕ is an isometric operator on $F(p,\alpha p-2,s)$. By Theorem 4.4, C_ϕ is an isometry on $F(p,\alpha p-2,s)$, if and only if the self-map ϕ such that $\phi(0)=0$ and for nearly every $z,a\in\mathbb{D}$

$$\begin{aligned} N_{p,\alpha,s}(\phi,w) &= \sum_{\phi(z)=w}\{|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}g^s(z,a)\} \\ &= (1-|w|^2)^{\alpha p-2}g^s(z,a). \end{aligned} \tag{4.10}$$

From Schwarz's lemma, we know that $|\phi'(0)|\leq 1$, and if $|\phi'(0)|=1$ then ϕ is a rotation map. And therefore, we may suppose that $|\phi'(0)|<1$. First, for $\alpha=\frac{2}{p}$, it will be shown that $|\phi'(0)|\neq 0$. By the continuity of ϕ' and (4.10), if $p>1$ then, we have that $|\phi'(z)|\leq 1$ for every $z,a\in\mathbb{D}$. Subsequently, by Lemma 3.3, we get $|\phi'(0)|\neq 0$. For the second time, by the continuity of ϕ' and (4.10), if $1<p<2$ then, we conclude that $|\phi'(z)|\geq 1$, for all $z,a\in\mathbb{D}$, mostly this means $|\phi'(0)|\neq 0$. Now, let's finish this part by Proposition 4.5, we conclude that ϕ is a rotation map.

Finally, let $\alpha\neq\frac{2}{p}$. By Lemma 4.8, if $|\phi'(z)|\leq 1$ then, we get $|\phi'(0)|\neq 0$, for all $z,a\in\mathbb{D}$ and again by Proposition 4.5, if $|\phi'(z)|\leq 1$ then, ϕ is a rotation map. Furthermore, if there exist $b\in\mathbb{D}$, which such that $|\phi'(b)|>1$. Since, $|\phi'(0)|<1$ then, also there exist $z_0\in\mathbb{D}$ such that $|\phi'(z_0)|=1$. Hence, the map ϕ is also univalent in a small disk D centered at z_0. By an assumption similar to the one for the relation of (4.7), and for the third time, by (4.10) and the continuity of ϕ', we can deduce that for almost every $z \in D$,

$$|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}g^s(z,a) = (1-|\phi(z)|^2)^{\alpha p-2}g^s(\phi(z),a);$$

if $z = z_0$ then, we get

$$(1-|z_0|^2)^{\alpha p-2}g^s(z_0,a) = (1-|\phi(z_0)|^2)^{\alpha p-2}g^s(\phi(z_0),a),$$

and ϕ is a rotation map by Schwarz's lemma. It is clear that rotations are isometries in each $F(p, \alpha p-2, s)$ spaces of α-Besov type, so the proof has been completed. □

At the end of this work, by Theorems 4.1 and 4.9 and for all other indicators, we obtain the next corollary:

Corollary 4.10. *Suppose that $\alpha > 1$ and let $p > 1$, except $p = 2$ and $\alpha p = 2$. Then, we get $\phi(0) = 0$ and also for almost every $z, a \in \mathbb{D}$, we get $N_{p,\alpha,s}(\phi, w) = (1-|w|^2)^{\alpha p-2}g^s(z,a)$, if and only if the ϕ is a rotation.*

Conclusions

The main purpose of this paper has been to study the question of when composition operators are an isometry in $F(p,q,s)$ spaces as well in $F(p, \alpha p-2, s)$ spaces. The researcher has found a generalization of a sufficient condition in the Banach spaces of the holomorphic functions that allow inferring that isometries among composition operators are caused by symbols that fix the origin. Isometric composition operators on $F(p,q,s)$ to isometric weighted composition operators on weighted Bergman type spaces $\mathcal{N}(p,q,s)$ have been fully distinguished. Moreover, it has been presented that (among the univalent maps) the only isometric composition operators acting on $F(p,q,s), p > 2$ arise from rotations of the unit disk $\mathbb{D}$. In most of the previous work, the additional hypothesis that the holomorphic self-map ϕ is univalent was a strong constraint. So, this limitation has been weakened and it was established that the result is still valid.

The researcher has classified isometric composition operators acting on all α-Besov type spaces $F(p, \alpha p-2, s)$, in terms of their counting function. Moreover, some more results have been on the isometries among composition operators C_ϕ on $B_{p,p-2}$ some Besov spaces to $F(p, \alpha p-2, s)$ spaces. Also, it has been shown that C_ϕ is an isometric operator on $F(p, \alpha p-2, s)$ if and only if the self-map ϕ is a rotation map, for some conditions on the symbols p, α.

The aforementioned results and those previously studied have shown that there are significant contrasts in symbols that prompt isometric composition operators within the scope of these spaces. It has been pointed out that the general issue in which linear operators are isometries of a Banach spaces of holomorphic functions still appear to be opened.

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M. A. Bakhit: Department of Mathematics, Faculty of Science, Jazan university, Jazan, Saudi Arabia
E-mail address: mabakhit@jaznu.edu.sa