Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 12 Issue 1 (2021), Pages 28-35.

CC-PARACOMPACTNESS AND CC_2-PARACOMPACTNESS.

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Abstract. A topological space X is called *CC-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. A topological space X is called *CC_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. We investigate these two properties.

We introduce two new topological properties, CC-paracompactness and CC_2-paracompactness. The purpose of this paper is to investigate these two properties. These two properties may subserve Topology in some advance subjects in the future, for example, L-normality was used to give a result in metrization theorems, [6]. Also, in these two *external* properties, we can discover some aspects about the first, the domain, space using the paracompactness of the second, the codomain, space as it will be seen in some theorems in this paper. We can consider CC_2-paracompactness as weaker version of normality.

Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a T_1 completely regular space. We do not assume T_2 in the definition of compactness, countable compactness, local compactness, and paracompactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 and the first uncountable ordinal is ω_1.

Definition 1. (Arhangel'skiĭ)
A topological space X is called *C-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

A topological space X is called *C_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

2000 *Mathematics Subject Classification.* 54D20, 54C10.
Key words and phrases. countably compact; CC-paracompact; CC_2-paracompact; CC-normal; submetrizable; epinormal; C-paracompact; C_2-paracompact.
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Submitted 14, 2020. Published January 23, 2021.
Communicated by P. Das.

C-Paracompactness and C_2-Paracompactness were studied in [7]. We use the idea of Arhangel'skiĭ's definition above and give the following definition.

Definition 2. A topological space X is called *CC-paracompact* if there exist a paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

A topological space X is called *CC_2-paracompact* if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

The Independence of CC-paracompactness and CC_2-paracompactness

(1) Since any T_2 paracompact space is normal, any CC_2-paracompact is CC-normal, where a topological space X is called *CC-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$, see [5]. But the converse is not true in general as Example 3 shows.
(2) Any paracompact space is CC-paracompact and any T_2-paracompact space is CC_2-paracompact. Just take $Y = X$ and the identity function. The converse is not true in general as Examples 5 and 6 illustrate.
(3) Since any compact set is countably compact, any CC-paracompact space is C-paracompact. Also any CC_2-paracompact space is C_2-paracompact. The converse may not be true as Example 8 illustrates.
(4) Clearly, any CC_2-paracompact space is CC-paracompact but the converse may not be true as Example 10 shows.

Example 3. Let $X = [0, \infty)$. Define $\tau = \{\emptyset, X\} \cup \{[0, x) : x \in \mathbb{R}, 0 < x\}$ which can be considered as a subspace of the left ray topology $(\mathbb{R}, \mathcal{L})$, [9, Example 50] *i.e.* $\tau = \mathcal{L}_{[0,\infty)}$. In this space, there are no two nonempty closed disjoint subsets. Thus (X, τ) is normal and hence is CC-normal.

On the other hand (X, τ) is not C-paracompact. This is because it is Fréchet and for each $U \in \tau$, we have $0 \in U$. So if (X, τ) is C-paracompact with witness Y and witness function f, f will be continuous bijection [7, Theorem 2.2]. Hence, for every $V \in \tau$, $f(0) \in V$. Therefore, any infinite open cover of Y will have no locally finite open refinement which contradicts the paracompactness of Y. Thus, (X, τ) is not C-paracompact, see [7, Example 2.11]. This implies that (X, τ) is not CC-paracompact too. ■

Theorem 4. *If X is a T_1 space such that the only countably compact subspaces of X are the finite subsets, then X is CC_2-paracompact.*

Proof. Let X be a T_1 space such that the only countably compact subspaces of X are the finite subsets. Since X is T_1, we conclude that any countably compact subspace of X is discrete. So, let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y works. □

Example 5. Consider $(\mathbb{R}, \mathcal{CC})$, where $\mathcal{CC}$ is the countable complement topology [9, Example 20]. The space $(\mathbb{R}, \mathcal{CC})$ is T_1 as any singleton is closed. Let C be any countably infinite subset of $\mathbb{R}$. For each $c \in C$, define $V_c = (\mathbb{R} \setminus C) \cup \{c\}$. Then the family $\{V_c : c \in C\}$ is a countable open cover for C which has no finite subcover, hence C is not countably compact. Thus the only countably compact subspaces are the finite sets. Therefore, by Theorem 4, $(\mathbb{R}, \mathcal{CC})$ is CC_2-paracompact which is not paracompact. This example shows also that a non Hausdorff space can be CC_2-paracompact. ■

Now, here is a Tychonoff separable zero-dimensional CC_2-paracompact space which is not paracompact.

Example 6. Consider $(\mathbb{R},\mathcal{RS})$ where $\mathcal{RS}$ is the rational sequence topology [9, Example 65]. We recall its definition, for each $x \in \mathbb{P}$ fix a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $x_n \longrightarrow x$, where the convergency is taken in $(\mathbb{R}, \mathcal{U})$, and $\mathcal{U}$ is the usual metric topology. For each $n \in \mathbb{N}$, let $A_n(x)$ denotes the n^{th}-tail of the sequence.

(1) For each $x \in \mathbb{P}$, let $\mathcal{B}(x) = \{U_n(x) : n \in \mathbb{N}\}$, where $U_n(x) = A_n(x) \cup \{x\}$.
(2) For each $x \in \mathbb{Q}$, let $\mathcal{B}(x) = \{\{x\}\}$.

The rational sequence topology $\mathcal{RS}$ on $\mathbb{R}$ is the unique topology which has $\{\mathcal{B}(x)\}_{x\in\mathbb{R}}$ as its neighborhood system. $(\mathbb{R},\mathcal{RS})$ is Tychonoff zero-dimensional separable but it is not normal by Jones' lemma, since $\mathbb{Q}$ is dense and $\mathbb{P}$ is closed uncountable discrete subspace. So, $(\mathbb{R},\mathcal{RS})$ is not paracompact.

Claim : $(\mathbb{R}, \mathcal{RS})$ is CC_2-paracompact.
Proof of the Claim: Consider the identity function $id : (\mathbb{R}, \mathcal{RS}) \longrightarrow (\mathbb{R},\mathcal{U})$ which is bijection and continuous. Let E be any countably compact subspace of $(\mathbb{R},\mathcal{RS})$. Then, $id|_E : E \longrightarrow E$ is a continuous bijection. It remains to prove that $id|_E$ is closed. So let $E' \subseteq E$ be any closed subset of E. Since E is countably compact, then E' is countably compact in $(\mathbb{R}, \mathcal{RS})$, see [1, 3.10.4]. Since $id|_E$ is continuous and $(\mathbb{R},\mathcal{U})$ is T_2, then $id|_E(E') = E'$ is countably compact in $(E,\mathcal{U}_E)$, see [1, 3.10.5]. Hence, E' is countably compact in $(\mathbb{R},\mathcal{U})$. Now since $(\mathbb{R},\mathcal{U})$ is metrizable, any subspace of it is compact if and only if it is countably compact [9]. Therefore, E' is compact and hence is closed in $(\mathbb{R},\mathcal{U})$ since $(\mathbb{R},\mathcal{U})$ is Hausdorff. Hence, E' is closed in $(E,\mathcal{U}_E)$. Therefore, $id|_E$ is homeomorphism and $(\mathbb{R}, \mathcal{RS})$ is CC_2-paracompact space. ■

Theorem 7. *Any T_2 countably compact non compact space is not CC-paracompact.*

Proof. Let X be any T_2 countably compact non compact space. Suppose, by contrary, that X is CC-paracompact. Then, there is a paracompact space Y and a bijection $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. But X itself is countably compact. So, X is homeomorphic to Y and Y is paracompact. Hence, X is compact being T_2 countably compact paracompact, which contradicts the assumption. Thus, X is not CC-Paracompact. □

The next example is an application of Theorem 7.

Example 8. Consider ω_1, the first uncountable ordinal, with its usual order topology. ω_1 is T_2 countably compact non compact space [9, Example 42]. Hence, by Theorem 7, ω_1 is not CC-paracompact and so is not CC_2-paracompact too.

Now ω_1 can be considered as an open subspace of its successor $(\omega_1 + 1)$, which is compact and hence is locally compact [9, Example 43]. Therefore, ω_1 is locally compact being an open subspace of a locally compact space [1, 3.3.8]. Also ω_1 is Hausdorff. Hence, it is C_2-paracompact by [7, Theorem 2.12]. ■

Theorem 9. *If X is countably compact but not T_2, then X cannot be CC_2-paracompact.*

Proof. Suppose, by contrary, that X is CC_2-paracompact. Then there is a T_2 paracompact space Y and a bijection $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. But X itself is countably compact. Hence, $f|_X = f : X \longrightarrow Y$ is homeomorphism which contradicts that Y is T_2 but X is not. □

Example 10. Let X be any finite set which has more than one element. Fix $p \in X$. Consider the particular point topology (X , τ_p), see [9, Example 8], where a subset U of X is open if and only if $U = \emptyset$ or $p \in U$. Since X is finite, it is compact. Hence, X is CC-paracompact and countably compact. But X is not T_2. So, using Theorem 9, X cannot be CC_2-paracompact. ■

Theorem 11. *CC-paracompactness (CC_2-paracompactness) is a topological property.*

Proof. Let X be a CC-paracompact (CC_2-paracompact) space and let $X \cong Z$. Let Y be a paracompact (Hausdorff paracompact) space and $f : X \longrightarrow Y$ be a bijective function such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. Let $g : Z \longrightarrow X$ be a homeomorphism. Then Y and $f \circ g : Z \longrightarrow Y$ satisfy the requirements. □

Theorem 12. *CC-paracompactness (CC_2-paracompactness) is an additive property.*

Proof. Let X_α be a CC-paracompact (CC_2-paracompact) space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is CC-paracompact (CC_2-paracompact). For each $\alpha \in \Lambda$, pick a paracompact (a Hausdorff paracompact) space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha_{|_{C_\alpha}}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each countably compact subspace C_α of X_α. Since Y_α is paracompact (Hausdorff paracompact) for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is paracompact (Hausdorff paracompact), [1, 2.2.7, 5.1.30]. Consider the function sum [1, 2.2.E], $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is countably compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is countably compact in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is countably compact, then $(\oplus_{\alpha\in\Lambda} f_\alpha)|_C$ is a homeomorphism because $f_{\alpha_{|_{C\cap X_\alpha}}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. □

If X is CC-paracompact (CC_2-paracompact) with a witness function f, then f need not be continuous. For example, the identity function in Example 5. Here is a case where a witness function will be continuous.

Theorem 13. *If X is a CC-paracompact (CC_2-paracompact) Fréchet space and $f : X \longrightarrow Y$ is a witness of the CC-paracompactness (CC_2-paracompactness) of X, then f is continuous.*

Proof. Let A be any non-empty subset of X. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that $f(x) = y$. Then $x \in \overline{A}$. Pick a sequence $(x_n) \subseteq A$ such that $x_n \longrightarrow x$. Let $B = \{x, x_n : n \in \mathbb{N}\}$; then B is a compact subspace of X, being a convergent sequence with its limit, and hence $f|_B : B \longrightarrow f(B)$ is a homeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of y; then $V \cap f(B)$ is open in the subspace $f(B)$ containing y. Thus $f^{-1}(V) \cap B$ is open in the subspace B containing x. Thus $(f^{-1}(V) \cap B) \cap \{x_n : n \in \mathbb{N}\} \neq \emptyset$. So $(f^{-1}(V) \cap B) \cap A \neq \emptyset$. Hence $\emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A)$. Thus $y \in \overline{f(A)}$. Therefore, f is continuous. □

For the next theorem, we recall the definition of a C-closed space. A topological space (X, τ) is called *C-closed* if and only if for each countably compact subset $A \subseteq X$, A is closed in X [4].

Theorem 14. *Let X be T_2 CC-paracompact space with C-closed witness. Let $A \subseteq X$ be arbitrary. Then, A is compact if and only if A is countably compact.*

Proof. Let A be arbitrary countably compact subset of X. By assumption, there is a C-closed paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism. This implies that $f(A)$ is countably compact subset of Y and hence is closed in Y as Y is C-closed. Thus $f(A)$ is paracompact [1, 5.1.29]. Thus A is paracompact being homeomorphic to $f(A)$. Since A is T_2 countably compact paracompact, A is compact. The other direction is clear. □

Theorem 15. *Let X be second countable topological space. Then, X is CC-paracompact if and only if X is C-paracompact*

Proof. Since second countability is hereditary, for every $A \subseteq X$, A is second countable. Hence, A is compact if and only if A is countably compact [9]. So, X is C-paracompact if and only if X is CC-paracompact. □

Theorem 16. *Let X be CC_2-paracompact topological space that has a countably compact non compact subset. Let Y be the witness of CC_2-paracompactness. Then, Y is not totally normal. (Particularly, Y is not perfectly normal).*

Proof. Assume the condition. Let $B \subseteq X$ be the countably compact non compact subset of X. Let $f : X \longrightarrow Y$ be the witness function. Since B is countably compact, $f|_B : B \longrightarrow f(B)$ is homeomorphism. So, $f(B)$ is a T_2 countably compact non compact subset of Y and hence is not paracompact. This implies that Y is T_2 paracompact but not hereditary paracompact. Since every T_2 totally normal paracompact space is hereditary paracompact [3, Theorem 2], Y is not totally normal. Since any perfectly normal space is totally normal [3], Y is not perfectly normal. □

Theorem 17. *Let (X, τ) be a topological space which has a coarser CC_2 - paracompact topology (X, τ') with sequential witness. Then,(X, τ) is also CC_2-paracompact with the same witness.*

Proof. Assume the condition. Since (X, τ') is CC_2- paracompact, there is a T_2 paracompact space (Y, τ^*) and a bijection $f : (X, \tau') \longrightarrow (Y, \tau^*)$ such that the restriction $f|_A : A \longrightarrow f(A)$ is a homeomorphism for every countably compact subset A in (X, τ'). Now Let B be arbitrary countably compact subset in (X, τ). Since $\tau' \subseteq \tau$, B is countably compact in (X, τ'). This implies that $f|_B : (B, \tau'_B) \longrightarrow (f(B), \tau^*_{f(B)})$ is a homeomorphism.

Now consider a function $g : (X, \tau) \longrightarrow (Y, \tau^*)$ defined by $g(x) = f(x)$ for every $x \in X$. Clearly g is a bijection. Since $f|_B$ is continuous and $\tau'_B \subseteq \tau_B$, $g|_B : (B, \tau_B) \longrightarrow (g(B), \tau^*_{g(B)})$ is continuous. It remains to show that $g|_B$ is closed.

Let $B^* \subseteq B$ be closed in τ_B. This implies that B^* is countably compact in τ_B [1, 3.10.4]. Since $g|_B$ is continuous and $g(B)$ is T_2, $g|_B(B^*) = g(B^*)$ is countably compact in $\tau^*_{g(B)}$ [1, 3.10.5]. Hence, $g|_B(B^*)$ is countably compact in τ^*. Since (Y, τ^*) is T_2 sequential, it is C-closed [4]. Hence, $g|_B(B^*)$ is closed in (Y, τ^*) and so is closed in $(g(B), \tau^*_{g(B)})$. This implies that $g|_B$ is closed and therefore is a homeomorphism for every countably compact subset B in (X, τ). Hence, (X, τ) is CC_2-paracompact with the same witness of (X, τ'). □

Corollary 18. *Let (X, τ') be CC_2-paracompact space with sequential witness. Let (X, τ) be arbitrary topological space such that $\tau' \subseteq \tau$. Then, for every countably compact subset A in (X, τ), τ_A is homeomorphic to τ'_A.*

Proof. Assume the condition. Let A be arbitrary countably compact subset in (X, τ). Since $\tau' \subseteq \tau$, A is countably compact in (X, τ'). Since (X, τ') is CC_2-paracompact, there exist a T_2 paracompact space (Y, τ^*) and a bijection $f : (X, \tau') \longrightarrow (Y, \tau^*)$ such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism $\Rightarrow (A, \tau'_A) \cong (f(A), \tau^*_{f(A)})$ ... (1) .

Now by assumption, the witness Y is sequential. Hence, using theorem 17, (X, τ) is CC_2-paracompact with the same witness and witness function $g : (X, \tau) \longrightarrow (Y, \tau^*)$ such that $g(x) = f(x)$ for every $x \in X$. Since A is countably compact in (X, τ), $g|_A : A \longrightarrow g(A)$ is a homeomorphism. Hence, $(A, \tau_A) \cong (g(A), \tau^*_{g(A)})$ and $(g(A), \tau^*_{g(A)}) \cong (f(A), \tau^*_{f(A)})$ using the identitty function. ... (2).

From (1) and (2), $(A, \tau_A) \cong (A, \tau'_A)$.

□

For the next theorem, we recall the definition of a submetrizable space. A space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than τ [2].

Theorem 19. *Any submetrizable space is CC_2-paracompact.*

Proof. Let (X, τ) be submetrizable space $\Rightarrow \tau' \subseteq \tau$ such that τ' is metrizable. Hence, τ' is T_2, sequential and paracompact $\Rightarrow (X, \tau')$ is CC_2-paracompact where (X, τ') itself is the sequential witness. Hence, using Theorem 17, (X, τ) is CC_2-paracompact. □

Observe that using Theorem 19, since any countable product of submetrizable spaces is submetrizable, submetrizable spaces are countable productively CC_2-paracompact.

Example 20. Observe that the rational sequence topology $(\mathbb{R}, \mathcal{RS})$, which was proved to be CC_2-paracompact in Example 6, is submetrizable being finer than $(\mathbb{R}, \mathcal{U})$.

The Sorgenfrey line square $\mathcal{S} \times \mathcal{S}$ is another example of a space that is neither normal nor paracompact but is CC_2-paracompact, being a submetrizable space, as it is finer than the usual topology on $\mathbb{R} \times \mathbb{R}$. ■

Now we need to recall the definition of Epinormality. A topological space (X, τ) is called *epinormal* if there is a coarser topology τ' on X such that (X, τ') is T_4 [8].

Theorem 21. *Every Lindelöf epinormal space with sequential witness of epinormality is CC_2-paracompact.*

Proof. Let (X, τ) be Lindelöf epinormal space, where (X, τ') is the witness of epinormality $\Rightarrow \tau' \subseteq \tau$ is sequential, T_4 and Lindelöf $\Rightarrow \tau' \subseteq \tau$ is sequential, T_2 paracompact space $\Rightarrow (X, \tau)$ has a coarser CC_2-paracompact topology with sequential witness. Thus, (X, τ) is CC_2-paracompact using Theorem 17. □

Recall also that a topological space (X, τ) is called *lower compact* if there is a coarser topology τ' on X such that (X, τ') is T_2 compact [7]. Now using Theorem 17, we get the following corollary.

Corollary 22. *Lower compact spaces with sequential witness of lower compactness is CC_2-paracompact.*

Theorem 23. *CC_2-paracompactness (CC-paracompactness) is not hereditary.*

Proof. $\omega_1 + 1$ with its order topology is T_2 compact and hence is CC_2-paracompact (CC-paracompact). But ω_1, which can be considered as a subspace of $\omega_1 + 1$, is not even CC-paracompact and hence is not CC_2 paracompact, see Example 8. So, we conclude that CC_2-paracompactness (CC-paracompactness) is not hereditary. □

Since T_2 totally normal paracompact spaces are hereditary paracompact [3, Theorem 2], it is easy to see that if a space X is CC_2-paracompact with totally normal witness, then it is hereditary CC_2-paracompact. Example 20 above shows that the Sorgenfrey line square and the rational sequence topology are examples of hereditary CC_2-paracompact spaces.

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