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FIXED POINT THEOREMS FOR RATIONAL CONTRACTIONS IN $\mathcal{F}$-METRIC SPACES

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Abstract. Jleli and Samet [J. Fixed Point Theory Appl. (2018) 20:128] introduced a new metric space named as $\mathcal{F}$-metric space. They established a new version of the Banach contraction principle in the setting of this generalized metric spaces. The aim of this article is to define rational contraction and obtain some fixed point theorems in the context of $\mathcal{F}$-metric spaces. Our results extend, generalize and unify several known results in the literature.

1. Introduction and Preliminaries

Metric fixed point theory plays a crucial role in the field of functional analysis. It was first introduced by the great Polish mathematician Banach [6] in this way.

Theorem 1.1. [6] *Let (X, d) be a complete metric space, and let $g : X \to X$ be a self mapping. If there exists $\beta \in [0, 1)$ such that*

$$d(g(x), g(y)) \leq \beta d(x, y)$$

for all $x, y \in X$, then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Over the years, due to its significance and application in different fields of science, a lot of generalizations have been done in different directions by several authors; see, for example, [1-14] and references therein.

Very recently, Jleli and Samet [10] introduced an interesting generalization of a metric space in the following way.

Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t \Longrightarrow f(s) \leq f(t)$.

($\mathcal{F}_2$) For every sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} f(t_n) = -\infty$;

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Definition 1.2. [10] *Let X be a nonempty set, and let $D : X \times X \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that*

(D$_1$) $(x, y) \in X \times X$, $D(x, y) = 0 \Longleftrightarrow x = y$.
(D$_2$) $D(x, y) = D(y, x)$, for all $(x, y) \in X \times X$.
(D$_3$) For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(u_i)_{i=1}^N \subset X$, with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \Rightarrow f(D(x, y)) \leq f(\sum_{i=1}^{N-1} D(u_i, u_{i+1})) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and (X, D) an $\mathcal{F}$-metric space.

Remark. *They showed that any metric space is an $\mathcal{F}$-metric space but the converse is not true in general, which confirms that this concept is more general than the standard metric concept.*

Example 1.3. [10] *The set of real numbers $\mathbb{R}$ is an $\mathcal{F}$-metric space if we define D by*

$$D(x, y) = \begin{cases} (x - y)^2 & \text{if } (x, y) \in [0, 3] \times [0, 3] \\ |x - y| & \text{if } (x, y) \notin [0, 3] \times [0, 3] \end{cases}$$

with $f(t) = \ln(t)$ and $\alpha = \ln(3)$.

Definition 1.4. [10] *Let (X, D) be an $\mathcal{F}$-metric space.*

(i) Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is $\mathcal{F}$-convergent to $x \in X$ if $\{x_n\}$ is convergent to x with respect to the $\mathcal{F}$-metric D.
(ii) A sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

(iii) We say that (X, D) is $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a certain element in X.

Theorem 1.5. [10] *Let (X, D) be an $\mathcal{F}$-metric space and $g : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) (X, D) is $\mathcal{F}$-complete,
(ii) there exists $k \in (0, 1)$ such that

$$D(g(x), g(y)) \leq kD(x, y)$$

Then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Afterward, Hussain et al. [7] considered the notion of α-ψ-contraction in the setting of $\mathcal{F}$-metric spaces and proved the following fixed point theorem.

Theorem 1.6. [7] *Let (X, D) be an $\mathcal{F}$-metric space and $T : X \to X$ be β-admissible mapping. Suppose that the following conditions are satisfied:*

(i) (X, D) is $\mathcal{F}$-complete,
(ii) there exist two functions $\beta : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\beta(x, y)D(T(x), T(y)) \leq \psi(M(x, y))$$

where

$$M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\}$$

for $x, y \in X$,

(iii) there exists $x_0 \in X$ such that $\beta(x_0, T(x_0)) \geq 1$. Then T has a unique fixed point $x^* \in X$.

In this paper, we first define rational contraction and then we prove a new fixed point theorem in the setting of $\mathcal{F}$-metric spaces. We also support our main result by providing a non trivial example.

2. Main Results

Definition 2.1. *Let (X, D) be an $\mathcal{F}$-metric space. A mapping $g : X \to X$ is said to be almost rational contraction if there exists $\beta_1, \beta_2 \in [0, 1)$ with $\beta_1 + \beta_2 < 1$ such that*

$$D(g(x), g(y)) \leq \beta_1 D(x, y) + \beta_2 \frac{D(x, g(x))D(y, g(y))}{1 + D(x, y)} \tag{2.1}$$

for $(x, y) \in X \times X$.

Theorem 2.2. *Let (X, D) be an $\mathcal{F}$-metric space, and let $g : X \to X$ be a rational contraction. If (X, D) is $\mathcal{F}$-complete, then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by*

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N}, \tag{2.2}$$

is $\mathcal{F}$-convergent to x^*.

Proof. First, observe that g has at most one fixed point. Indeed, if $(u, v) \in X \times X$ are two fixed points of g with $u \neq v$, i.e.

$$D(u, v) > 0, \ g(u) = u, \ g(v) = v.$$

As $g : X \to X$ be a rational contraction, so

$$\begin{aligned} D(u, v) &= D(g(u), g(v)) \leq \beta_1 D(u, v) + \beta_2 \frac{D(u, g(u))D(v, g(v))}{1 + D(u, v)} \\ &= \beta_1 D(u, v) + \beta_2 \frac{D(u, u)D(v, v)}{1 + D(u, v)} \\ &= \beta_1 D(u, v) \\ &< D(u, v) \end{aligned}$$

which is a contradiction.

Next, let $(f, \alpha) \in F \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \alpha. \tag{2.3}$$

Let $x_0 \in X$ be an arbitrary element. Let $\{x_n\} \subset X$ be the sequence defined by (2.2). Without restriction of the generality, we may suppose that $D(x_0, x_1) > 0$.

Otherwise, x_0 will be a fixed point of g. As $g : X \to X$ is a rational contraction, we have

$$\begin{aligned} D(x_n, x_{n+1}) &= D(g(x_{n-1}), g(x_n)) \leq \beta_1 D(x_{n-1}, x_n) \\ &\quad + \beta_2 \frac{D(x_{n-1}, g(x_{n-1}))D(x_n, g(x_n))}{1 + D(x_{n-1}, x_n)} \\ &= \beta_1 D(x_{n-1}, x_n) \\ &\quad + \beta_2 \frac{D(x_{n-1}, x_n)D(x_n, x_{n+1})}{1 + D(x_{n-1}, x_n)} \\ &\leq \beta_1 D(x_{n-1}, x_n) + \beta_2 D(x_n, x_{n+1}) \\ &\leq \frac{\beta_1}{1 - \beta_2} D(x_{n-1}, x_n) \end{aligned}$$

as $\frac{D(x_{n-2}, x_{n-1})}{1+D(x_{n-2}, x_{n-1})} < 1$. Which further yields that

$$D(x_n, x_{n+1}) \leq \frac{\beta_1}{1 - \beta_2} D(x_{n-1}, x_n). \tag{2.4}$$

Similarly

$$\begin{aligned} D(x_{n-1}, x_n) &= D(g(x_{n-2}), g(x_{n-1})) \leq \beta_1 D(x_{n-2}, x_{n-1}) \\ &\quad + \beta_2 \frac{D(x_{n-2}, g(x_{n-2}))D(x_{n-1}, g(x_{n-1}))}{1 + D(x_{n-2}, x_{n-1})} \\ &= \beta_1 D(x_{n-2}, x_{n-1}) \\ &\quad + \beta_2 \frac{D(x_{n-2}, x_{n-1})D(x_{n-1}, x_n)}{1 + D(x_{n-2}, x_{n-1})} \\ &\leq \beta_1 D(x_{n-2}, x_{n-1}) + \beta_2 D(x_{n-1}, x_n) \end{aligned}$$

as $\frac{D(x_{n-2}, x_{n-1})}{1+D(x_{n-2}, x_{n-1})} < 1$. Which further yields that

$$D(x_{n-1}, x_n) \leq \frac{\beta_1}{1 - \beta_2} D(x_{n-2}, x_{n-1}). \tag{2.5}$$

Let $\lambda = \frac{\beta_1}{1-\beta_2} < 1$. Then from (2.4) and (2.5) and continuing the process, we get

$$D(x_n, x_{n+1}) \leq \lambda^n D(x_0, x_1), \qquad n \in N. \tag{2.6}$$

This implies that

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{\lambda^n}{1 - \lambda} D(x_0, x_1), \quad m > n.$$

Since

$$\lim_{n \to \infty} \frac{\lambda^n}{1 - \lambda} D(x_0, x_1) = 0$$

so there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1 - \lambda} D(x_0, x_1) < \delta, \ n \geq N. \tag{2.7}$$

Hence, by (2.3) and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) \leq f(\frac{\lambda^n}{1 - \lambda} D(x_0, x_1)) < f(\epsilon) - \alpha \tag{2.8}$$

for $m > n \geq N$. Using (D_3) and (2.8), we obtain $D(x_n, x_m) > 0$, $m > n \geq N$ implies

$$f(D(x_n, x_m)) \leq f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) + \alpha < f(\epsilon)$$

which implies by ($\mathcal{F}_1$) that $D(x_n, x_m) < \epsilon$, $m > n \geq N$. This proves that $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since (X, D) is $\mathcal{F}$-complete, there exists $x^* \in X$ such that $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, i.e.

$$\lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.9}$$

We shall prove that x^* is a fixed point of g. We argue by contradiction by supposing that $D(g(x^*), x^*) > 0$. By (D_3), we have

$$f(D(g(x^*), x^*)) \leq f(D(g(x^*), g(x_n)) + D(g(x_n), x^*)) + \alpha,\ n \in N$$

As $g : X \to X$ is said to be a rational contraction, so using ($\mathcal{F}_1$), we obtain

$$\begin{aligned} f(D(g(x^*), x^*)) \quad &\leq \quad f(\beta_1 D(x^*, x_n) + \beta_2 \frac{D(x^*, g(x^*))D(x_n, g(x_n))}{1 + D(x^*, x_n)} \\ &\quad + D(x_{n+1}, x^*)) + \alpha, \end{aligned}$$

for $n \in N$. On the other hand, using ($\mathcal{F}_2$) and (2.9), we have

$$\lim_{n\to\infty} f(\beta_1 D(x^*, x_n) + \beta_2 \frac{D(x^*, g(x^*))D(x_n, g(x_n))}{1 + D(x^*, x_n)} + D(x_{n+1}, x*)) + \alpha = -\infty,$$

which is a contradiction. Therefore, we have $D(g(x^*), x^*) = 0$, i.e. $g(x^*) = x^*$. As consequence, $x^* \in X$ is the unique fixed point of g. □

Example 2.3. *Let $X = [0, 2)$ endowed with $\mathcal{F}$-complete $\mathcal{F}$-metric D given by*

$$D(x, y) = \begin{cases} 2^{|x-y|}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Take $f(t) = \frac{-1}{t}$ and $\alpha = 1$. Define $g : X \to X$ by

$$g(x) = \begin{cases} \frac{1}{2}(x + 1), & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Case 1: *If $x \in [0, 1]$. Then $D(x, y) = 2^{|x-y|}$ and $D(g(x), g(y)) = 2^{\frac{1}{2}|x-y|}$. Now for $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{3}$, we have*

$$2^{\frac{1}{2}|x-y|} \leq \frac{1}{2} 2^{|x-y|} + \frac{1}{3} \frac{2^{|x-\frac{1}{2}|} 2^{|y-\frac{1}{2}|}}{1 + 2^{|x-y|}}$$

which implies that

$$D(g(x), g(y)) \leq \beta_1 D(x, y) + \beta_2 \frac{D(x, g(x))D(y, g(y))}{1 + D(x, y)}$$

Hence g is a rational contraction.

Case II: If $x \notin [0, 1]$. Then trivially g is rational contraction. Thus all the hypotheses of Theorem 2.1 are satisfied. Consequently g has a unique fixed point which is $x^* = 1$.

Remark. *It can be easily observed that the result of Jleli and Samet cannot be applied to Example 2.3, as*

$$D(g(x), g(y)) > kD(x, y)$$

for all $x, y \in X$.

Corollary 2.4. [10]*Let* (X, D) *be an* $\mathcal{F}$*-metric space, and let* $g : X \to X$ *be a given mapping. Suppose that the following conditions are satisfied:*

(i) (X, D) is $\mathcal{F}$-complete.
(ii) There exists $\beta_1 \in (0, 1)$ such that

$$D(g(x), g(y)) \leq \beta_1 D(x, y), \quad (x, y) \in X \times X.$$

Then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Proof. Taking $\beta_2 = 0$ in Theorem 2.2. □

Corollary 2.5. *Let* (X, D) *be an* $\mathcal{F}$*-metric space, and let* $g : X \to X$ *be a given mapping. Suppose that the following conditions are satisfied:*

(i) (X, D) is $\mathcal{F}$-complete.
(ii) There exists $\beta_2 \in (0, 1)$ such that

$$D(g(x), g(y)) \leq \beta_2 \frac{D(x, g(x))D(y, g(y))}{1 + D(x, y)}$$

for all $(x, y) \in X \times X$.Then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Proof. Setting $\beta_1 = 0$ in Theorem 2.2. □

Now, we consider some special cases, when our result deduces several well-known fixed point theorems of the existing literature.

By Remark 1, we can have the following theorem of Brian Fisher from our Main Result.

Theorem 2.6. *Let* (X, d) *be a complete metric space, and let* $g : X \to X$ *be a self mapping. If there exists* $\beta_1, \beta_2 \in [0, 1)$ *with* $\beta_1 + \beta_2 < 1$ *such that*

$$d(g(x), g(y)) \leq \beta_1 d(x, y) + \beta_2 \frac{d(x, g(x))d(y, g(y))}{1 + d(x, y)}$$

for all $x, y \in X$*, then* g *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$*-convergent to* x^*.

Remark. *By setting* $\beta_1 = \beta$ *and* $\beta_2 = 0$ *in above Theorem, we can obtain the classical Banach Contraction Principle 1.1.*

If we take $\beta_1 = 0$ in 2.6, then we have the following result in complete metric space.

Corollary 2.7. *Let (X, d) be a complete metric space, and let $g : X \to X$ be a self mapping. If there exists $\beta_2 \in [0, 1)$ such that*

$$d(g(x), g(y)) \leq \beta_2 \frac{d(x, g(x))d(y, g(y))}{1 + d(x, y)}$$

for all $x, y \in X$, then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

3. Conclusions

Recently, Jleli and Samet [J. Fixed Point Theory Appl. (2018) 20:128] introduced the concept of $\mathcal{F}$-metric space and proved a new version of the Banach contraction principle in the setting of this $\mathcal{F}$-metric spaces. In this article we define rational contraction and establish some fixed point theorems in the context of $\mathcal{F}$-metric spaces. Our results extend, generalize and unify several known results in the literature. Also, we give an example to illustrate the significance of the main result.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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