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SYMMETRIZATION METHOD FOR THE LAWS OF LARGE NUMBERS UNDER SUB-LINEAR EXPECTATION

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Abstract. In this study, two laws of large numbers are established for sub-linear expectation using the symmetrization method for the row sums of arrays from row-wise independent random variables.

1. Introduction

Limit theorems are the fundamental concepts in the probability theory, especially in the discussion of limit theorems, the Strong Law of Large Numbers (SLLN) and Weak Law of Large Numbers (WLLN) are well known as essential outcomes in probability theory. Sub-linear expectations are used in the fields when we cannot completely model by classical theory. The notation of WLLN and SLLN for sub-linear expectation has been generalized and extended in different ways as mentioned in previous studies (cf. [1, 2, 3, 6, 7, 8, 9]).

In this paper, an array of row-wise independent random variables was considered with no independence conditions on the random variables from different rows of arrays. The remainder of the present study is organized as follows. In Section 2, some fundamental concepts are expressed related to sub-linear expectations. In Section 3, the main result is stated and proved, which is the WLLN and SLLN using the symmetrization method under sub-linear expectation.

2. Preliminaries

Suppose $(\Omega, \mathcal{F})$ is a measurable space. Assume $\mathcal{L}$ is a linear space from real functions on Ω.

To prove the main results of this paper, we need the following definitions and lemmas.

Definition 2.1 ([6]). *The sub-linear expectation E on $\mathcal{L}$ is a function $E : \mathcal{L} \to \mathbb{R}$, which for every $\xi, \eta \in \mathcal{L}$ satisfying*

(i) $E[\xi] \geq E[\eta], \quad$ *if* $\quad \xi \geq \eta$.
(ii) $E[k] = k, \quad$ *for* $k \in \mathbb{R}$.
(iii) $E[\xi + \eta] \leq E[\xi] + E[\eta], \quad$ *for* $\xi, \eta \in \mathcal{L}$,

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(iv) $E[\alpha\xi] = \alpha E[\xi], \quad$ *for* $\alpha \geq 0$.

The triple $(\Omega, \mathcal{L}, E)$ *is called a sub-linear expectation space.*

Remark. *By combining (ii) and (iii), we see that:*

(v) $E[\xi + k] = E[\xi] + k, \quad$ *for* $k \in \mathbb{R}$.

Definition 2.2 ([1])**.** *The function* $V : F \to [0, 1]$ *is said a capacity if it satisfying to the following properties:*

(i) $V(\emptyset) = 0, \quad V(\Omega) = 1$.

(ii) $V(E) \leq V(F) \quad$ *if* $E \subset F$ *and* $E, F \in \mathcal{F}$.

Remark. *A capacity* V *is called to be sub-additive if it reads,*

$$V(E \cup F) \leq V(E) + V(F), \qquad E, F \in \mathcal{F}.$$

Definition 2.3 ([2])**.** *Let* $(\Omega, \mathcal{L}, E)$ *be a sub-linear expectation for every* $B \in \mathcal{F}$*, as we describe* $V(B) = E(I_B)$*. Define the conjugate capacity* $v(B) = 1 - V(B^c)$*,* $\forall B \in \mathcal{F}$*, where,* V *is a sub-additive capacity and* $v(B) = \varepsilon(I_B)$ *wherein* $\varepsilon(\xi) = -E(-\xi)$.

Definition 2.4 ([6])**.** *Suppose* $\xi_1, \xi_2, \ldots, \xi_{m+1}$ *is a sequence of random variables on* $(\Omega, \mathcal{F})$*.* ξ_{m+1} *is called independent from* $(\xi_1, \ldots, \xi_m)$ *under* $E(\cdot)$*, if for every nonnegative measurable function* $\psi(\cdot)$ *on* R *with* $E[\psi_i(\xi_i)] < \infty$*,* $i = 1, \ldots, m + 1$*, we have*

$$E\left[\prod_{i=1}^{m+1} \psi_i(\xi_i)\right] = E\left[\prod_{i=1}^{m} \psi(\xi_i)\right] \cdot E\left[\psi_{m+1}(\xi_{m+1})\right].$$

Definition 2.5 ([5])**.** *Let* ξ *and* ξ' *are two independent and same distributed random variables. Then* $\xi^s = \xi - \xi'$ *is called the symmetrized random variables.*

Lemma 2.6. *Suppose* $(\Omega, \mathcal{L}, E)$ *is a sub-linear expectation space and* E *is sub-linear expectation and* V *is a capacity. Let* $\{\xi_i\}_{i=1}^{\infty}$ *is a symmetric sequence of random variables, for each* m*, let* $S_m = \sum_{i=1}^{m} \xi_i$*. Then for each integer* n *and every* $\beta > 0$*,*

$$V(\max_{m \leq n} |S_m| > \beta) \leq 2V(|S_n| > \beta), \tag{2.1}$$

and

$$V(\max_{i \leq n} |\xi_i| > \beta) \leq 2V(|S_n| > \beta). \tag{2.2}$$

If S_m *converges to* S *in probability, we have*

$$V(\sup_m |S_m| > \beta) \leq 2V(|S| > \beta).$$

Proof. To prove the Inequality (2.1), supose $\eta = \inf\{m \leq n, \ \ |S_m| > \beta\}$. Then

$$V(|S_n| > \beta) = \sum_{m=1}^{n} V(|S_n| > \beta, \eta = m),$$

since for each m, $\{\eta = m\}$ depends on $\xi_1, \ldots, \xi_m$, let $D_m = S_n - S_m$ for $m \leq n$,

$$V(|S_n| > \beta) = \sum_{m=1}^{n} V(|S_m - D_m| > \beta, \ \ \eta = m),$$

by the triangle inequality, we have:

$$\begin{aligned} 2|S_m| &= |S_m + S_m + D_m - D_m| \leq |S_m + D_m| + |S_m - D_m| \\ &= |S_n| + |S_m - D_m|. \end{aligned}$$

By definition 2.2,

$$2V(|S_n| > \beta) \geq \sum_{m=1}^{n} V(\eta = m) = V(\max_{m \geq n} |S_m| > \beta).$$

Thus the proof of (2.1) is completed. The proof of (2.2) is similar. □

Lemma 2.7. *Suppose $(\Omega, \mathcal{L}, E)$ is a sub-linear expectation space and E is a sub-linear expectation and V is a capacity. Assume $\{\xi_i, i \leq n\}$ is a sequence of independent random variables. Let $S_n = \sum_{i=1}^{m} \xi_i$, $m \leq n$. For each $\alpha, \beta > 0$, we have*

$$V(\max_{m \leq n} |S_m| > \alpha + 3\beta) \leq V(\max_{i \leq n}(|\xi_i| > \alpha) + \left(V \max_{m \leq n} |S_m| > \beta)\right)^2. \tag{2.3}$$

If the random variables are symmetric, hence for $\alpha, \beta > 0$,

$$V(|S_n| > \alpha + 2\beta) \leq V(\max_{i \leq n} |\xi_i| > \alpha) + 4(V(|S_n| > \beta))^2. \tag{2.4}$$

Proof. Set $\eta = \inf\{l \leq n, |S_l| > \beta\}$, then, $\{\eta = l\}$ depends on the random variables $\xi_1, \ldots, \xi_l$, and by disjoint union

$$\{\max_{m \leq n} ||S_m| > \beta\} = \sum_{l=1}^{n} \{\eta = l\}.$$

Also on $\{\eta = l\}$, we have $|S_m| < \beta$ for $m < l$, and for $m \geq l$, we have:

$$|S_m| \leq |S_m - S_l| + |\xi_l| + \beta.$$

Then,

$$\max_{m \leq n} |S_m| \leq \max_{l < m \leq n} |S_m - S_l| + \max_{i \leq n} |\xi_i| + \beta.$$

And, by independence,

$$\begin{aligned} V(\eta = l, \ \max_{m \leq n} |S_m| > \alpha + 3\beta) \leq & V(\eta = l) \cdot V(\max_{l < m \leq n} |S_m - S_l| > 2\beta) \\ & + V(\eta = l, \ \max_{i \leq n} |\xi_i| > \alpha). \end{aligned}$$

since $\max_{l<m\leq n} |S_m - S_l| \leq 2 \max_{m \leq n} |S_m|$; for $l = 1, \ldots, n$, Inequality (2.3) is established. To prove (2.4), we have

$$|S_n| \leq |S_n - S_l + S_l| \leq |S_n - S_l| + |\xi_l| + |S_{l-1}|,$$

for $l = 1, \ldots, n$. Hence,

$$V(\eta = l, |S_n| > \alpha + 2\beta) \leq V(\eta = l) \cdot V(|S_n - S_l| > \beta) + V(\eta = l, \ \max_{i \leq n} |\xi_i| > \alpha\}.$$

By applying (2.1) for symmetric random variables, the proof of (2.4) is obtained. □

Lemma 2.8. *Suppose $\{\xi_i, i \leq n\}$ is a sequence of independent and symmetric random variables. Let $S_m = \sum_{i=1}^{m} \xi_i$, $m \leq n$. Thus, for $\beta_0 = \inf\{\beta > 0, \ V(|S_n| > \beta \leq (8 \times 3)^{-1}\}$, we have*

$$E|S_n| \leq 6E \max_{i \leq n} |\xi_i| + 6\beta_0. \tag{2.5}$$

Proof. Assume $\gamma \geq \beta_0$, by integration by two parts and (2.4), we have

$$\begin{aligned}
E|S_n| &= 3\int_0^\infty V(|S_n| > 3\beta)d\beta \\
&= 3\left(\int_0^\gamma V(|S_n| > 3\beta)d\beta + \int_\gamma^\infty V(|S_n| > 3\beta)d\beta\right) \\
&\leq 3\gamma + 4 \times 3\int_\gamma^\infty (V(|S_n| > \beta))^2 d\beta + 3\int_\gamma^\infty V(\max_{i\leq n}|\xi_i| > \beta)d\beta \\
&\leq 3\gamma + 4 \times 3V(|S_n| > \gamma)\int_0^\gamma V(|S_n| > \beta)d\beta + 3E\max_{i\leq n}|\xi_i| \\
&\leq 2 \times 3\gamma + 2 \times 3E\max_{i\leq n}|\xi_i|.
\end{aligned}$$

By choosing γ, we have $4\times 3V(|S_n| > \gamma) < \frac{1}{2}$. Then the proof of (2.5) complete. □

3. Main results

The following theorems are our main results.

Theorem 3.1. *Suppose $\{b_j,\ j \geq 1\}$, is a sequence of positive integers, suppose $\{\xi_{i,j},\ 1 \leq i \leq b_j,\ j \geq 1\}$ is an array of random variables such that for every $j \geq 1$, the random variable $\xi_{i,j}, \ldots, \xi_{b_j,j}$ are independent. Suppose $\{\xi'_{i,j},\ 1 \leq i \leq b_j,\ j \geq 1\}$ is an independent copy of $\{\xi_{i,j},\ 1 \leq i \leq b_j,\ j \geq 1\}$, let $\xi^s_{i,j} = \xi_{i,j} - \xi'_{i,j}$, $1 \leq i \leq b_j,\ j \geq 1$. If q is a real number such that:*

$$\lim_{j\to\infty}\max_{1\leq i\leq b_j} V(|\xi_{i,j}| > q) < \frac{1}{2}. \tag{3.1}$$

Then, symmetrization method for the WLLN hold:

$$\sum_{i=1}^{b_j}\xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j}I(|\xi_{i,j}| < 2q)\right) \xrightarrow{P} 0, \tag{3.2}$$

if and only if

$$\sum_{i=1}^{b_j}\xi^s_{i,j} \xrightarrow{P} 0. \tag{3.3}$$

Proof. Because of that $\{\xi'_{i,j}, 1 \leq i \leq b_j, j \geq 1\}$ is an independent copy of $\{\xi_{i,j}, 1 \leq i \leq b_j, j \geq 1\}$, we have for $j \geq 1$,

$$\begin{aligned}
\sum_{i=1}^{b_j}\xi^s_i = &\left(\sum_{i=1}^{b_j}\xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j}I(|\xi_{i,j}| < 2q)\right)\right) \\
&- \left(\sum_{i=1}^{b_j}\xi'_{i,j} - \sum_{i=1}^{b_j} E\left(\xi'_{i,j}I(|\xi'_{i,j}| < 2q)\right)\right).
\end{aligned}$$

Then (3.3) follows from (3.2). Conversly, let (3.3) hold since the random variables $\hat{\xi}_{i,j}$, $1 \leq i \leq b_j$ are independent and symmetric, thus by Lemma 2.6 and for each $j \geq 1$, we have

$$V(\max_{1\leq i\leq b_j}|\xi^s_{i,j}| > \beta) \leq 2V(|\sum_{i=1}^{b_j}\xi^s_{i,j}| > \beta) \quad \forall\beta > 0, \tag{3.4}$$

by (3.3) and (3.4) we have

$$\max_{1\le i\le b_j} |\xi^s_{i,j}| \xrightarrow{P} 0. \tag{3.5}$$

By (3.4) and independence condition, it can be seen that,

$$V(\max_{1\le i\le b_j} |\xi^s_{i,j}| > \beta) = 1 - \prod_{i=1}^{b_j}(1 - v(|\xi^s_{i,j}| > \beta)) \to 0 \quad \text{as } j \to \infty,\ \forall \beta > 0.$$

Therefore

$$\sum_{i=1}^{b_j} V(|\xi^s_{i,j}| > \beta) \to 0 \quad \text{as } j \to \infty,\ \forall \beta > 0. \tag{3.6}$$

By applying (3.1), there is $j_0 \ge 1$,

$$\max_{1\le i\le b_j} V(|\xi_{i,j}| > q) < \frac{1}{2}, \quad \forall j \ge j_0.$$

Furthermore,

$$\left\{|\xi_{i,j}| \le q,\ \ |\xi'_{i,j}| > 2q\right\} \subseteq \left\{|\xi_{i,j} - \xi'_{i,j}| > q\right\} = \left\{|\xi^s_{i,j}| > q\right\}, \quad 1 \le i \le b_j.$$

Then, for $1 \le i \le b_j$, $j > j_0$, we get

$$\begin{aligned} V(|\xi^s_{i,j}| > q) &\ge V(|\xi_{i,j}| \le q,\ \ |\xi'_{i,j}| > 2q) \\ &= V(|\xi_{i,j}| \le q) \cdot V(|\xi'_{i,j}| > 2q) \ge \frac{1}{2} V(|\xi'_{i,j}| > 2q) \end{aligned}$$

hence

$$V(|\xi'_{i,j}| > 2q) \le 2V(|\xi^s_{i,j}| > q).$$

Then, we have

$$\sum_{i=1}^{b_j} V(|\xi_{i,j}| > 2q) = \sum_{i=1}^{b_j} V(|\xi'_{i,j}| > 2q) \le 2 \sum_{i=1}^{b_j} V(|\xi^s_{i,j}| > q), \quad \forall j \ge j_0.$$

By (3.6), we get

$$\sum_{i=1}^{b_j} V(|\xi_{i,j}| > 2q) \to 0, \quad \text{as } j \to \infty. \tag{3.7}$$

For $i = 1, \dots, b_j$, $j \ge 1$, assume

$$\tau_{i,j} = \xi_{i,j} I(|\xi_{i,j}| \le 2q), \quad \tau'_{i,j} = \xi'_{i,j} I(|\xi'_{i,j}| \le 2q), \quad \tau^s = \tau_{i,j} - \tau'_{i,j}.$$

For $j \ge 1$, we have

$$V(|\sum_{i=1}^{b_j} \tau^s_{i,j} - \sum_{i=1}^{b_j} \xi^s_{i,j}| \ne 0) \le 2 \sum_{i=1}^{b_j} V(|\xi_{i,j}| > 2q) \to 0 \quad \text{as } j \to \infty.$$

Then, by (3.3), we have

$$\sum_{i=1}^{b_j} \tau^s_{i,j} \xrightarrow{P} 0. \tag{3.8}$$

Hence, by (3.8),

$$\max_{1\le i\le b_j} |\tau^s_{i,j}| \xrightarrow{P} 0. \tag{3.9}$$

Note that

$$\max_{1\le i\le b_j} |\tau^s_{i,j}| \le 4q \qquad \forall j \ge 1.$$

By applying (3.9) and Lebesgue's dominated convergence theorem, we have

$$\lim_{j\to\infty} E(\max_{1\le i\le b_j} |\tau_{i,j}^s|) = 0. \tag{3.10}$$

By Lemma 2.8, for each $j \ge 1$, we get

$$E(|\sum_{i=1}^{b_j} \tau_{i,j}^s|) \le 6E(\max_{1\le i\le b_j} |\tau_{i,j}^s|) + 6\beta_j,$$

wherein

$$\beta_j = \inf\left\{\beta > 0 :\ V(|\sum_{i=1}^{b_j} \tau_{i,j}^s| > \beta) \le \frac{1}{24}\right\}.$$

By (3.8), we conclude $\lim_{j\to\infty} \beta_j = 0$. It follows from (3.10), that

$$\lim_{j\to\infty} E(|\sum_{i=1}^{b_j} \tau_{i,j}^s|) = 0. \tag{3.11}$$

For each $j \ge 1$, $\sum_{i=1}^{b_j} \tau_{i,j}$ and $\sum_{i=1}^{b_j} \tau'_{i,j}$, are i.i.d random variables, by (3.8), we have

$$E\left(\left|\sum_{i=1}^{b_j} \tau_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right)\right|\right) \le E\left(\left|\sum_{i=1}^{b_j} \tau_{i,j}^s\right|\right).$$

By (3.11), we have

$$\lim_{j\to\infty} E\left(\left|\sum_{i=1}^{b_j} \tau_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right)\right|\right) = 0,$$

therefore

$$\sum_{i=1}^{b_j} \tau_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right) \xrightarrow{P} 0. \tag{3.12}$$

Since for $j \ge 1$, we have

$$\sum_{i=1}^{b_j} \xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right) = \left(\sum_{i=1}^{b_j} \tau_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right)\right) + \sum_{i=1}^{b_j} \xi_{i,j} I(|\xi_{i,j}| > 2q).$$

By (3.12) and (3.7), the proof of (3.2) is completed. □

Theorem 3.2. *Suppose $\{b_j, j \ge 1\}$, is a sequence of positive integers, assume $\{\xi_{i,j},\ 1 \le i \le b_j,\ j \ge 1\}$ is an array of random variables such that for every $j \ge 1$, the random variables $\xi_{i,j}, \ldots, \xi_{b_j,j}$ are independent. Suppose $\{\xi'_{i,j},\ 1 \le i \le b,\ j \ge 1\}$ is an independent copy of $\{\xi_{i,j},\ 1 \le i \le b_j,\ j \ge 1\}$, let $\xi_{i,j}^s = \xi_{i,j} - \xi'_{i,j}$, $1 \le i \le b_j$, $j \ge 1$. If q is a real number that we have:*

$$\lim_{j\to\infty} \max_{1\le i\le b_j} V(|\xi_{i,j}| > q) < \frac{1}{2}.$$

Then, symmetrization method for the SLLN hold:

$$\sum_{i=1}^{b_j} \xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| < 2q)\right) \xrightarrow{\text{a.s.}} 0, \tag{3.13}$$

if and only if

$$\sum_{i=1}^{b_j} \xi_{i,j}^s \xrightarrow{\text{a.s.}} 0. \tag{3.14}$$

Proof. Assume that (3.14) is hold. We show that (3.13) is hold. Assume

$$\tau_j = \sum_{i=1}^{b_j} \xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right),$$
$$\tau'_j = \sum_{i=1}^{b_j} \xi'_{i,j} - \sum_{i=1}^{b_j} E\left(\xi'_{i,j} I(|\xi'_{i,j}| \le 2q)\right), \quad j \ge 1.$$

Since, $\{\xi'_{i,j}, 1 \le i \le b_j, j \ge 1\}$ is an independent copy of $\{\xi_{i,j}, 1 \le i \le b_j, j \ge 1\}$. Then $\{\tau'_j, j \ge 1\}$ is an independent copy of $\{\tau_j, j \ge 1\}$.

(3.14) implies (3.3), thus by Theorem 3.1, (3.2) holds and we have

$$\tau_j = \sum_{i=1}^{b_j} \xi_{i,j} - \sum_{i=1}^{b_j} E\left(\xi_{i,j} I(|\xi_{i,j}| \le 2q)\right) \xrightarrow{P} 0.$$

By (3.14),

$$\tau_j - \tau'_j = \sum_{i=1}^{b_j} \xi_{i,j}^s \xrightarrow{\text{a.s.}} 0 \quad \text{as} \ \ j \to \infty. \tag{3.15}$$

By applying (3.13), (3.15), (3.2), the proof is completed. □

Corollary 3.3. *Suppose $\{\xi, \xi_j, j \ge 1\}$ is a sequence of i.i.d random variables. Assume $\{\xi'_j, j \ge 1\}$ is an independent copy of $\{\xi_j, j \ge 1\}$. Let $S_j = \sum_{i=1}^{j} \xi_i$ and $S'_j = \sum_{i=1}^{j} \xi'_i$, $j \ge 1$. Suppose $\{c_j, j \ge 1\}$ is an increasing sequence of real and positive numbers so that; $\lim_{j\to\infty} c_j = \infty$. Then*

$$\frac{S_j - jE\left(\xi I(|\xi| < c_j)\right)}{c_j} \xrightarrow{\text{a.s.}} 0, \quad \textit{(or in P.)}$$

if and only if

$$\frac{S_j - S'_j}{c_j} \xrightarrow{\text{a.s.}} 0. \quad \textit{(or in P.)}$$

Proof. Assume $m_j = j$, for $j \ge 1$. Suppose $\xi_{i,j} = \frac{\xi_j}{c_j}$, $1 \le i \le j$, $j \ge 1$. Thus $\{\xi_{i,j},\ 1 \le i \le j,\ j \ge 1\}$ is an arrays of random variables so that for every $j \ge 1$, the random variables $\xi_{i,j}, \ldots, \xi_{j,j}$ are independent. Because of that $\lim_{j\to\infty} c_j = \infty$ and $\{\xi,\ \xi_j,\ j \ge 1\}$ is a sequence of i,i,d random variables, we have:

$$\max_{1\le i\le j} V(|\xi_{i,j}| > \frac{1}{2}) = V(|\xi| > \frac{c_j}{2}) \to 0, \quad \text{as} \ \ n \to \infty.$$

Therefore, (3.1) is satisfied for $q = \frac{1}{2}$. By Theorem 3.1 we have:

$$\sum_{i=1}^{j} \xi_{i,j} - \sum_{i=1}^{j} E\left(\xi_{i,j} I(|\xi_{i,j}| \leq 2q)\right) = \frac{S_j - jE\left(|\xi| I(|\xi| \leq c_j)\right)}{c_j}, \quad j \geq 1.$$

Then, proof is completed. □

4. Conclusion

The results of this paper can be applied for time dynamic and risk measures in the field of finance, and the problem of probability model uncertainty arising from stochastic controls. For further studies, it is suggested to obtain central limit theorems in this frame.

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