EPI-ALMOST NORMALITY

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Abstract. A space \((X, \tau)\) is called epi-almost normal if there exists a coarser topology \(\tau'\) on \(X\) such that \((X, \tau')\) is Hausdorff \((T_2)\) almost normal. We investigate this property and present some examples to illustrate the relationships between epi-almost normality and other weaker kinds of normality.

1. Introduction

In this paper, we define a new topological property called epi-almost normality. Unlike epi-mild normality [9] and epinormality [8]. We investigate this property and we using the technique of semi-regularization to illustrate the relationships between epi-almost normality and partial normality [2], and to improve the Theorem ”Any almost normal; almost regular is almost completely Hausdorff” [11]. Throughout this paper, we denote an ordered pair by \(\langle x, y \rangle\), the set of positive integers by \(\mathbb{N}\), the set of rational numbers by \(\mathbb{Q}\), the set of irrational numbers by \(\mathbb{P}\), and the set of real numbers by \(\mathbb{R}\). A \(T_4\) space is a \(T_1\) normal space, a Tychonoff \((T_{3\frac{1}{2}})\) space is a \(T_1\) completely regular space, and a \(T_3\) space is a \(T_1\) regular space. A space \(X\) is called completely Hausdorff if for each distinct elements \(a, b \in X\) there exist two open sets \(U\) and \(V\) such that \(a \in U, b \in V\), and \(U \cap V = \emptyset\). We do not assume \(T_2\) in the definition of compactness, countable compactness and paracompactness. For a subset \(A\) of a space \(X\), \(\text{int} A\) and \(\overline{A}\) denote the interior and the closure of \(A\), respectively. \((X, \tau_s)\) is the semi-regularization of \((X, \tau)\).

Definition 1.1. A subset \(A\) of a space \(X\) is called regular closed [5], if \(A = \text{int} \overline{A}\). A subset \(A\) of a space \(X\) is called regular open [5], if \(A = \text{int}(A)\). A subset \(A\) of a space \(X\) is said to be \(\pi\)-closed if it is a finite intersection of regular closed sets. A subset \(A\) of a space \(X\) is said to be \(\pi\)-open if it is a finite union of regular open sets. A space \(X\) is called almost normal [11], if for any two disjoint closed sets \(A\) and \(B\) of \(X\) one of them regular closed there exist two disjoint open sets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\) and \(B \subseteq V\). A space \(X\) is called mildly normal [13], called also \(\kappa\)-normal [15], if for any two disjoint regular closed sets \(A\) and \(B\) of \(X\) there exist two disjoint open sets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\) and \(B \subseteq V\). A space \(X\) is said to be an almost regular if for any closed domain subset \(A\) and any \(x \not\in A\),
there exist two disjoint open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$ [14]. A space $X$ is said to be an almost completely regular if for any regular closed subset $A$ and any $x \notin A$, there is a continuous function $f$ on $X$ into the closed interval $[0, 1]$ such that $f(x) = 1$ and $f(A) = 0$ [14]. A space $X$ is said to be semi-normal if for every closed set $A$ contained in an open set $U$, there exists a regularly open set $V$ such that $A \subset V \subset U$.

2. Epi-Almost Normality

Definition 2.1. A space $(X, \tau)$ is called epi-almost normal if there exists a coarser topology $\tau'$ on $X$ such that $(X, \tau')$ is $T_2$ (Hausdorff) almost normal.

Note that if we do not presume $T_2$ in Definition 2 above, then any space will be epi-almost normal as the indiscreet topology will refine. Also, if we assume $(X, \tau')$ to be $T_1$ almost normal in Definition 2 above, then any $T_1$ space will be epi-almost normal as the finite complement topology, see [16], will refine.

Observe that if $\tau'$ and $\tau$ are two topologies on $X$ such that $\tau'$ is coarser than $\tau$ and $(X, \tau')$ is $T_i$, $i \in \{0, 1, 2\}$, then so is $(X, \tau)$. So, we conclude the following:

Theorem 2.2. Every epi-almost normal space is $T_2$.

Recall that a topological space $X$ is called completely Hausdorff, $T_{2\frac{1}{2}}$ [16], if for each distinct elements $a, b \in X$ there exist two open sets $U$ and $V$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. The space $(X, \tau_s)$ is called the semi-regularization of $(X, \tau)$ if the topology on $(X, \tau_s)$ generated by the family of all regularly open sets in $(X, \tau)$.

Theorem 2.3. Every epi-almost normal space is completely Hausdorff.

Proof. Let $(X, \tau)$ be any epi-almost normal space and $\tau'$ is the witness of epi-almost normality. Then $\tau'$ is almost regular. Let $\tau_s$ be the semi-regularization of $\tau'$. Since $(X, \tau')$ is Hausdorff almost regular space, $\tau_s$ is Hausdorff regular space [10], and hence it is a completely Hausdorff. By using Proposition 3 [10], $(X, \tau')$ is completely Hausdorff. Thus $(X, \tau)$ is completely Hausdorff. $\square$

We note that any space which is not completely Hausdorff cannot be epi-almost normal, e.g., Prime Integer topology and Double origin topology [16].

A topological space $(X, \tau)$ is called epinormal if there is a coarser topology $\tau'$ on $X$ such that $(X, \tau')$ is $T_4$ [8]. A topological space $(X, \tau)$ is called epi-mildly normal if there is a coarser topology $\tau'$ on $X$ such that $(X, \tau')$ is $T_2$ (Hausdorff) mildly normal [8]. From the definitions we conclude the following implications.

epinormality $\implies$ epi-almost normality $\implies$ epi-mild normality.

Since every $T_1$ almost normal is almost regular and every almost regular semi-regular is regular [10], then we have the following theorem.

Theorem 2.4. In semiregular space, epi-almost normality and epinormality are equivalent.

Note that epi-almost normality does not imply epi-mild normality and there is an example.
**Example 2.5.** Let \((\mathbb{N}, \mathcal{RPI})\) be Relatively Prime Integer topology \(\mathcal{RPI}\) [16] which generated by the basis \(\mathcal{B} = \{B_a(n) : a, b \in X, \gcd(a, b) = 1\}\), where \(B_a(b) = \{b + na \in X : n \in \mathbb{Z}\}\). Note that \((\mathbb{N}, \mathcal{RPI})\) is Hausdorff but not completely Hausdorff [16]. Hence it is not epi-almost normal.

Claim: \((\mathbb{N}, \mathcal{RPI})\) is mildly normal.

**Proof Claim:** Let \(F\) and \(E\) be two arbitrary disjoint regular closed sets. Hence if \(x \in \text{int}(F)\) and \(y \in \text{int}(E)\), then \(x \neq y\). Since \(X\) is Hausdorff, there exist \(B_x(b)\) and \(B_y(d)\) disjoint neighborhoods of \(x\) and \(y\) respectively such that \(x \in B_x(b) \in \text{int}(F)\) and \(y \in B_y(d) \in \text{int}E\). \(x \in B_x(b) \subseteq F\) and \(y \in B_y(d) \subseteq E\). But the closures of \(B_x(b)\) and \(B_y(d)\) contain in common all multiples of \([x, y]\). Hence \(B_y(d)\) and \(B_x(b)\) are intersects, contradiction with \(E\) and \(F\) are disjoint. Thus, any two non-empty regular closed must intersect. Therefore \(\mathcal{RPI}\) is mildly normal.

Since \(\mathcal{RPI}\) is Hausdorff mildly normal, then \(\mathcal{RPI}\) is epi-mildly normal which is not epi-almost normal. 

**Theorem 2.6.** Epi-almost normal is topological property.

**Proof.** Let \((X, \tau)\) be any epi-almost normal space. Assume that \((X, \tau) \cong (Y, S)\).

Let \(\tau'\) be a coarser topology on \(X\) such that \((X, \tau')\) is Hausdorff almost normal space. Let \(f : (X, \tau) \rightarrow (Y, S)\) be a homeomorphism and define \(S'\) on \(Y\) by \(S' = \{f(U) : U \in \tau'\}\). Then \(S'\) is a topology on \(Y\) coarser than \(S\) and \((Y, S')\) is Hausdorff almost normal. 

**Theorem 2.7.** The sum \(\bigoplus_{\alpha \in \Lambda} X_\alpha\), where \(X_\alpha\) is a space for each \(\alpha \in \Lambda\), is epi-almost normal if and only if all spaces \(X_\alpha\) are epi-almost normal.

**Proof.** If the sum \((\bigoplus_{\alpha \in \Lambda} X_\alpha, \bigoplus_{\alpha \in \Lambda} \tau_\alpha)\) is epi-almost normal, then there exist \(\tau'\) topology on \(\bigoplus_{\alpha \in \Lambda} X_\alpha\), coarser than \(\bigoplus_{\alpha \in \Lambda} \tau_\alpha\) such that \((\bigoplus_{\alpha \in \Lambda} X_\alpha, \tau')\) is a Hausdorff almost normal space. Since \(X_\alpha\) is clopen in \(\bigoplus_{\alpha \in \Lambda} X_\alpha\) for each \(\alpha \in \Lambda\), \((X_\alpha, \tau'_\alpha)\), where \(\tau'_\alpha = \{U \cap X_\alpha : U \in \tau'\}\), is a Hausdorff almost normal space. Thus all spaces \(X_\alpha\) are epi-almost normal as \((X_\alpha, \tau'_\alpha)\) is coarser topology than \((X_\alpha, \tau_\alpha)\). Conversely, if all the \(X_\alpha\)'s are epi-almost normal, then there exists a topology \(\tau'_\alpha\) on \(X_\alpha\) for each \(\alpha \in \Lambda\), coarser than \(\tau_\alpha\) such that \((X_\alpha, \tau'_\alpha)\) is a Hausdorff almost normal space. Since \(T_2\) and almost normality are both additive [5], then \((\bigoplus_{\alpha \in \Lambda} X_\alpha, \bigoplus_{\alpha \in \Lambda} \tau_\alpha)\) is a Hausdorff almost normal space. Therefore \((\bigoplus_{\alpha \in \Lambda} X_\alpha, \bigoplus_{\alpha \in \Lambda} \tau_\alpha)\) is epi-almost normal. 

**Theorem 2.8.** If \(X\) is epi-almost normal countably compact and \(M\) is Hausdorff paracompact first countable, then \(X \times M\) is epi-almost normal.

**Proof.** Let \((X, \tau)\) be any epi-almost normal countably compact space. Then there exits coarser topology \(\tau'\) on \(X\) such that \((X, \tau')\) is Hausdorff almost normal space. Since the coarser topology of countably compact is countably compact, so \((X, \tau') \times M\) is Hausdorff almost normal, by Theorem 9 of [7]. Thus \(X \times M\) is epi-almost normal. 

Epi-almost normal version of Stones theorem.

**Corollary 2.9.** If \(X\) is epi-almost normal countably compact and \(M\) is metrizable, then \(X \times M\) is epi-almost normal.

Let us recall the following definition from [2]

**Definition 2.10.** A topological space \(X\) is called partially normal if any two disjoint subsets \(A\) and \(B\) of \(X\), where \(A\) is regular closed and \(B\) is \(\pi\)-closed, are separated.
Theorem 2.11. $(X, \tau)$ is partial normality if and only if the semi-regularization of $X$ is almost normal.

Proof. Let $(X, \tau)$ be partial normal space, $A$ and $B$ be disjoint closed sets in $\tau_s$ such that $A$ is regular closed in $\tau_s$. Hence, $B = E \cap F$ where $E$ and $F$ are regular closed sets in $\tau_s$. Thus, $A$ and $B$ are disjoint closed sets in $\tau$ such that $A$ is regular closed and $B$ is $\pi$-closed. By partial normality, there exist two open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Then there exist two disjoint regular open sets $U_s$ and $V_s$ containing $U$ and $V$ respectively, see [4]. Thus, $A$ and $B$ are separated by two open sets in $\tau_s$. Therefore, $(X, \tau_s)$ is almost normality.

Corollary 2.12. almost normal is not a semiregular property, but partial normality is.

Proof. The Half disc topology is not almost normal [6]. But its semi-regularization $\tau_s$ is the usual topology on the closed upper half plane. Hence the semi-regularization of half disc topology is almost normal, indeed metrizable. Applying Theorem 2.11 and using [10], Lemma 5] show that $(X, \tau_s)$ is partial normal if and only if it is almost normal. But $(X, \tau)$ is partial normality, since $(X, \tau_s)$ is almost normal, using Theorem 2.11. Therefore partial normality is a semiregular property. □

It is clear from the definition that any $T_2$ almost normal space is epi-almost normal, just take the coarser topology equal the same topology. But here we have something stronger.

Corollary 2.13. Every Hausdorff partial normal space is epi-almost normal.

Corollary 2.14. A semiregular space is partial normal if and only if it is almost normal.

Corollary 2.15. If $(X, \tau)$ is seminormal partial normal space, then $(X, \tau_s)$ is normal space.

Proof. Let $B$ be any open set containing a closed set $A$ in $(X, \tau_s)$. By seminormality, there exists an open set $U$ such that $A \subseteq U \subseteq \text{int}(U) \subseteq B$. Since $\text{int}(U)$ is regular open and using Theorem 2.11 there exists an open set $V$ in $(X, \tau_s)$ such that $A \subseteq V \subseteq V_s \subseteq \text{int}(U) \subseteq B$. Thus $(X, \tau_s)$ is normal space. □

Corollary 2.16. If $(X, \tau)$ is seminormal partial normal space and $\tau_s$ is $T_1$, then $(X, \tau)$ is epinormal.

In [11] proved that "Any almost normal; almost regular is almost completely Hausdorff", we use this theorem to prove the following Corollary.

Corollary 2.17. If $X$ is almost normal almost regular then $X$ is completely Hausdorff space.

Corollary 2.18. If $X$ is almost normal almost regular then $X$ is epi-almost normal.
Corollary 2.19. If \((X, \tau)\) is partial normal and \(\tau_s\) is \(T_1\), then \((X, \tau)\) is completely Hausdorff space.

Proof. Since \((X, \tau)\) is partial normal space, then the semi-regularization of \(X\) is almost normal, hence almost regular. Moreover, almost regular almost normal is completely Hausdorff space, by Corollary 2.17 so the semi-regularization of \(X\) is completely Hausdorff space. Therefore \((X, \tau)\) is completely Hausdorff. \qed

Corollary 2.20. If \((X, \tau)\) is partial normal almost compact and \(\tau_s\) is \(T_1\), then \((X, \tau)\) is epi-mildly normal.

Proof. If \((X, \tau)\) is partial normal, then the semi-regularization of \(X\) is almost regular. Moreover, the coarser topology of almost compact is almost compact. So, \((X, \tau_s)\) is almost compact. But every almost regular almost compact is mildly normal \[13\]. Thus \((X, \tau_s)\) is mildly normal. By Theorem 2.17 we conclude that \((X, \tau)\) is epi-mildly normal. \qed

Recall that a topological space \((X, \tau)\) is called eipregular if there is a coarser topology \(\tau'\) on \(X\) such that \((X, \tau')\) is \(T_3\) \[3\]. Since the coarser topology of a semiregular space is also semiregular and every almost regular semiregular is regular \[10\], so we conclude.

Theorem 2.21. Every epi-almost normal semiregular is eipregular.

As every weakly regular, paracompact space is an almost normal \[12\], then we have the following corollary.

Corollary 2.22. If \((X, \tau)\) is epiregular and the witness of epiregularity \((X, \tau')\) is paracompact, then \((X, \tau)\) is epi-almost normal.

3. Relation of epi-almost normality with other separation axioms

Note that epi-almost normality does not imply normality, for example Niemytzki Plan topology is epi-almost normal being Hausdorff almost normal \[11\], but not normal by Jones’ lemma. Consider Either-Or Topology \[16\]. It is normal because the only disjoint closed sets are the ground set and the empty set. But it is not epi-almost normal because it is not completely Hausdorff.

Also, epi-almost normality does not imply almost normality, agood example for this is Right Order Topology \[16\]. It is almost normal because there are no non-empty disjoint closed sets. But it is not epi-almost normal because it is not completely Hausdorff space. The Heath’s V-space \[16\] is an example of a Tychonoff zero-dimensional scattered epi-almost normal space, since the Niemytzki topology is coarser than its, which is almost normal.

Moreover, epi-almost normality does not imply partial normality. For example: let \((\mathbb{R}, M)\) denote the Michael line, the irrational points are isolated, and a basic open neighborhood for a rational point is the same as in the usual topology and consider \(M \times P\) denote the Michael product, where \(P\) with the usual topology, see \[3\]. Since \(M \times P\) is not almost normal \[7\] but it is semiregular \[16\], hence it is not partial normality by Theorem 2.11. Observed that \(M \times P\) is epi-almost normal, by taking the coarser topology \(U \times P\) of \(M \times P\). Then \(U \times P\) is Hausdorff almost normal, since it is metrizable.
Recall that a topological space \((X, \tau)\) is called \(\pi\)-normal if for any two disjoint closed subsets \(A\) and \(B\) of \(X\) one of which is \(\pi\)-closed, there exist two disjoint open subsets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\) and \(B \subseteq V\) [6]. A topological space \((X, \tau)\) is called extremally disconnected if it is \(T_1\) and the closure of any open set is open [5]. Since every extremally disconnected semiregular space is a Tychonoff space [6] and every \(\pi\)-normal is extremally disconnected space [6], then we have the following corollary.

**Corollary 3.1.** Every extremally disconnected semiregular is epi-almost normal.

The following problems are still open:
1. Is epi-almost normality is invariant under quotient mapping. Given an example?
2. When the Aleksandrov duplicate is epi almost normal?
3. Is epi-almost normality hereditary with respect to closed subspaces?
4. Is a almost \(\beta\)-normal [1] epi-almost normal space normal?

**References**


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