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SOME COMMON FIXED POINT RESULTS FOR T-CONTRACTIONS IN CONE METRIC SPACES OVER A BANACH ALGEBRA

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Abstract. The category of cone metric spaces over Banach algebra is wider than the metric spaces. In this paper, we prove some common fixed point theorems for T-Contraction mappings in cone metric spaces over a Banach algebra. We also use our results to prove the existence of a solution for integral equations.

1. Introduction

In 2009, Beiranvand et al. ([3]) defined T-contractions. Afterward, some results dealing with Kannan contraction, Zamfirescu and another types of fixed point theorems were proved for T-contractions (see [23], [24], [13], [25], [26], [29], [30] and [31]). The key point in T-contractions is that mapping like $f(x) = 2x$ (which does not have a Banach contraction property) can be composed with mapping like $Tx = \frac{1}{x}$, which Tof has Banach contraction property. Since By taking $Tx = x$ T-contractions and contractions are equivalent, then T-contractions are vital improvement.

Cone metric spaces were first introduced by D. Kurepa ([17]) in 1934. Afterward, E. Popa ([27]), L.B. Ciric ([6]), Rzepecki ([36]) and later Long-Guang and Xian ([20]) studied the contraction type mappings in these spaces. Since then, many authors have been interested in the study of fixed point results in the setting of cone metric spaces (see [1], [8], [15], [16], [28], [34]). Recently, some authors investigated whether cone metric spaces are equivalent to metric spaces in terms of the existence of fixed points of the mappings involved (see [4], [10], [9] and [16])). Actually, they showed that any cone metric space (X, d) is equivalent to a usual metric space (X, d^*), where the real-valued metric function d^* is defined by a nonlinear scalarization function ξ_e (see [9]), or by Minkowski functional q_e(see [16]). In the former, d^* is defined as $d^* = \xi_e od$ and in the latter, it is defined as $d^* = q_e od$. Liu and Xu ([18]) introduced the concept of cone metric spaces over Banach algebras replacing Banach spaces with Banach algebras as the underlying space. The

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key point in their paper was that they provided an example of cone metric spaces over a Banach algebra that was not equivalent to the usual metrics $d^* = \xi_e od$, or $d^* = q_e od$ (Example 2.1 of [18]). Not long ago, Xu and Radenovic ([41]) omitted the normality of cones and greatly generalized the main results of [18]. To study more other results see [11], [21], [22], [32] and [38]. Since some cone metric spaces over Banach algebras is not necessary equivalent with usual metric spaces and because the particular importance of cone metric spaces in numerical analysis ([37]), then we prefer to use the cone metric spaces over Banach algebras instead of usual metric spaces.
In this paper, we obtain some common fixed-point results for T-contractions in cone metric spaces over a Banach algebra without normality condition.

2. Some basic notations and definitions

Let E always be a real Banach algebra. That is, E is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in E, \lambda \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\lambda(xy) = (\lambda x)y = x(\lambda y)$;
(4) $||xy|| \leq ||x||||y||$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in E$. An element $x \in E$ is said to be invertible if there is an inverse element $y \in E$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [35]. The following proposition is well known (see [35]).

Proposition 2.1. *Let E be a Banach algebra with a unit e and $x \in A$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||x^n||^{\frac{1}{n}} < 1,$$

then $(e - x)$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark. *From* [35] *we see that the spectral radius $r(x)$ of x satisfies*

$$r(x) \leq ||x||$$

for all $x \in E$, where E is a Banach algebra with a unit e.

Remark. *(*[16]*) In Proposition 2, if the condition "$r(x) < 1$" is replaced by $||x|| \leq 1$, then the conclusion remains true.*

Now let us recall the concepts of cone and partial order for a Banach algebra E. A subset P of E is called a cone of E if

(1) P is non-empty closed and $\{\theta, e\} \subseteq P$;
(2) $\lambda P + \gamma P \subseteq P$ for all non-negative real numbers λ, γ;
(3) $P^2 = PP \subseteq P$;
(4) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra E. For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \subseteq intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$ then P is called a solid cone. The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow ||x|| \leq M||y||,$$

The last positive number satisfying above is called the normal constant of P [20]. In the following we always assume that P is a cone in E with $intP \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 2.2. *(see* [20, 18] *and* [19]*) Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(1) *$0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(3) *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X and (X, d) is called a cone metric space over a Banach algebra E.

For convenience, we give an example of cone metric space over Banach algebra as follows. For more examples, see [18].

Example 2.3. *(*[18]*) Let $E = M_n(\mathbb{R}) = \{a = (a_{ij})_{n\times n} | a_{ij} \in \mathbb{R},$ for all $1 \leq i, j \leq n\}$ be the algebra of all $n-$square real matrices and define the norm*

$$||a|| = \sum_{1\leq i,j\leq n} |a_{ij}|.$$

Then E is a real Banach algebra with the unit e the identity matrix. Let $P = \{a \in E | a_{ij} \geq 0,$ for all $1 \leq i, j \leq n\}$. Then $P \subseteq A$ is a normal cone with normal constant $M = 1$.
Let $X = M_n(\mathbb{R})$ and define the metric $d : X \times X \to E$ by

$$d(x, y) = d((x_{ij})_{n\times n}, (y_{ij})_{n\times n}) = (|x_{ij} - y_{ij}|)_{n\times n} \in E.$$

Then (X, d) is a cone metric space over Banach algebra E with normality.

Example 2.4. *(*[34]*) Let $E = C^1_{\mathbb{R}}[0, 1]$ and define a norm on E by $||x|| = ||x||_\infty + ||x'||_\infty$ for $x \in E$. Define multiplication in E as just pointwise multiplication. Then E is a real unital Banach algebra with unit $e = 1$. The set $P = \{x \in E : x \geq 0\}$ is a cone in E. Moreover, P is not normal.*
Let $X = \{1, 2, 3\}$. Define $d : X \times X \to A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = d(1, 3)(t) = d(3, 1)(t) = 2e^t$, $d(x, x)(t) = 0$. We see that (X, d) is a cone metric space over Banach algebra E without normality.

Definition 2.5. *(*[1]*) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then*
(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 2.6. *(see* [15]*) If E is a real Banach space with a cone P and if $a \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Lemma 2.7. *(see* [28]*) If E is a real Banach space with a solid cone P and $\theta \leq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.8. *(see* [28]*) If E is a real Banach space with a solid cone P and $||x_n|| \to 0$ $(n \to \infty)$, then for any $\theta \ll c$ there exists $N \in \mathbb{N}$ such that for any $n > N$ we have $x_n \ll c$.*

Remark. *(*[35]*) If $r(\alpha) < 1$, then $r(\alpha^n) < 1$ (for any $n \in \mathbb{N}$) and $||\alpha^n|| \to 0$ as $n \to \infty$. Also for any $x \in E$ and $\lambda \geq 0$ we have $r(\lambda x) \leq \lambda r(x)$.*

Definition 2.9. *(*[16] , [8]*) Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subseteq P$ is a c-sequence if for each $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.*

Proposition 2.10. *(*[16]*) Let P be a solid cone in a Banach space E and $\{x_n\}$ and $\{y_n\}$ be two sequences in P. If $\{x_n\}$ and $\{y_n\}$ be c-sequences and $\lambda, \gamma > 0$, then $\{\lambda x_n + \gamma y_n\}$ is a c-sequence.*

The following propositions are crucial for the proof of our main theorems.

Proposition 2.11. *(*[16]*) Let P be a solid cone in a Banach algebra E and $\{x_n\}$ be a sequence in P. Then the following conditions are equivalent:*

1. *$\{x_n\}$ is a c-sequence;*
2. *For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for $n \geq n_0$;*
3. *For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $x_n \leq c$ for $n \geq n_1$.*

Proposition 2.12. *(*[41]*) Let P be a solid cone in a Banach algebra E and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Proposition 2.13. *(*[41]*) Let E be a Banach algebra with a unit e, P be a cone in E and $\leq$ be the partial order generated by the cone P. The following assertions hold true:*

1. *For any $x, y \in E$ with $x \leq y$ and $a \in P$, we have $ax \leq ay$.*
2. *For any sequences $\{x_n\}$, $\{y_n\} \subseteq E$ with $x_n \to x (n \to \infty)$ and $y_n \to y$ $(n \to \infty)$, where $x, y \in E$, we have $x_n y_n \to xy (n \to \infty)$.*

Proposition 2.14. *(*[41]*) Let E be a Banach algebra with a unit e, P be a cone in E and $\leq$ be the partial order generated by the cone P. Let $\alpha \in P$. If the spectral radius $r(\alpha)$ of α is less than 1, then the following assertions hold true:*

1. *Suppose that $x \in E$ is invertible and that $x^{-1} > \theta$ implies $x > \theta$, then for any integer $n \geq 1$, we have $\alpha^n \leq \alpha \leq \theta$.*
2. *For any $u > \theta$, we have $u \not\leq \alpha u$, i.e., $\alpha u - u \notin P$.*
3. *If $\alpha \geq \theta$, then we have $(e - \alpha)^{-1} \geq \theta$.*

Remark. *Proposition 2.14(ii) is the version of Lemma 2.6 in the setting of cone metric spaces over Banach algebras.*

Proposition 2.15. *(*[41]*) Let (X, d) be a complete cone metric space over a Banach algebra E, and let P be the underlying solid cone in Banach algebra E. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then we have*

(1) $\{d(x_n, x)\}$ *is a c-sequence.*
(2) *For any* $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ *is a c-sequence.*

Lemma 2.16. *([41]) Let* E *be a Banach algebra and let* α *be a vector in* E*. If* $0 \leq r(\alpha) < 1$*, then we have* $r((e-\alpha)^{-1}) \leq (1-r(\alpha))^{-1}$*. Also we have* $r((e-\alpha)^{-1}\alpha) < 1$*.*

Lemma 2.17. *Let* (X, d) *be a cone metric space over a Banach algebra* E *with a solid cone* $P \subseteq E$*. Let* $\{x_n\}$ *be a sequence in* X *such that*

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}),$$

for all $n \in \mathbb{N}$ *and* $\alpha \in P$ *where* $0 \leq r(\alpha) < 1$*. Then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For any $n \in \mathbb{N}$ we have

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) \leq \cdots \leq \alpha^n d(x_1, x_0).$$

Thus, for $n < m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\alpha^n + \cdots + \alpha^{m-1}) d(x_1, x_0) \\ &\leq \alpha^n (\sum_{i=0}^{\infty} \alpha^i) d(x_1, x_0) \\ &= \alpha^n (e-\alpha)^{-1} d(x_1, x_0). \end{aligned}$$

Therefore, $||\alpha^n (e-\alpha)^{-1}|| \to 0$ $(n \to \infty)$ and so by Lemma 2.8, for any $c \gg \theta$ there exists $N \in \mathbb{N}$ such that for all $m > n > N$

$$d(x_n, x_m) \leq \alpha^n (e-\alpha)^{-1} d(x_1, x_0) \ll c,$$

which imply that $\{x_n\}$ is a Cauchy sequence. □

Definition 2.18. *([1]) Let* f *and* g *be two self maps on a set* X*. If* $w = fx = gx$ *for some* $x \in X$*, then* x *is called a coincidence point of* f *and* g*, and* w *is called a point of coincidence of* f *and* g*.*

The mappings f and g are weakly compatible if they commute at coincidence points, that is, for each point $x \in X$ such that $fx = gx$ we have $fgx = gfx$.

Proposition 2.19. *([1]) Let* f *and* g *be weakly compatible self maps of a set* X*. If* f *and* g *have a unique point of coincidence* $w = fx = gx$*, then* w *is the unique common fixed point of* f *and* g*.*

Definition 2.20. *([3]) Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is said sequentially convergent if for every sequence* $\{y_n\}$ *such that* $\{Ty_n\}$ *be convergent, then* $\{y_n\}$ *also is convergence.*

Definition 2.21. *([3]) Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is said subsequentially convergent if for every sequence* $\{y_n\}$ *such that* $\{Ty_n\}$ *is convergent, then* $\{y_n\}$ *has a convergent subsequence.*

Proposition 2.22. *([3]) If* (X, d) *is a compact metric space, then every function* $T : X \to X$ *is subsequentially convergent and every continuous function* $T : X \to X$ *is sequentially convergent.*

For a given cone $P \subseteq E$, let $F : P \to P$ satisfy that

(1) $F(\theta) = \theta$ and $F(t) \gg \theta$ for each $t \in P - \{\theta\}$;

(2) F is non-decreasing on P;
(3) F is continuous.

Define $\Im[P] = \{F|F \quad \text{satisfies} \quad 1-3\}$.

Lemma 2.23. *Let $F \in \Im[P]$. If $\lim_{n\to\infty} F(\epsilon_n) = \theta$ for the sequence $\{\epsilon_n\} \subseteq P$, then $\lim_{n\to\infty} \epsilon_n = \theta$*

Proof. Suppose that the result is false. Then there exist $\delta \gg \theta$ and a subsequence $\{\epsilon_{n_i}\}$ of $\{\epsilon_n\}$ such that $\epsilon_{n_i} \gg \delta$ for each i. Hence

$$\theta \ll F(\delta) \leqslant F(\epsilon_{n_i}) \to \theta, \quad n \to \infty,$$

that is a contradiction. □

For a given cone $P \subseteq E$, let $\psi : P \to P$ satisfy that

(1) $\psi(t) \ll t$ for each $t \in P - \{\theta\}$;
(2) ψ is non-decreasing continuous;
(3) for each $t \in P - \{\theta\}$, $\lim_{n\to\infty} \psi^n(t) = \theta$.

Define $\Psi[P] = \{\psi|\psi \quad \text{satisfies} \quad 1-3\}$.

Lemma 2.24. *Let $\psi \in \Psi[P]$, then $\psi(\theta) = \theta$.*

Proof. For any $t \in P - \{\theta\}$ we have

$$\theta \leq \psi(\theta) \leq \psi(t) \ll t.$$

Therefore, $\psi(\theta) = \theta$. □

3. Main Results

Theorem 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra E and let P be the underlying solid cone with $\alpha \in P$ where $r(\alpha) < 1$. Also, suppose*
(i) $T : X \to X$ is a one-to-one, continuous and subsequentially convergent mapping;
(ii) $f(X) \subseteq g(X)$;
(iii) $Tg(X)$ is a complete subspace of X, and $g(X)$ is a closed subset of X;
(iv) $f, g : X \to X$ are two mappings that satisfy

$$d(Tfx, Tfy) \leq \alpha d(Tgx, Tgy) \tag{3.1}$$

for all $x, y \in X$. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ is an arbitrary point in X. By condition (ii) we may choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Therefore

$$d(Tgx_{n+1}, Tgx_n) = d(Tfx_n, Tfx_{n-1}) \leq \alpha d(Tgx_n, Tgx_{n-1})$$
$$\leq \cdots \leq \alpha^n d(Tgx_1, Tgx_0).$$

So by Lemma 2.17, $\{Tgx_n\}_{n\geq 1}$ is a Cauchy sequence in $Tg(X)$. By (iii), there exists $w \in Tg(X)$ such that $Tgx_n \to w$ as $n \to \infty$. Since T is subsequentially convergent, gx_n has a convergent subsequence. So, there exist $q \in X$ and subsequence $\{n_k\}$ of $\{n\}$ such that $gx_{n_k} \to q$ as $k \to \infty$. Now, choose $p \in X$ such that $gp = q$. Further,

$$d(Tgx_{n_k}, Tfp) = d(Tfx_{n_k-1}, Tfp) \leq \alpha d(Tgx_{n_k-1}, Tgp).$$

But, $Tgx_{n_k-1} \to gp$ as $k \to \infty$.

By proposition 2.12 for each $\varepsilon \gg \theta$, there exists a natural number N such that for all $k \geq N$, $d(Tgx_{n_k}, Tfp) \ll \varepsilon$ and so $Tgx_{n_k} \to Tfp$. The uniqueness of a limit in a cone metric space implies that $Tfp = Tgp$. Since T is one-to-one, we get $fp = gp$. Now if $fz = gz$, then

$$\theta \leq d(Tgz, Tgp) = d(Tfz, Tfp) \leq \alpha d(Tgz, Tgp).$$

By proposition 2.14 we get $d(Tgz, Tgp) = \theta$. Hence, $Tgz = Tgp$ and $gz = gp$. This implies that f and g have a unique point of coincidence in X. By proposition 2.19, f and g have a unique common fixed point. □

Remark. *In theorem 3.1, we only assume that the spectral radius of α is less than 1. This is a vital improvement. In fact, the condition $r(\alpha) < 1$ is weaker than that $\alpha < e$ and $||\alpha|| < 1$, as is illustrated by Example 2.1 of* [18].

All the conditions of theorem 3.1 are satisfied in the following example.

Example 3.2. *Let $X = [0,1]$ and $E = C^1_{\mathbb{R}}[0,1]$ with the norm $||x|| = ||x||_\infty + ||x'||_\infty$. Consider cone $P = \{\varphi \in E | \varphi \geq 0\}$. By example 2.4 we know that P is not normal. Define $d(x,y) = |x-y|e^t$ where $e^t \in E$ on X(in this example, suppose e is the Napier number). Then (X,d) is a cone metric space over the Banach algebra E. Suppose that $T : X \to X$ is defined by $Tx = \frac{1}{x}$. Also suppose that $f, g : X \to X$ are defined by $f(x) = 30x$ and $g(x) = ex$. Consider the function $\alpha(t) = \frac{e^{-t}}{3}$ on X. We have $\alpha(t) \in E$ and*

$$\begin{aligned} r(\alpha(t)) &= \lim_{n\to\infty} ||(\alpha(t))^n||^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{3}(||e^{-nt}||_\infty + n||e^{-t}||_\infty)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{3}(1+n)^{\frac{1}{n}} = \frac{e}{3} < 1. \end{aligned}$$

Since for any $t \in [0,1]$, $\alpha(t) \geq \frac{1}{3e}$ we have

$$\begin{aligned} d(Tfx, Tfy) = |\frac{1}{30x} - \frac{1}{30y}|e^t &\leq \frac{1}{3e}\frac{1}{e}|\frac{1}{x} - \frac{1}{y}|e^t \\ &\leq \alpha(t)|\frac{1}{ex} - \frac{1}{ey}|e^t = \alpha(t)d(Tgx, Tgy). \end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. Hence, 0 is a unique common fixed point of f and g.

Theorem 3.3. *Let (X,d) be a complete cone metric space over a Banach algebra E, and let P be the underlying solid cone with $\alpha \in P$ where $r(\alpha) < \frac{1}{2}$. Also, suppose*
(i) $T : X \to X$ is a one-to-one, continuous and subsequentially convergent mapping;
(ii) $f(X) \subseteq g(X)$;
(iii) $Tg(X)$ is a complete subspace of X, and $g(X)$ is a closed subset of X;
(iv) $f, g : X \to X$ are two mappings that satisfy

$$d(Tfx, Tfy) \leq \alpha(d(Tfx, Tgx) + d(Tfy, Tgy))$$

for all $x, y \in X$. Then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By condition (ii) we may choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen x_n in X, we

obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(Tgx_{n+1}, Tgx_n) &= d(Tfx_n, Tfx_{n-1}) \\ &\leq \alpha(d(Tfx_n, Tgx_n) + d(Tfx_{n-1}, Tgx_{n-1})) \\ &= \alpha(d(Tgx_{n+1}, Tgx_n) + d(Tgx_n, Tgx_{n-1})). \end{aligned}$$

Hence, $d(Tgx_{n+1}, Tgx_n) \leqslant \beta d(Tgx_n, Tgx_{n-1})$, where $\beta = \alpha(e-\alpha)^{-1}$. By lemma 2.16 we have $r(\beta) < 1$. So by Lemma 2.17, $\{Tgx_n\}_{n\geq 1}$ is a Cauchy sequence in $Tg(X)$. By (iii), there exists $w \in Tg(X)$ such that $Tgx_n \to w$ as $n \to \infty$. Since T is subsequentially convergent, then gx_n has a convergent subsequence. So, there exist $q \in X$ and the subsequence $\{n_k\}$ of $\{n\}$ such that $gx_{n_k} \to q$ as $k \to \infty$. Now, choose $p \in X$ such that $gp = q$. Further,

$$\begin{aligned} d(Tgx_{n_k}, Tfp) &= d(Tfx_{n_k-1}, Tfp) \\ &\leq \alpha(d(Tfx_{n_k-1}, Tgx_{n_k-1}) + d(Tfp, Tgp)) \\ &\leq \alpha(d(Tgx_{n_k}, Tgp) + d(Tgp, Tgx_{n_k-1}) \\ &\quad + d(Tfp, Tgx_{n_k}) + d(Tgx_{n_k}, Tgp)), \end{aligned}$$

and

$$(e-\alpha)d(Tgx_{n_k}, Tfp) = 2\alpha d(Tgx_{n_k}, Tgp) + \alpha d(Tgp, Tgx_{n_k-1}).$$

Therefore

$$d(Tgx_{n_k}, Tfp) \leq 2\beta d(Tgx_{n_k}, Tgp) + \beta d(Tgp, Tgx_{n_k-1}).$$

But, $Tgx_{n_k-1} \to Tgp$. Hence, by propositions 2.12, 2.13, 2.14 and 2.15 for each $\varepsilon \gg 0$, there exists a natural number N such that for all $k \geq N$, $d(Tgx_{n_k}, Tfp) \ll \varepsilon$ and so $Tgx_{n_k} \to Tfp$. Hence, $Tfp = Tgp$. Since T is one-to-one, we get $fp = gp$. Now if $fz = gz$, then

$$\theta \leq d(Tgz, Tgp) = d(Tfz, Tfp) \leq \alpha(d(Tfz, Tgz) + d(Tfp, Tgp)) = \theta.$$

Therefore, $Tgz = Tgp$ and $gz = gp$. This implies that f and g have a unique point of coincidence in X. By Proposition 2.19, f and g have a unique common fixed point. □

Theorem 3.4. *Let (X, d) be a complete cone metric space over a Banach algebra E and let P be the underlying solid cone with $\alpha \in P$ where $r(\alpha) < \frac{1}{2}$. Also, suppose*
(i) $T : X \to X$ is a one-to-one, continuous and subsequentially convergent mapping;
(ii) $f(X) \subseteq g(X)$;
(iii) $Tg(X)$ is a complete subspace of X, and $g(X)$ is a closed subset of X;
(iv) $f, g : X \to X$ are two mappings that satisfy

$$d(Tfx, Tfy) \leq \alpha(d(Tfx, Tgy) + d(Tfy, Tgx))$$

for all $x, y \in X$. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By condition (ii) we may choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen x_n in X, we

obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(Tgx_{n+1}, Tgx_n) &= d(Tfx_n, Tfx_{n-1}) \\ &\leq \alpha(d(Tfx_n, Tgx_{n-1}) + d(Tfx_{n-1}, Tgx_n)) \\ &= \alpha(d(Tgx_{n+1}, Tgx_{n-1}) + d(Tgx_n, Tgx_n)) \\ &= \alpha d(Tgx_{n+1}, Tgx_{n-1}) \\ &\leq \alpha(d(Tgx_{n+1}, Tgx_n) + d(Tgx_n, Tgx_{n-1})). \end{aligned}$$

Hence, $d(Tgx_{n+1}, Tgx_n) \leqslant \beta d(Tgx_n, Tgx_{n-1})$, where $\beta = \alpha(e-\alpha)^{-1}$ with $r(\beta) < 1$. Therefore by Lemma 2.17, $\{Tgx_n\}_{n\geq 1}$ is a Cauchy sequence in $Tg(X)$. by (iii), there exists $w \in Tg(X)$ such that $Tgx_n \to w$ as $n \to \infty$. Since T is subsequentially convergent, gx_n has a convergent subsequence. So, there exist $q \in X$ and the subsequence $\{n_k\}$ of $\{n\}$ such that $gx_{n_k} \to q$ as $k \to \infty$. Now, choose $p \in X$ such that $gp = q$. Further,

$$\begin{aligned} d(Tgx_{n_k}, Tfp) &= d(Tfx_{n_k-1}, Tfp) \\ &\leq \alpha(d(Tfx_{n_k-1}, Tgp) + d(Tfp, Tgx_{n_k-1})) \\ &\leq \alpha(d(Tgx_{n_k}, Tgp) + d(Tgp, Tgx_{n_k-1}) \\ &\quad + d(Tfp, Tgx_{n_k}) + d(Tgx_{n_k}, Tgp)). \end{aligned}$$

Therefore

$$d(Tgx_{n_k}, Tfp) \leq 2\beta d(Tgx_{n_k-1}, Tgp) + \beta d(Tgp, Tgx_{n_k-1}).$$

But, $Tgx_{n_k-1} \to Tgp$. Hence by propositions 2.12, 2.13, 2.14 and 2.15 for each $\varepsilon \gg 0$, there exists a natural number N such that for all $k \geq N$, $d(Tgx_{n_k-1}, Tgp) \ll \varepsilon$ and so $Tgx_{n_k} \to Tfp$. Hence, $Tfp = Tgp$. Since T is one-to-one, we get $fp = gp$. Now if $fz = gz$, then

$$\begin{aligned} \theta \leq d(Tgz, Tgp) &= d(Tfz, Tfp) \\ &\leq \alpha(d(Tfz, Tgp) + d(Tfp, Tgz)) = 2\alpha d(Tgz, Tgp). \end{aligned}$$

which (by proposition 2.14 (2)) implies that $d(Tgz, Tgp) = \theta$, $Tgz = Tgp$ and so $gz = gp$. This implies that f and g have a unique point of coincidence in X. By Proposition 2.19, f and g have a unique common fixed point. □

Here, we provide another result related to common fixed point of self maps on cone metric spaces.

Theorem 3.5. *Let (X, d) be a complete cone metric space over a Banach algebra E and let P be the underlying solid cone with $\alpha \in P$ where $r(\alpha) < 1$. Also, suppose (i) $T : X \to X$ is a one-to-one, continuous and subsequentially convergent mapping; (ii) $f, g : X \to X$, $F \in \Im[P]$, $\psi \in \Psi[C]$ and there exists*

$$u \in M(Tx, Ty) = \{d(Tx, Ty), d(Tfx, Tx), d(Tgy, Ty)\}$$

such that

$$F(d(Tfx, Tgy)) \leq \psi(F(u)),$$

for all $x, y \in X$. Then f and g have a unique common fixed point. Moreover, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ has a convergent subsequence to the common fixed point of f and g.

Proof. We first prove that any fixed point of f is also a fixed point of g, and vice versa. If $fx^* = x^*$ but $gx^* \neq x^*$, then we have

$$\begin{aligned} M(Tx^*, Tx^*) &= \{d(Tx^*, Tx^*), d(Tfx^*, Tx^*), d(Tgx^*, Tx^*)\} \\ &= \{d(Tgx^*, Tx^*), \theta\}. \end{aligned}$$

Since F is non-decreasing, we get

$$\begin{aligned} F(d(Tx^*, Tgx^*)) &= F(d(Tfx^*, Tgx^*)) \\ &\leq \psi(F(d(Tgx^*, Tx^*))) \ll F(d(Tgx^*, Tx^*)). \end{aligned}$$

This is a contradiction, so $Tgx^* = Tx^*$. Since T is one-to-one, we get $gx^* = x^*$. Similarly if $gx^* = x^*$, then $fx^* = x^*$.
Now we show that if f and g have a common fixed point, then the fixed point is unique. Let $fx^* = gx^* = x^*$ and $fy^* = gy^* = y^*$. If $x^* \neq y^*$, then we have

$$\begin{aligned} M(Tx^*, Ty^*) &= \{d(Tx^*, Ty^*), d(Tfx^*, Tx^*), d(Tgy^*, Ty^*)\} \\ &= \{d(Tx^*, Ty^*), \theta\}. \end{aligned}$$

Since F is non-decreasing, we get

$$\begin{aligned} F(d(Tx^*, Ty^*)) &= F(d(Tfx^*, Tgy^*)) \\ &\leq \psi(F(d(Tx^*, Ty^*))) \ll F(d(Tx^*, Ty^*)), \end{aligned}$$

and so we get a contradiction. Thus $x^* = y^*$.
Given $x_0 \in X$, let

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2} = gx_{2n+1}, \quad n = 0, 1, 2, ...$$

Suppose that for any $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Otherwise f or g have a common fixed point and the proof is complete.

$$\begin{aligned} M(Tx_{2n}, Tx_{2n-1}) &= \{d(Tx_{2n}, Tx_{2n-1}), d(Tx_{2n+1}, Tx_{2n}), d(Tx_{2n}, Tx_{2n-1})\} \\ &= \{d(Tx_{2n+1}, Tx_{2n}), d(Tx_{2n}, Tx_{2n-1})\}. \end{aligned}$$

If

$$\begin{aligned} F(d(Tx_{2n+1}, Tx_{2n})) &= F(d(Tfx_{2n}, Tgx_{2n-1})) \\ &\leq \psi(F(d(Tx_{2n+1}, Tx_{2n}))) \ll F(d(Tx_{2n+1}, Tx_{2n})). \end{aligned}$$

We get a contradiction. So we have

$$F(d(Tx_{2n+1}, Tx_{2n})) \ll F(d(Tx_{2n}, Tx_{2n-1})).$$

Similarly we have

$$F(d(Tx_{2n+2}, Tx_{2n+1})) \ll F(d(Tx_{2n+1}, Tx_{2n})).$$

Generally, for each $n \in \mathbb{N}$ we get

$$F(d(Tx_{n+1}, Tx_n)) \ll F(d(Tx_n, Tx_{n-1}))$$

and

$$F(d(Tx_{n+1}, Tx_n)) \leq \psi(F(d(Tx_n, Tx_{n-1}))) \leq \cdots \leq \psi^n(F(d(Tx_0, Tx_1))).$$

Let $n \to \infty$, we have $F(d(Tx_{n+1}, Tx_n)) \to \theta$. By lemma 2.23, $d(Tx_{n+1}, Tx_n) \to \theta$ as $n \to \infty$. Now we prove that $\{Tx_n\}$ is a Cauchy sequence. Suppose not, then there exist $\epsilon \gg \theta$ and subsequences $\{Tx_{n_i}\}$ and $\{Tx_{m_i}\}$ of $\{Tx_n\}$ with $n_i < m_i$ such that

$$d(Tx_{n_i}, Tx_{m_i}) \geq \epsilon$$

for each i. Since $d(Tx_{n+1}, Tx_n) \to \theta$, when i is large enough, $d(Tx_{n_i}, Tx_{n_i+1}) \ll \frac{\epsilon}{2}$ and $d(Tx_{m_i}, Tx_{m_i+1}) \ll \frac{\epsilon}{2}$. By inequalities

$$\begin{aligned} d(Tx_{n_i+1}, Tx_{m_i}) &\geq d(Tx_{n_i}, Tx_{m_i}) - d(Tx_{n_i+1}, Tx_{n_i}) \gg \epsilon, \\ d(Tx_{n_i}, Tx_{m_i-1}) &\geq d(Tx_{n_i}, Tx_{m_i}) - d(Tx_{m_i-1}, Tx_{m_i}) \gg \epsilon, \\ d(Tx_{n_i+1}, Tx_{m_i-1}) &\geq d(Tx_{n_i}, Tx_{m_i}) - d(Tx_{m_i-1}, Tx_{m_i}) - d(Tx_{n_i+1}, Tx_{n_i}) \gg \epsilon. \end{aligned}$$

We can assume that n_i are even numbers, m_i are odd numbers and

$$d(Tx_{n_i}, Tx_{m_i}) \gg \epsilon$$

for all i. Set $k_i = min\{m_i | d(Tx_{n_i}, Tx_{m_i}) \gg \epsilon, \quad m_i \quad \text{is odd number}\}$. We have

$$\begin{aligned} \epsilon &\ll d(Tx_{n_i}, Tx_{k_i}) \\ &\leq d(Tx_{n_i}, Tx_{k_i-2}) + d(Tx_{k_i-2}, Tx_{k_i-1}) + d(Tx_{k_i-1}, Tx_{k_i}) \\ &\leq \epsilon + d(Tx_{k_i-2}, Tx_{k_i-1}) + d(Tx_{k_i-1}, Tx_{k_i}) \to \epsilon \quad (i \to \infty) \end{aligned}$$

By proposition 2.10 we have $d(Tx_{n_i}, Tx_{k_i}) \to \epsilon$ as $i \to \infty$. By inequality

$$\begin{aligned} d(Tx_{n_i}, Tx_{k_i}) - d(Tx_{n_i}, Tx_{n_i+1}) - d(Tx_{k_i}, Tx_{k_i+1}) \\ \leq d(Tx_{n_i+1}, Tx_{k_i+1}) \\ \leq d(Tx_{n_i}, Tx_{k_i}) + d(Tx_{n_i}, Tx_{n_i+1}) + d(Tx_{k_i}, Tx_{k_i+1}), \end{aligned}$$

let $i \to \infty$, we get $d(Tx_{n_i+1}, Tx_{k_i+1}) \to \epsilon$. On the other hand we have

$$M(Tx_{n_i}, Tx_{k_i}) = \{d(Tx_{n_i}, Tx_{k_i}), d(Tx_{n_i+1}, Tx_{n_i}), d(Tx_{k_i+1}, Tx_{k_i})\}.$$

Since

$$\begin{aligned} d(Tx_{n_i}, Tx_{k_i}) &\leq d(Tx_{n_i}, Tx_{k_i}) + d(Tx_{n_i}, Tx_{n_i+1}), \\ d(Tx_{n_i}, Tx_{n_i+1}) &\leq d(Tx_{n_i}, Tx_{k_i}) + d(Tx_{n_i}, Tx_{n_i+1}), \\ d(Tx_{n_i}, Tx_{k_i}) &\leq d(Tx_{n_i}, Tx_{k_i}) + d(Tx_{n_i}, Tx_{n_i+1}) \end{aligned}$$

and because F is non-decreasing, we get

$$F(d(Tx_{n_i+1}, Tx_{k_i+1})) = F(d(Tfx_{n_i}, Tgx_{k_i})) \leq \psi(F(d(Tx_{n_i}, Tx_{k_i}) + \delta_i))$$

where $\delta_i \to \theta$ as $i \to \infty$. Let $i \to \infty$, we get

$$F(\epsilon) \leq \psi(F(\epsilon)) \ll F(\epsilon).$$

This is a contradiction. Therefore $\{Tx_n\}$ is a Cauchy sequence. By completeness of X, there exist $a \in X$ such that $Tx_n \to a \in X$ as $n \to \infty$. Since T is subsequentially convergent, x_n has a convergent subsequence. So, there exist $x \in X$ and the subsequence $\{n_j\}$ of $\{n\}$ such that $x_{n_j} \to x$ as $j \to \infty$. Since

$$M(Tx_{2n_j}, Tx) = \{d(Tx_{2n_j}, Tx), d(Tx_{2n_j+1}, Tx_{2n_j}), d(Tgx, Tx)\}.$$

We have

$$F(d(Tx_{2n_j}, Tx) \leq \psi(F(d(Tx_{2n_j}, Tx)))$$

or

$$F(d(Tx_{2n_j}, Tx) \leq \psi(F(d(Tx_{2n_j+1}, Tx_{2n_j})))$$

or

$$F(d(Tx_{2n_j}, Tx) \leq \psi(F(d(Tgx, Tx)))$$

Then if $j \to \infty$, we get

$$F(d(Tx, Tgx)) \leq \psi(F(d(Tx, Tgx))) \ll F(d(Tx, Tgx)).$$

Therefore $d(Tx, Tgx) = \theta$, $Tx = Tgx$ and $x = gx$. By the proof above, x is the unique common fixed point of f and g. □

4. Application to nonlinear integral equations

The purpose of this section is to study the existence of a solution for a system of nonlinear integral equations using the results we obtained.
Let $X = E = C([0,T],\mathbb{R})$ (the set of continuous functions defined on $[0,T]$ and taking values in $\mathbb{R}$) with the norm $||x|| = ||x||_\infty + ||x'||_\infty$, and $P = \{x \in E : x \geq 0\}$. Define $d : X \times X \to E$ by

$$d(x,y) = (\sup_{t\in[0,T]} |x(t) - y(t)|)e^t.$$

Then d is a cone metric space on X. It is easy to see that (X,d) is a complete cone metric space over Banach algebra E.
Now Consider the following system of integral equations

$$F(x)(t) = \int_0^T G(t,s)u(t,x(s))ds + w(t), \tag{4.1}$$

for $t \in [0,T], T > 0$.

Theorem 4.1. *Suppose that the following hold:*
(a) $G : [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function;
(b) $w \in C([0,T],\mathbb{R})$;
(c) $u : [0,T] \to \mathbb{R}$ is a continuous function;
(d) For all $x, y \in X$ and $t \in [0,T]$, we have

$$|u(t,x(t)) - u(t,y(t))| \leq |x(t) - y(t)|$$

(e) $\int_0^T |G(t,s)|ds < \alpha(t)$ where $r(\alpha) < 1$.
Then the integral equation (4.1) has a unique solution in $C([0,T],\mathbb{R})$.

Proof. It is easy to see that x is a solution for the integral equation (4.1) if and only if x is a fixed point of F. Existence of such a point is deducted from theorem 3.1, by assuming G as identity mapping. So we have to check that all the conditions of theorem 3.1 are met. For all $x, y \in X$ and $t \in [0,T]$, we have

$$\begin{aligned} |F(x)(t) - F(y)(t)| &\leq \int_0^T |G(t,s)||u(t,x(s)) - u(t,y(s))|ds \\ &\leq \int_0^T |G(t,s)||x(t) - y(t)|ds \\ &\leq (\int_0^T |G(t,s)|ds)d(x,y). \end{aligned}$$

which proves that

$$d(Fx, Fy) \leq \alpha(t)d(x,y), \quad x, y \in X.$$

This shows that the contractive condition of theorem 3.1 is met. It is easy to see that the other conditions of theorem 3.1 are also met. Therefore, F has a unique fixed point $x \in C([0,T],\mathbb{R})$ which is the unique solution of the integral equation (4.1). □

5. Conclusion

In this paper, we extend the theorems 2.1, 2.3 and 2.4 of Abbas and Jungck([1]) for T-contractions in cone metric spaces over a Banach algebra, and we proved Theorem 3.5 that extends the special case of Theorem 1 of Zhang ([42]) for T-contractions in cone metric spaces over a Banach algebra, and finally we prove the existence of a solution for some integral equations.

Remark. *One of the closest developments to our results is the study of the best proximity point theorems (*[5]*) for T-contractions. Another useful developments is the investigation of common fixed point theorems for cyclic mappings (*[39]*) in the case of T-contractions. Furthermore, it seems that, it is interesting to prove the common fixed point theorems in $G-$metric spaces (*[2]*) and partially ordered cone metric spaces (*[40]*) in the case of T-contractions.*

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