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A REMARK ON BLOW-UP SOLUTIONS FOR NONLINEAR WAVE EQUATION WITH WEIGHTED NONLINEARITIES

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Abstract. We investigate the blow-up of solutions to the nonlinear wave equation with weighted nonlinearities $u_{tt} - \Delta u = \omega(x)f(u)g(u_t)$ where $\omega : \mathbb{R}^N \to \mathbb{R}$ is a given function and $f, g : \mathbb{R} \to \mathbb{R}$ having super-linear growth. Sufficient conditions of blow-up in finite time of solutions are obtained.

1. Introduction

In this paper we study the Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u = \omega(x)f(u)g(u_t), \\ u(t=0,x) = u_0(x), \\ u_t(t=0,x) = u_1(x), \end{cases} \tag{1.1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $\omega : \mathbb{R}^N \to \mathbb{R}$ is a given function and $f, g : \mathbb{R} \to \mathbb{R}$ having super-linear growth. The given functions ω, f, g and the initial data u_0, u_1 are so smooth that the Cauchy problem (1.1) has a classical local (in time) solution. This follows from Duhamel's formula via the usual fixed point argument in the space $H^s_{loc} \times H^{s-1}_{loc}$, $s > N/2 + 1$. Such an s guarantee that $u, u_t, \nabla u$ are in L^∞. Note that $u \in H^s_{loc}$ means that the H^s norm over a ball centered at x_0 and with radius 1 is uniformly bounded by a constant independent of x_0 [9].

The initial value problem (1.1) has attracted considerable attention in the mathematical community and the well-posedness theory in the Sobolev spaces, especially for $\omega(x) = 1$ and for polynomial type nonlinearities, has been extensively studied. The case of exponential nonlinearity was recently investigated (see [3] and references therein). We refer the reader to [10] and references therein for more properties and information on nonlinear wave equations.

Our main interest is the non-existence of global solutions for (1.1). We will show that for some class of initial data the local classical solutions to (1.1) blow-up in finite time.

There have been many works devoted to the questions of global non-existence and blow-up of solutions to initial value problems for nonlinear wave equations (see, e.g., [1, 2, 4, 5, 6] and references therein.)

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We first investigate the case $g = 1$ so that the equation reads $u_{tt} - \Delta u = \omega(x)f(u)$. To state the main result in a clear way, we define $F(u) = \int_0^u f(s)ds$ and we make the following assumptions:

$$\exists \quad \lambda, \alpha > 0 \quad \text{such that} \quad (2+\lambda)F(u) \leq uf(u) + \alpha\, u^2\,. \tag{1.2}$$

$$\begin{cases} 0 \lneqq \omega \in L^{N/2}(\mathbb{R}^N) & \text{if} \quad N \geq 3, \\ 0 \lneqq \omega \in L^{\infty}(\mathbb{R}^N) & \text{if} \quad N \leq 2. \end{cases} \tag{1.3}$$

$$\begin{cases} \lambda \geq 2\alpha\, C_*^2\, \|\omega\|_{N/2} & \text{if} \quad N \geq 3, \\ \lambda > 2\alpha \|\omega\|_{\infty} & \text{if} \quad N \leq 2, \end{cases} \tag{1.4}$$

where C_* stands for the sharp constant in the Sobolev embedding (2.10).

The energy of a solution u to (1.1) is given by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|u_t(t,x)|^2 + |\nabla u(t,x)|^2\right)\, dx - \int_{\mathbb{R}^N} \omega(x)F(u(t,x))\, dx\,. \tag{1.5}$$

For smooth and sufficiently decaying at infinity solutions u, one can easily verify that the energy is constant, that is,

$$\frac{dE(u(t))}{dt} = 0\,.$$

Hence $E(u(t)) = E_0$ for all t, where E_0 is the initial energy defined by

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^N} \left(|u_1(x)|^2 + |\nabla u_0(x)|^2\right)\, dx - \int_{\mathbb{R}^N} \omega(x)F(u_0(x))\, dx\,. \tag{1.6}$$

Our first main result can be formulated as follows.

Theorem 1.1. *Suppose that assumptions* (1.2)-(1.3)-(1.4) *are fulfilled. Assume that the initial data* u_0, u_1 *are smooth and let* u *be the corresponding maximal solution of* (1.1)*. Then*

(i) *if* $N \geq 3$ *and* $E_0 < 0$*, the solution* u *blows-up in finite time;*
(ii) *if* $N \leq 2$*,* $E_0 < 0$ *and*

$$\int_{\mathbb{R}^N} u_0(x)u_1(x)dx > \sqrt{\frac{2\alpha\|\omega\|_{\infty}}{\lambda}} \int_{\mathbb{R}^N} u_0^2(x)dx, \tag{1.7}$$

the solution u *blows-up in finite time.*

The main tool used in the proof of Theorem 1.1 is the concavity method and its modification. See Proposition 2.1 below. The concavity method introduced in [5] is based on a construction of some positive function $y(t) = y(u(t))$, which is defined in terms of the local solution u of the problem and obtaining some differential inequality satisfied by y leading to the blow-up in finite time.

Our next interest is the nonexistence of global solutions for (1.1) when $\omega = f = 1$. We will assume that $f = 1$ and $\omega = 1$ so that the equation reads

$$u_{tt} - \Delta\, u = g(u_t)\,. \tag{1.8}$$

We make the following assumption on the nonlinearity g.

$$g \geq 0, \quad \int^{\infty} \frac{ds}{g(s)} < \infty \quad \text{and} \quad \int_0 \frac{ds}{g(s)} = \infty\,. \tag{1.9}$$

As an example of function satisfying (1.9), we have

$$g(u) = |u|^p \ \ (1 < p < \infty), \quad g(u) = u^2\, \mathrm{e}^{u^2}, \quad g(u) = \frac{u^2 \ln(1+u^2)}{\sin^2 u}\,.$$

We look at the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = g(u_t), \\ u(t=0,x) = 0, \\ u_t(t=0,x) = u_1(x). \end{cases} \tag{1.10}$$

Theorem 1.2. *For any $T > 0$, there exists $u_1 \in C_c^\infty(\mathbb{R}^N)$ such that* (1.10) *does not admit a C^2-solution past time T.*

The proof of this theorem is based on an ODE approach together with Theorem 2.4.

The rest of this paper is organized as follows. In the next section, we recall some basic facts and useful tools. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we give the proof of Theorem 1.2.

2. Background material

In this section we collect some mathematical tools needed for the proofs of the main theorems. In many nonlinear evolution equations we can obtain the blow-up of the solutions in a finite time as a consequence of the blow-up of some differential inequalities.

Now, we state a differential inequality leading to non-existence of global solutions.

Proposition 2.1. *Let $y : [0,T) \to [0,\infty)$ be a C^2-function satisfying for all $0 \le t < T$ the inequality*

$$y(t)y''(t) - (1+\gamma)\left(y'(t)\right)^2 \ge -\mu\left(y(t)\right)^2, \tag{2.1}$$

where $\gamma > 0$ and $\mu \ge 0$. Then

(i) *if $\mu = 0$ and*

$$y(t_0) > 0, \ \ y'(t_0) > 0, \tag{2.2}$$

for some $0 \le t_0 < T$, we have

$$T < t_1 := t_0 + \frac{y(t_0)}{\gamma y'(t_0)} < \infty; \tag{2.3}$$

(ii) *if $\mu > 0$ and*

$$y(t_0) > 0, \ \ y'(t_0) - \sqrt{\frac{\mu}{\gamma}}\, y(t_0) > 0, \tag{2.4}$$

for some $0 \le t_0 < T$, we have

$$T < t_2 := t_0 + \frac{1}{2\sqrt{\gamma\mu}} \ln\left(\frac{y'(t_0) + \sqrt{\frac{\mu}{\gamma}}\, y(t_0)}{y'(t_0) - \sqrt{\frac{\mu}{\gamma}}\, y(t_0)}\right) < \infty. \tag{2.5}$$

We refer to [7] for some differential inequalities which yield the nonexistence of global solutions to certain evolutionary partial differential equations. For reader convenience we give the proof of Proposition 2.1 here.

Proof of Proposition 2.1.

(i) Assume that $\mu = 0$ and define $z(t) = (y(t))^{-\gamma}$. We deduce from (2.1) that

$$z''(t) = -\gamma\,(y(t))^{-\gamma-2}\left[y(t)y''(t) - (\gamma+1)\,(y'(t))^2\right] \le 0.$$

Hence

$$\left(\frac{y(t)}{y(t_0)}\right)^{-\gamma} \le 1 - \gamma(t-t_0)\frac{y'(t_0)}{y(t_0)}.$$

It follows that

$$1 - \gamma(t-t_0)\frac{y'(t_0)}{y(t_0)} \ge 0.$$

This leads to the desired conclusion, that is

$$T < t_1 < \infty.$$

(ii) Assume that $\mu > 0$ and let $z(t) = (y(t))^{-\gamma}$. From (2.1) it follows that z solves the differential inequality

$$z'' - \gamma\mu\, z \le 0. \tag{2.6}$$

Let φ solve the Cauchy problem

$$\begin{cases} \varphi'' - \gamma\mu\,\varphi = 0, \\ \varphi(t_0) = z(t_0), \\ \varphi'(t_0) = z'(t_0), \end{cases} \tag{2.7}$$

that is,

$$\varphi(t) = a\,\mathrm{e}^{t\sqrt{\gamma\mu}} + b\,\mathrm{e}^{-t\sqrt{\gamma\mu}}, \tag{2.8}$$

with

$$\begin{aligned} a &= -\frac{1}{2}\sqrt{\frac{\gamma}{\mu}}\,y(t_0)^{-\gamma-1}\left[y'(t_0) - \sqrt{\frac{\mu}{\gamma}}y(t_0)\right]\mathrm{e}^{-t_0\sqrt{\gamma\mu}} < 0, \\ b &= \frac{1}{2}\sqrt{\frac{\gamma}{\mu}}\,y(t_0)^{-\gamma-1}\left[y'(t_0) + \sqrt{\frac{\mu}{\gamma}}y(t_0)\right]\mathrm{e}^{t_0\sqrt{\gamma\mu}} > 0. \end{aligned}$$

By a simple comparison principle, we have

$$0 \le z(t) \le \varphi(t).$$

It is easy to verify that

$$\varphi(t_2) = 0.$$

Hence (2.5) follows immediately.

□

A useful consequence is the following.

Corollary 2.2. *There is no C^2-function $y : [0,\infty) \to [0,\infty)$ satisfying* (2.1) *(with $\mu = 0$) and*

$$y'' \ge c > 0, \tag{2.9}$$

for some constant $c > 0$.

Proof.
Let us argue by contradiction. Suppose that there exists a C^2-function $y : [0,\infty) \to [0,\infty)$ satisfying (2.1) and (2.9). Since $y'' \geq c$, then

$$y'(t) \geq c\,t + y'(0) \quad \text{and} \quad y(t) \geq \frac{c}{2}\,t^2 + y'(0)t + y(0).$$

Clearly we can pick a time $t_0 \geq 0$ such that

$$c\,t_0 + y'(0) > 0 \quad \text{and} \quad \frac{c}{2}\,{t_0}^2 + y'(0)t_0 + y(0) > 0.$$

Hence the condition (2.2) is fulfilled and the conclusion follows. □

We also recall the following sharp Sobolev embedding [11].

Theorem 2.3. *We have, for all $u \in H^1(\mathbb{R}^N), N \geq 3$,*

$$\|u\|_{\frac{2N}{N-2}} \leq C_* \, \|\nabla u\|_2, \tag{2.10}$$

where

$$C_* = \frac{1}{\sqrt{\pi\, N(N-2)}} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{1/N}.$$

Moreover, the constant C_* is sharp.

The following uniqueness result will be needed in the proof of Theorem 1.2. For any fixed $(T, x_0) \in (0,\infty) \times \mathbb{R}^N$, we introduce

$$\Omega = \Big\{ (t,x); \quad 0 \leq t \leq T, \;\; |x - x_0| \leq T - t \Big\} \tag{2.11}$$

which is called the backward light cone trough (T, x_0). The base B_0 of Ω is defined as

$$B_0 = \{\, x; \quad |x - x_0| \leq T \,\} = B(x_0, T).$$

Theorem 2.4. *If $u \in C^2(\Omega)$ solves* (1.8) *in Ω. Then u is uniquely determined by its data u and u_t on B_0. In other words, if $u, v \in C^2(\Omega)$ are two solutions of* (1.8) *in Ω, with the same data in B_0, then $u = v$ in Ω.*

The proof of this theorem uses energy method and can be found in many references. See for example [10, 8]. For completeness, we give a detailed proof here. Let us first prove the following result which yields easily Theorem 2.4. Consider the linear wave equation

$$\mathbf{w}_{tt} - \Delta \mathbf{w} = a(t,x)\mathbf{w}_t, \tag{2.12}$$

where a is a continuous function in (t,x).

Proposition 2.5. *Let $\mathbf{w}$ be a C^2 solution of* (2.12) *in Ω. If $\mathbf{w} = \mathbf{w}_t = 0$ in B_0, then $\mathbf{w} = 0$ in Ω.*

Proof of Proposition 2.5. Consider for $0 \leq t \leq T$ the function

$$\begin{aligned} E(t) &= \int_{B(x_0, T-t)} \left(|\mathbf{w}(t,x)|^2 + |\mathbf{w}_t(t,x)|^2 + |\nabla \mathbf{w}(t,x)|^2 \right) dx \\ &= \int_0^{T-t} \int_{\partial B(x_0,\tau)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla \mathbf{w}|^2 \right) d\sigma \, d\tau. \end{aligned}$$

We have

$$\begin{aligned}\frac{dE}{dt}(t) &= 2\int_{B(x_0,T-t)} \left(\mathbf{w}\mathbf{w}_t + \mathbf{w}_t\mathbf{w}_{tt} + \nabla\mathbf{w}\cdot\nabla\mathbf{w}_t\right)\,dx \\ &- \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla\mathbf{w}|^2\right)\,d\sigma \\ &= 2\int_{B(x_0,T-t)} \mathbf{w}_t\left(\mathbf{w} + \mathbf{w}_{tt} - \Delta\mathbf{w}\right)\,dx + 2\int_{B(x_0,T-t)} \operatorname{div}\left(\mathbf{w}_t\nabla\mathbf{w}\right)\,dx \\ &- \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla\mathbf{w}|^2\right)\,d\sigma,\end{aligned}$$

where we have used the fact that

$$\operatorname{div}\left(\mathbf{w}_t\nabla\mathbf{w}\right) = \nabla\mathbf{w}_t\cdot\nabla\mathbf{w} + \mathbf{w}_t\Delta\mathbf{w}.$$

Using $\mathbf{w}_{tt} = \Delta\mathbf{w} + a\mathbf{w}_t$ and the divergence theorem, we get

$$\begin{aligned}\frac{dE}{dt}(t) &= 2\int_{B(x_0,T-t)} \left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}_t^2\right)\,dx \\ &+ 2\int_{\partial B(x_0,T-t)} \mathbf{w}_t\nabla\mathbf{w}\cdot\nu\,d\sigma - \int_{\partial B(x_0,T-t)} \left(|\mathbf{w}|^2 + |\mathbf{w}_t|^2 + |\nabla\mathbf{w}|^2\right)\,d\sigma,\end{aligned}$$

where ν denotes the outward unit normal to $\partial B(x_0, T-t)$. Since

$$2|\mathbf{w}_t\nabla\mathbf{w}\cdot\nu| \le |\mathbf{w}_t|^2 + |\nabla\mathbf{w}|^2,$$

we deduce that

$$\frac{dE}{dt}(t) \le 2\int_{B(x_0,T-t)} \left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}_t^2\right)\,dx.$$

Recalling that a is continuous, there exists a constant $M > 0$ such that $|a(t,x)| \le M$ for all $(t,x) \in \Omega$. Hence

$$2\left(\mathbf{w}\mathbf{w}_t + a\mathbf{w}_t^2\right) \le 2\mathbf{w}\mathbf{w}_t + 2M\mathbf{w}_t^2 \le \mathbf{w}^2 + \mathbf{w}_t^2 + 2M\mathbf{w}_t^2 \le (2M+1)(\mathbf{w}^2 + \mathbf{w}_t^2).$$

It follows that

$$\frac{dE}{dt}(t) \le (2M+1)E(t).$$

This implies that $\frac{d}{dt}\left[E(t)\mathrm{e}^{-(2M+1)t}\right] \le 0$. Therefore $E(t) \le E(0)\mathrm{e}^{(2M+1)t}$ for $0 \le t \le T$. Since $E(0) = 0$, we conclude that $\mathbf{w} = 0$ in Ω. □

A straightforward consequence of Proposition 2.5 is Theorem 2.4.

Proof of Theorem 2.4. Suppose that $u, v \in C^2(\Omega)$ both solve (1.8), and with identical data in $B(x_0, T)$. Let $\mathbf{w} = u - v$. Then

$$\mathbf{w}_{tt} - \Delta\mathbf{w} = g(u_t) - g(v_t), \quad \mathbf{w} = \mathbf{w}_t = 0 \quad \text{in} \quad B(x_0, T).$$

Let us write

$$\begin{aligned}g(u_t) - g(v_t) &= \int_0^1 \frac{d}{d\tau}\Big[g((1-\tau)v_t + \tau u_t)\Big]\,d\tau \\ &= \left(\int_0^1 g'((1-\tau)v_t + \tau u_t)\,d\tau\right)(u_t - v_t) \\ &= a(t,x)\mathbf{w}_t,\end{aligned}$$

where

$$a(t,x) = \int_0^1 g'((1-\tau)v_t(t,x) + \tau u_t(t,x))\, d\tau.$$

Using the fact that $u, v \in C^2$ and g is smooth, we deduce that a is a continuous function in (t,x). Applying Proposition 2.5 we conclude the proof of Theorem 2.4. □

3. Proof of Theorem 1.1

Define

$$y(t) = \int_{\mathbb{R}^N} (u(t,x))^2\, dx. \tag{3.1}$$

The first and second derivatives are

$$\begin{aligned} y'(t) &= 2\int_{\mathbb{R}^N} u_t(t,x)\, u(t,x)\, dx, \\ y''(t) &= -2\int_{\mathbb{R}^N} |\nabla u(t,x)|^2\, dx + 2\int_{\mathbb{R}^N} |u_t(t,x)|^2\, dx + 2\int_{\mathbb{R}^N} \omega(x) u(t,x) f(u(t,x))\, dx. \end{aligned}$$

By Cauchy-Schwarz inequality we have

$$(y'(t))^2 \le 4y(t)\left(\int_{\mathbb{R}^N} |u_t(t,x)|^2\, dx\right). \tag{3.2}$$

In order to use Proposition 2.1, we have to find a relation between $y''(t)$ and $\int_{\mathbb{R}^N} |u_t|^2\, dx$. Write

$$\begin{aligned} y''(t) &= (\lambda+4)\int_{\mathbb{R}^N} |u_t|^2\, dx + \lambda\int_{\mathbb{R}^N} |\nabla u|^2\, dx + 2\int_{\mathbb{R}^N} \omega(x)\, u\, f(u)\, dx \\ &- (\lambda+2)\int_{\mathbb{R}^N} \left(|u_t|^2 + |\nabla u|^2\right)\, dx\,. \end{aligned}$$

By the energy conservation, we get

$$\begin{aligned} y''(t) &= (\lambda+4)\int_{\mathbb{R}^N} |u_t|^2 dx + \lambda\int_{\mathbb{R}^N} |\nabla u|^2\, dx - 2\,(\lambda+2)\, E_0 \\ &+ 2\int_{\mathbb{R}^N} \omega(x)\,(u\, f(u) - (2+\lambda)F(u))\, dx. \end{aligned}$$

The assumption (1.2) implies that

$$\begin{aligned} y''(t) &\ge (\lambda+4)\int_{\mathbb{R}^N} |u_t|^2\, dx + \lambda\int_{\mathbb{R}^N} |\nabla u|^2\, dx - 2\,(\lambda+2)\, E_0 \\ &- 2\alpha\int_{\mathbb{R}^N} \omega(x)\, u^2\, dx. \end{aligned} \tag{3.3}$$

For the rest of the proof, we will treat separately the cases $N \ge 3$ and $N \le 2$.

(i) Assume that $N \ge 3$. Hölder's inequality together with (2.10) yields

$$\int_{\mathbb{R}^N} \omega(x)\, u^2\, dx \le \|\omega\|_{N/2}\|u\|^2_{\frac{2N}{N-2}} \le C_*^2\|\omega\|_{N/2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx\,. \tag{3.4}$$

It follows that

$$y''(t) \ge (\lambda+4)\int_{\mathbb{R}^N} |u_t|^2\, dx + \left(\lambda - 2\alpha\, C_*^2\|\omega\|_{N/2}\right)\int_{\mathbb{R}^N} |\nabla u|^2\, dx - 2\,(\lambda+2)\, E_0\,. \tag{3.5}$$

As $E_0 < 0$, and thanks to (1.4), we get

$$y''(t) \geq (\lambda + 4) \int_{\mathbb{R}^N} |u_t|^2 \, dx \, .$$

Multiplying both sides by $y(t)$ gives

$$y(t)y''(t) \geq (\lambda + 4) \, y(t) \int_{\mathbb{R}^N} |u_t|^2 \, dx \, .$$

Applying (3.2), we obtain that

$$y(t)y''(t) \; \geq \; \left(\frac{\lambda}{4} + 1 \right) \, (y'(t))^2 \, .$$

Since $E_0 < 0$ and using (1.4), we find that $y''(t) \geq -2\,(\lambda + 2)\, E_0 > 0$. Hence we can apply Corollary 2.2 to conclude that y blows-up in finite time. This finishes the proof of Theorem 1.1 (Part (i)).

(ii) Assume now that $N \leq 2$. Clearly we cannot use Sobolev embedding to obtain inequality like (3.4). From assumption (1.4) and thanks to (3.3) it follows that

$$y'' \geq (\lambda + 4) \int_{\mathbb{R}^N} |u_t|^2 \, dx - 2\alpha \|\omega\|_\infty y,$$

where we have used the fact that $E_0 < 0$. Hence

$$y \, y'' \geq (1 + \gamma) \, y'^2 - \mu \, y^2,$$

where $\gamma = \frac{\lambda}{4}$ and $\mu = 2\alpha \|w\|_\infty$. This means that y satisfies the differential inequality (2.1). Since the assumption (1.7) implies that $y(0) > 0$ and $y'(0) - \sqrt{\frac{\mu}{\gamma}} y(0) > 0$, we can apply Proposition 2.1 to conclude that y blows-up in finite time. This finishes the proof of Part (ii) of Theorem 1.1.

□

Note that one can construct initial data u_0, u_1 such that $E_0 < 0$. To this end, we use the following result dealing with the behavior of the nonlinearity f.

Proposition 3.1. *Suppose that* (1.2) *is fulfilled and assume that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *having super linear growth i.e.*

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty. \tag{3.6}$$

Then, there exists $A > 0$ *such that*

$$F(u) \geq A u^{2+\lambda} + \frac{\alpha}{\lambda} u^2 \tag{3.7}$$

for $u > 0$ *large enough.*

Proof.
The assumption (1.2) implies that

$$uF'(u) - (2 + \lambda) F(u) \geq -\alpha u^2. \tag{3.8}$$

By virtue of (3.8), we get

$$\big(u^{-(2+\lambda)} F(u)\big)' \geq -\alpha u^{-(1+\lambda)} = \big(\frac{\alpha}{\lambda} u^{-\lambda}\big)'.$$

Hence

$$\big(u^{-(2+\lambda)} F(u) - \frac{\alpha}{\lambda} u^{-\lambda}\big)' \geq 0$$

which implies that

$$\left[u^{-\lambda}\big(\frac{F(u)}{u^2}-\frac{\alpha}{\lambda}\big)\right]' \geq 0.$$

Therefore, the function $u \longmapsto u^{-\lambda}(\frac{F(u)}{u^2}-\frac{\alpha}{\lambda})$ is increasing. The conclusion follows easily. □

Corollary 3.2. *Suppose that* (1.2) *and* (3.6) *are fulfilled. Then, there exist smooth initial data* u_0, u_1 *such that* $E_0 < 0$.

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a non trivial function and set $u_0 = u_1 = k\phi$ where $k > 0$ to be chosen suitability. Then,

$$\begin{aligned} E_0 &= \frac{1}{2}\int \left(|\nabla u_0|^2 + |u_1|^2\right) dx - \int \omega(x)F(u_0(x))dx \\ &= \frac{k^2}{2}\int \left(|\nabla \phi|^2 + |\phi|^2\right) dx - \int \omega(x)F(k\phi(x))dx. \end{aligned}$$

By virtue of (3.7) we get

$$\begin{aligned} E_0 &\leq \frac{k^2}{2}\int \left(|\nabla \phi|^2 + |\phi|^2\right) dx - \int A\omega(x)k^{2+\lambda}\phi^{2+\lambda}(x)dx - \int \frac{\alpha}{\lambda}\omega(x)k^2\phi^2(x)dx \\ &\leq k^2\left[\frac{1}{2}\int \left(|\nabla \phi|^2 + |\phi|^2\right) dx - \frac{\alpha}{\lambda}\int \omega(x)\phi^2(x)dx - Ak^\lambda \int \omega(x)\phi^{2+\lambda}(x)dx\right]. \end{aligned}$$

Since,

$$\lim_{k\longrightarrow\infty} k^2\left[\frac{1}{2}\int \left(|\nabla \phi|^2 + |\phi|^2\right) dx - \frac{\alpha}{\lambda}\int \omega(x)\phi^2(x)dx - Ak^\lambda \int \omega(x)\phi^{2+\lambda}(x)dx\right] = -\infty,$$

one can choose k large enough so that $E_0 < 0$. □

Remark. *By taking* $u_0 = u_1 = k\phi$ *and using* (1.4), *we reach* (1.7) *in the lower dimensions* $N \leq 2$.

4. Proof of Theorem 1.2

The proof is divided in two steps.

Step1. Take u_1 in (1.10) to be a constant $a > 0$. Then the equation $u_{tt} - \Delta u = g(u_t)$ reduces to an *ODE* in time,

$$y' = g(y), \quad y(0) = a, \tag{4.1}$$

where $y = u_t$.

By the assumption (1.9) made on g, we can see that the solution of the initial value problem (4.1) blows up as $t \to T_a := \int_a^\infty \frac{ds}{g(s)}$. Thus $u(t,x) = \displaystyle\int_0^t y(\tau)\,d\tau$ solves (1.10) and $u \to \infty$ as $t \to T_a$.

Step2. By applying Theorem 2.4 to the solution obtained in Step 1, we conclude that if $u \in C^2(\Omega)$ solves (1.8) in Ω, with initial data $u = 0$ and $u_t = \frac{1}{T}$ in B_0, then u blows-up at time T. To conclude the proof of Theorem 1.2, we make a cut-off the constant $\frac{1}{T}$ smoothly outside a sufficiently large ball to produce $u_1 \in C_c^\infty(\mathbb{R}^n)$.

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