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EPI-ALMOST NORMALITY

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Abstract. A space $(X , \mathcal{T})$ is called *epi-almost normal* if there exists a coarser topology $\mathcal{T}'$ on X such that $(X , \mathcal{T}')$ is Hausdorff (T_2) almost normal. We investigate this property and present some examples to illustrate the relationships between epi-almost normality and other weaker kinds of normality.

1. Introduction

In this paper, we define a new topological property called epi-almost normality. Unlike epi-mild normality [9] and epinormality [8]. We investigate this property and we using the technique of semi-regularization to illustrate the relationships between epi-almost normality and partial normality [2], and to improve the Theorem "Any almost normal; almost regular is almost completely Hausdorff " [11].
Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, the set of rational numbers by $\mathbb{Q}$, the set of irrational numbers by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space, a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space, and a T_3 space is a T_1 regular space. A space X is called *completely Hausdorff* if for each distinct elements $a, b \in X$ there exist two open sets U and V such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. We do not assume T_2 in the definition of compactness, countable compactness and paracompactness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. (X ,τ_s) is the semi-regularization of (X , τ).

Definition 1.1. *A subset A of a space X is called regular closed* [5], if $A = \overline{\text{int}A}$. A subset A of a space X is called *regular open* [5], if $A = \text{int}(\overline{A})$. A subset A of a space X is said to be *π-closed* if it is a finite intersection of regular closed sets. A subset A of a space X is said to be *π-open* if it is a finite union of regular open sets. A space X is called *almost normal* [11], if for any two disjoint closed sets A and B of X one of them regular closed there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is called *mildly normal* [13], called also *κ-normal* [15], if for any two disjoint regular closed sets A and B of X there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is said to be an *almost regular* if for any closed domain subset A and any $x \notin A$,

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there exist two disjoint open sets U and V such that $x \in U$ and $A \subseteq V$ [14]. A space X is said to be an *almost completely regular* if for any regular closed subset A and any $x \notin A$, there is a continuous function f on X into the closed interval $[0, 1]$ such that $f(x) = 1$ and $f(A) = 0$ [14]. A space X is said to be *semi-normal* if for every closed set A contained in an open set U, there exists a regularly open set V such that $A \subset V \subset U$.

2. Epi-Almost Normality

Definition 2.1. *A space* $(X, \mathcal{T})$ *is called epi-almost normal* if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_2 (Hausdorff) almost normal.

Note that if we do not presume T_2 in Definition 2 above, then any space will be epi-almost normal as the indiscreet topology will refine. Also, if we assume $(X, \mathcal{T}')$ to be T_1 almost normal in Definition 2 above, then any T_1 space will be epi-almost normal as the finite complement topology, see [16], will refine.

Observe that if $\mathcal{T}'$ and $\mathcal{T}$ are two topologies on X such that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and $(X, \mathcal{T}')$ is T_i, $i \in \{0, 1, 2, \}$, then so is $(X, \mathcal{T})$. So, we conclude the following:

Theorem 2.2. *Every epi-almost normal space is* T_2*.*

Recall that a topological space X is called *completely Hausdorff,* $T_{2\frac{1}{2}}$ [16], if for each distinct elements $a, b \in X$ there exist two open sets U and V such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. The space $(X, \mathcal{T}_s)$ is called the semi-regularization of $(X, \mathcal{T})$ if the topology on $(X, \mathcal{T}_s)$ generated by the family of all regularly open sets in $(X, \mathcal{T})$.

Theorem 2.3. *Every epi-almost normal space is completely Hausdorff.*

Proof. Let $(X, \mathcal{T})$ be any epi-almost normal space and $\mathcal{T}'$ is the witness of epi-almost normality. Then $\mathcal{T}'$ is almost regular. Let $\mathcal{T}_s$ be the semi-regularization of $\mathcal{T}'$. Since $(X, \mathcal{T}')$ is Hausdorff almost regular space, $\mathcal{T}_s$ is Hausdorff regular space [10], and hence it is a completely Hausdorff. By using Proposition 3 [10], $(X, \mathcal{T}')$ is completely Hausdorff. Thus $(X, \mathcal{T})$ is completely Hausdorff. □

We note that any space which is not completely Hausdorff cannot be epi-almost normal, e.g., Prime Integer topology and Double origin topology [16].

A topological space $(X, \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4 [8]. A topological space $(X, \mathcal{T})$ is called *epi-mildly normal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_2 (Hausdorff) mildly normal [8]. From the definitions we conclude the following implications.

$$\text{epinormality} \Longrightarrow \text{epi-almost normality} \Longrightarrow \text{epi-mild normality.}$$

Since every T_1 almost normal is almost regular and every almost regular semiregular is regular [10], then we have the following theorem.

Theorem 2.4. *In semiregular space, epi-almost normality and epinormality are equivalent.*

Note that epi-almost normality does not imply epi-mild normality and there is an example.

Example 2.5. *Let* ($\mathbb{N}$, $\mathcal{RPI}$) *be Relatively Prime Integer topology* $\mathcal{RPI}$ [16] *which generated by the basis* $\mathcal{B} = \{B_a(b) : a, b \in X, gcd(a, b) = 1\}$*, where* $B_a(b) = \{b+na \in X : n \in \mathbb{Z}\}$*. Note that* ($\mathbb{N}$, $\mathcal{RPI}$) *is Hausdorff but not completely Hausdorff* [16]*. Hence it is not epi-almost normal.*
Claim: ($\mathbb{N}$, $\mathcal{RPI}$) *is mildly normal.*
Proof Claim: Let F *and* E *be two arbitrary disjoint regular closed sets. Hence if* $x \in \text{int}(F)$ *and* $y \in \text{int}(E)$*, then* $x \neq y$*. Since* X *is Hausdorff, there exist* $B_x(b)$ *and* $B_y(d)$ *disjoint neighborhoods of* x *and* y *respectably such that* $x \in B_x(b) \in \text{int}(F)$ *and* $y \in B_y(d) \in \text{int}E$*.* $x \in \overline{B_x(b)} \in F$ *and* $y \in \overline{B_y(d)} \in E$*. But the closures of* $B_y(d)$ *and* $B_x(b)$ *contain in common all multiples of* $[x, y]$*. Hence* $B_y(d)$ *and* $B_x(b)$ *are intersects, contradiction with* E *and* F *are disjoint. Thus, any two non-empty regular closed must intersect. Therefore* $\mathcal{RPI}$ *is mildly normal.*
Since $\mathcal{RPI}$ *is Hausdorff mildly normal, then* $\mathcal{RPI}$ *is epi-mildly normal which is not epi-almost normality* ■

Theorem 2.6. *Epi-almost normal is topological property.*

Proof. Let $(X,\mathcal{T})$ be any epi-almost normal space. Assume that $(X,\mathcal{T}) \cong (Y,\mathcal{S})$. Let $\mathcal{T}'$ be a coarser topology on X such that $(X,\mathcal{T}')$ is Hausdorff almost normal space. Let $f : (X,\mathcal{T}) \longrightarrow (Y,\mathcal{S})$ be a homeomorphism and define $\mathcal{S}'$ on Y by $\mathcal{S}'=\{f(U) : U \in \mathcal{T}'\}$. Then $\mathcal{S}'$ is a topology on Y coarser than $\mathcal{S}$ and $(Y,\mathcal{S}')$ is Hausdorff almost normal. □

Theorem 2.7. *The sum* $\oplus_{\alpha\in\Lambda}X_\alpha$*, where* X_α *is a space for each* $\alpha \in \Lambda$*, is epi-almost normal if and only if all spaces* X_α *are epi-almost normal.*

Proof. If the sum $(\oplus_{\alpha\in\Lambda}X_\alpha, \oplus_{\alpha\in\Lambda}\mathcal{T}_\alpha)$ is epi-almost normal, then there exist $\mathcal{T}^{'}$ topology on $\oplus_{\alpha\in\Lambda}X_\alpha$, coarser than $\oplus_{\alpha\in\Lambda}\mathcal{T}_\alpha$ such that $(\oplus_{\alpha\in\Lambda}X_\alpha,\mathcal{T}')$ is a Hausdorff almost normal space. Since X_α is clopen in $\oplus_{\alpha\in\Lambda}X_\alpha$ for each $\alpha \in \Lambda$, $(X_\alpha,\mathcal{T}^{'}_\alpha)$, where $\mathcal{T}'_\alpha=\{U \cap X_\alpha : U \in\mathcal{T}'\}$, is a Hausdorff almost normal space. Thus all spaces X_α are epi-almost normal as $(X_\alpha,\mathcal{T}^{'}_\alpha)$ is coarser topology than $(X_\alpha,\mathcal{T}_\alpha)$. Conversely, if all the X_α's are epi-almost normal, then there exists a topology $\mathcal{T}^{'}_\alpha$ on X_α for each $\alpha \in \Lambda$, coarser than $\mathcal{T}_\alpha$ such that $(X_\alpha,\mathcal{T}^{'}_\alpha)$ is a Hausdorff almost normal space. Since T_2 and almost normality are both additive [5], then $(\oplus_{\alpha\in\Lambda}X_\alpha, \oplus_{\alpha\in\Lambda}\mathcal{T}_\alpha)$ is a Hausdorff almost normal space. Therefore $(\oplus_{\alpha\in\Lambda}X_\alpha, \oplus_{\alpha\in\Lambda}\mathcal{T}^{'}_\alpha)$ is epi-almost normal. □

Theorem 2.8. *If* X *is epi-almost normal countably compact and* M *is Hausdorff paracompact first countable, then* $X \times M$ *is epi-almost normal.*

Proof. Let $(X,\mathcal{T})$ be any epi-almost normal countably compact space. Then there exits coarser topology $\mathcal{T}'$ on X such that $(X,\mathcal{T}')$ is Hausdorff almost normal space. Since the coarser topology of countably compact is countably compact, so $(X,\mathcal{T}') \times M$ is Hausdorff almost normal, by Theorem 9 of [7]. Thus $X \times M$ is epi-almost normal. □

Epi-almost normal version of Stones theorem.

Corollary 2.9. *If* X *is epi-almost normal countably compact and* M *is metrizable, then* $X \times M$ *is epi-almost normal.*

Let us recall the following definition from [2]

Definition 2.10. *A topological space* X *is called partially normal if any two disjoint subsets* A *and* B *of* X*, where* A *is regular closed and* B *is* π*-closed, are separated.*

Theorem 2.11. *(X, τ) is partial normality if and only if the semi-regularization of X is almost normal.*

Proof. Let (X, τ) be partial normal space, A and B be disjoint closed sets in τ_s such that A is regular closed in τ_s. Hence, $B = E \cap F$ where E and F are regular closed sets in τ_s. Thus, A and B are disjoint closed sets in τ such that A is regular closed and B is π-closed. By partial normality, there exist two open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Then there exist two disjoint regular open sets U_s and V_s containing U and V respectively, see [4]. Thus, A and B are separated by two open sets in τ_s. Therefore, (X, τ_s) is almost normality.
Conversely, let A and B be disjoint closed sets in τ such that A is π-closed and B is regular closed. Hence A and B are disjoint closed sets in τ_s such that B is regular closed. Since (X, τ_s) is almost normal, there exist disjoint open sets U and V containing A and B ,respectively. Therefore (X, τ) is partial normal. □

Corollary 2.12. *almost normal is not a semiregular property, but partial normality is.*

Proof. The Half disc topology is not almost normal [6]. But its semi-regularization τ_s is the usual topology on the closed upper half plane. Hence the semi-regularization of half disc topology is almost normal, indeed metrizable. Applying Theorem 2.11 and using [[10], Lemma 5] show that (X, τ_s) is partial normal if and only if it is almost normal. But (X, τ) is partial normality, since (X, τ_s) is almost normal, using Theorem 2.11. Therefore partial normality is a semiregular property. □

It is clear from the definition that any T_2 almost normal space is epi-almost normal, just take the coarser topology equal the same topology. But here we have something stronger.

Corollary 2.13. *Every Hausdorff partial normal space is epi-almost normal.*

Corollary 2.14. *A semiregular space is partial normal if and only if it is almost normal.*

Corollary 2.15. *If (X, τ) is seminormal partial normal space, then (X, τ_s) is normal space.*

Proof. Let B be any open set containing a closed set A in (X, τ_s). By seminormality, there exists an open set U such that $A \subseteq U \subseteq int(\overline{U}) \subseteq B$. Since $int(\overline{U})$ is regular open and using Theorem 2.11, there exists an open set V in (X, τ_s) such that $A \subseteq V \subseteq \overline{V} \subseteq int(\overline{U}) \subseteq B$.Thus (X, τ_s) is normal space. □

Corollary 2.16. *If (X, τ) is seminormal partial normal space and τ_s is T_1, then (X, τ) is epinormal.*

In [11] proved that "Any almost normal; almost regular is almost completely Hausdorff ", we use this theorem to prove the following Corollary.

Corollary 2.17. *If X is almost normal almost regular then X is completely Hausdorff space.*

Corollary 2.18. *If X is almost normal almost regular then X is epi-almost normal.*

Corollary 2.19. *If (X , τ) is partial normal and τ_s is T_1, then (X , τ) is completely Hausdorff space.*

Proof. Since (X , τ) is partial normal space,then the semi-regularization of X is almost normal, hence almost regular. Moreover, almost regular almost normal is completely Hausdorff space, by Corollary 2.17, so the semi-regularization of X is completely Hausdorff space. Therefore (X , τ) is completely Hausdorff. □

Corollary 2.20. *If (X , τ) is partial normal almost compact and τ_s is T_1, then (X , τ) is epi-mildly normal.*

Proof. If (X , τ) is partial normal, then the semi-regularization of X is almost regular. Moreover, the coarser topology of almost compact is almost compact. So, (X ,τ_s) is almost compact. But every almost regular almost compact is mildly normal [13]. Thus (X ,τ_s) is mildly normal. By Theorem 2.17, we conclude that (X, τ) is epi-mildly normal. □

Recall that a topological space (X , τ) is called *eipregular* if there is a coarser topology τ' on X such that (X , τ') is T_3 [3]. Since the courser topology of a semiregular space is also semiregular and every almost regular semiregular is regular[10], so we conclude.

Theorem 2.21. *Every epi-almost normal semiregular is eipregular.*

As every weakly regular, paracompact space is an almost normal [12], then we have the following corollary.

Corollary 2.22. *If (X , τ) is epiregular and the witness of epiregularity (X , τ') is paracompact, then (X , τ) is epi-almost normal.*

3. Relation of epi-almost normality with other separation axioms

Note that epi-almost normality does not imply normality, for example Niemytzki Plan topology is epi-almost normal being Hausdorff almost normal [11], but not normal by Jones' lemma. Consider Either-Or Topology [16]. It is normal because the only disjoint closed sets are the ground set and the empty set. But it is not epi-almost normal because it is not completely Hausdorff.

Also, epi-almost normality does not imply almost normality , agood example for this is Right Order Topology [16]. It is almost normal because there are no non-empty disjoint closed sets. But it is not epi-almost normal because it is not completely Hausdorff space. The Heath's V-space [16] is an example of a Tychonoff zero-dimensional scattered epi-almost normal space, since the Niemytzki topology is coarser than its, which is almost normal.

Moreover, epi-almost normality does not imply partial normality.For example: let $(\mathbb{R}, \mathcal{M})$ denote the Michael line, the irrational points are isolated, and a basic open neighborhood for a rational point is the same as in the usual topology and consider $\mathcal{M} \times \mathbb{P}$ denote the Michael product, where $\mathbb{P}$ with the usual topology, see [5]. Since $\mathcal{M} \times \mathbb{P}$ is not almost normal [7] but it is semiregular [16], hence it is not partial normality by Theorem 2.11. Observed that $\mathcal{M} \times \mathbb{P}$ is epi-almost normal, by taking the coarser topology $\mathcal{U} \times \mathbb{P}$ of $\mathcal{M} \times \mathbb{P}$. Then $\mathcal{U} \times \mathbb{P}$ is Hausdorff almost normal, since it is metrizable.

Recall that a topological space (X , τ) is called π-normal if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$[6]. A topological space (X , τ) is called extremally disconnected if it is T_1 and the closure of any open set is open [5]. Since every extremally disconnected semiregular space is a Tychonoff space [5] and every π-normal is extremally disconnected space [6], then we have the following corollary.

Corollary 3.1. *Every extremally disconnected semiregular is epi-almost normal.*

The following problems are still open:

(1) Is epi-almost normality is invariant under quotient mapping. Given an example?
(2) When the Alecsandroff duplicate is epi almost normal?
(3) Is epi-almost normality hereditary with respect to closed subspaces ?
(4) Is a almostβ-normal [1] epi-almost normal space normal ?

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