SOME CONTRACTION MAPPINGS WITH NONUNIQUE FIXED POINT THEOREMS

SEHER SULTAN SEPET, CAFER AYDIN

Abstract. In this article, we obtain nonunique fixed point theorems for certain type of self mappings in partial metric spaces. Our results generalize and prove many previously obtained results. Also we give a example showing that our main theorem is applicable.

1. Introduction

In 1922 Banach [2], introduced the famous fundamental fixed point theorem, also known as the Banach contraction principle. The Banach contraction principle is the simplest and one of the most adaptable elementary results in fixed point theory. Banach proved that every contraction in a complete metric space has a unique fixed point. Throughout the years, several extensions and generalizations of this principle have appeared on various types of spaces such as metric spaces, cone metric spaces, partial metric spaces, $b$-metric spaces. Matthews [15], introduced the partial metric spaces and presented a fixed point theorem on partial metric space. It is widely recognized that partial metric space plays an important role in constructing models in the theory of computation. Matthews proved the contraction principle of Banach in this new framework and also discussed some properties of convergence of sequences. Following this various fixed point results were proved in these spaces, for more details, see [1, 7, 8, 9, 10, 11, 12, 13, 14, 16, 19].

Now, we mention briefly some fundamental definitions.

Definition 1.1. [15] A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative reals) such that, for all $x, y, z \in X$:

1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$ ($T_0$-separation axiom),
2. $p(x, x) \leq p(x, y)$ (small self-distance axiom),
3. $p(x, y) = p(y, x)$ (symmetry),
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (modified triangular inequality).

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. It is clear that if $p(x, y) = 0$, then, from (p1) and (p2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For another example,
Then are interesting from a computational point of view may be found in \[1, \[2, 3\]. \]

Let \( p : I \times I \to \mathbb{R}^+ \) be the function such that \( p([a, b], [c, d]) = \max \{b, d\} - \min \{a, c\} \). Then \((I, p)\) is a partial metric space. Other examples of partial metric space which are interesting from a computational point of view may be found in \([3, 15, 16]\).

Each partial metric \( p \) on \( X \) generates a \( T_0 \) topology \( \tau_p \) on \( X \) which has a base as the family open \( p \)-balls

\[ \{B_p(x, \epsilon) : x \in X, \epsilon > 0\}, \]

where

\[ B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}, \]

for all \( x \in X \) and \( \epsilon > 0 \).

**Lemma 1.2.** \([15, 16]\) Let \((X, p)\) be a complete partial metric space. Then,

(A) If \( p(x, y) = 0 \), then \( x = y \).

(B) If \( x \neq y \), then \( p(x, y) > 0 \).

**Example 1.3.** \([13, 20]\) Let \((X, d)\) and \((X, p)\) be a metric space and a partial metric space, respectively. Functions \( p : X \times X \to \mathbb{R}^+ (i \in \{1, 2, 3\}) \) given by

\[
\begin{align*}
\rho_1(x, y) &= d(x, y) + p(x, y), \\
\rho_2(x, y) &= d(x, y) + \max \{u(x), u(y)\}, \\
\rho_3(x, y) &= d(x, y) + a,
\end{align*}
\]

define partial metrics on \( X \), where \( u : X \to \mathbb{R}^+ \) is an arbitrary function and \( a \geq 0 \).

**Definition 1.4.** \([15, 16]\) Let \((X, p)\) be a partial metric space. Then,

i. a sequence \( \{x_n\} \) in a partial metric space \((X, p)\) converges with respect to \( \tau_p \) to a point \( x \in X \) if \( p(x, x_n) = \lim_{n \to \infty} p(x, x_n) \),

ii. a sequence \( \{x_n\} \) in a partial metric space \((X, p)\) is called a Cauchy sequence if there exists \( \lim_{n,m \to \infty} p(x_n, x_m) \) (and is finite),

iii. a partial metric space \((X, p)\) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \( X \) converges to a point \( x \in X \); that is, \( p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m) \).

If \( p \) is a partial metric on \( X \), then the functions \( d_p : X \times X \to \mathbb{R}^+ \) given by

\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}
\]

are ordinary metrics on \( X \). The functions \( d_m^p \) and \( p_0 \) defined on \( X \times X \) by

\[
\begin{align*}
d_m^p(x, y) &= \max \{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \\
&= p(x, y) - \min \{p(x, x), p(y, y)\} \tag{1.2}
\end{align*}
\]

and by \( p_0(x, x) = 0 \) for all \( x \in X \) and \( p_0(x, y) = p(x, y) \) for \( x \neq y \), are metrics \( X \). The following topological inclusions are well-known and easy to check: \( \tau_p \subseteq \tau_{d_p} = \tau_{d_m^p} \subseteq \tau_{p_0} \).

It is easy to see that \( d_p \) and \( d_m^p \) are equivalent metrics on \( X \).

**Lemma 1.5.** \([15, 16]\) Let \((X, p)\) be a partial metric space. Then,

i. \( \{x_n\} \) is a Cauchy sequence in \((X, p)\) if and only if it is Cauchy sequence in the metric space \((X, d_p)\),

ii. \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete. Furthermore,

\[
\lim_{n \to \infty} d_p(x_n, x) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m).
\]
The following lemma proved by Karapınar [13] is very useful.

**Lemma 1.6.** Let \((X, p)\) be a partial metric space. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \((X, p)\) is fundamental (Cauchy) sequence in \((X, p)\) if and only if it satisfies the following condition:

\((^*)\) for each \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(p(x_n, x_m) - p(x_n, x_n) < \varepsilon\) whenever \(n_0 \leq n \leq m\).

**Definition 1.7.** (cf. [3]) Let \((X, p)\) be a partial metric space and \(F\) a self map of \(X\).

1. \(F\) is called orbitally continuous if

\[
\lim_{i,j \to \infty} p(F^{n_i}x, F^{n_j}x) = \lim_{i \to \infty} p(F^{n_i}x, z) = p(z, z) \tag{1.3}
\]

implies

\[
\lim_{i,j \to \infty} p(F^{n_i}x, F^{n_j}x) = \lim_{i \to \infty} p(F^{n_i}x, Fz) = p(Fz, Fz) \tag{1.4}
\]

for each \(x \in X\). Equivalently, \(F\) is orbitally continuous provided that if \(F^{n_i}x \to z\) with respect to \(\tau_{dp}\), then \(F^{n_i+1}x \to Fz\) with respect to \(\tau_{dp}\), for each \(x \in X\).

2. A partial metric space \((X, p)\) is called orbitally complete if every Cauchy sequence \(\{F^{n_i}x\}_{i \in \mathbb{N}}\) converges with respect to \(\tau_{dp}\), that is if there is \(z \in X\) such that

\[
\lim_{i,j \to \infty} p(F^{n_i}x, F^{n_j}x) = \lim_{i \to \infty} p(F^{n_i}x, z) = p(z, z). \tag{1.5}
\]

2. **The results**

In this section we give nonunique fixed point theorems for certain type of self mappings in partial metric spaces and present some results from the work Ciric and Jotic [4]. Also we give a example showing that our main theorems are applicable.

**Theorem 2.1.** Let \(F\) be an orbitally continuous self map of a \(F\)-orbitally complete partial metric space \((X, p)\). If there is \(\alpha \geq 0\) and \(\lambda \in [0, 1)\) is such that

\[
\min\{p(x, y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x,Fx)[1+p(y,Fy)]}{1+p(x,y)}, \frac{p(y,Fy)[1+p(x,Fx)]}{1+p(x,y)}\}, \min\left\{\min\left\{\frac{p^2(Fx,Fy),p^2(x,Fx),p^2(y,Fy)}{p(x,y)}\right\}\right\} - \alpha \min\left\{d^p_m(x,Fy), d^p_m(y,Fx)\right\} \leq \lambda \max\{p(x, y), p(x, Fx)\} \tag{2.1}
\]

for all \(x, y \in X\), then for each \(x_0 \in X\) the sequence \(\{F^{n_i}x_0\}_{n \in \mathbb{N}}\) converges with respect to \(\tau_{dp}\) to a fixed point of \(F\).

**Proof.** Let \(x_0\) be an arbitrary point of \(X\) and choose \(x_n \in X\) such that

\[x_{n+1} = Fx_n.\]

If for some \(n_0 \in \{1, 2, \ldots n\}\) we have \(x_{n_0} = x_{n_0+1}\) then

\[x_{n_0+1} = x_{n_0} = Fx_{n_0} \]
that is $x_{n_0}$ is a fixed point of $F$ which completes the proof. We suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. From (2.1) for $x = x_{n}$ and $y = x_{n+1}$ we obtain

$$
\min \{p(x_{n},x_{n+1}), p(Fx_{n},Fx_{n+1}), p(x_{n+1},Fx_{n+1})\},
$$

where

$$
p(x_{n},Fx_{n})[1 + p(x_{n+1},Fx_{n+1})],
$$

$$
\min \{p^{2}(Fx_{n},Fx_{n+1}), p^{2}(x_{n},Fx_{n}), p^{2}(x_{n+1},Fx_{n+1})\}
$$

$$
p(x_{n},x_{n+1})
$$

$$
- \alpha \min \{d_{m}^{p}(x_{n},Fx_{n+1}), d_{m}^{p}(x_{n+1},Fx_{n})\} \leq \lambda \max \{p(x_{n},x_{n+1}), p(x_{n},Fx_{n})\}
$$

which imply that

$$
\min \{p(x_{n},x_{n+1}), p(x_{n+1},x_{n+2}), p(x_{n},x_{n+1}), p(x_{n+1},x_{n+2})\},
$$

where

$$
\alpha \min \{d_{m}^{p}(x_{n},x_{n+2}), d_{m}^{p}(x_{n+1},x_{n+1})\} = 0.
$$

Then we obtain

$$
\min \{p(x_{n},x_{n+1}), p(x_{n+1},x_{n+2}), p(x_{n+1},x_{n+2})\},
$$

$$
\frac{p(x_{n},x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_{n},x_{n+1})},
$$

$$
\min \{p^{2}(x_{n},x_{n+1}), p^{2}(x_{n+1},x_{n+2})\}
$$

$$
\leq \lambda p(x_{n},x_{n+1}).
$$

We will consider the following four cases:

Case 1 : If

$$
\min \{p(x_{n},x_{n+1}), p(x_{n+1},x_{n+2}), p(x_{n+1},x_{n+2})\},
$$

$$
\frac{p(x_{n},x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_{n},x_{n+1})},
$$

$$
\min \{p^{2}(x_{n},x_{n+1}), p^{2}(x_{n+1},x_{n+2})\}
$$

$$
\leq p(x_{n},x_{n+1})
$$

for some $n \in \mathbb{N}_0$, then we obtain that

$$
p(x_{n},x_{n+1}) \leq \lambda p(x_{n},x_{n+1}) < p(x_{n},x_{n+1})
$$

a contraction.

Case 2 : If

$$
\min \{p(x_{n},x_{n+1}), p(x_{n+1},x_{n+2}), p(x_{n+1},x_{n+2})\},
$$

$$
\frac{p(x_{n},x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_{n},x_{n+1})},
$$

$$
\min \{p^{2}(x_{n},x_{n+1}), p^{2}(x_{n+1},x_{n+2})\}
$$

$$
\leq p(x_{n+1},x_{n+2})
$$

for some $n \in \mathbb{N}_0$, then we obtain

$$
p(x_{n+1},x_{n+2}) \leq \lambda p(x_{n},x_{n+1}).
$$
Case 3 : If
\[
\min\left\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{p(x_n,x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_n,x_{n+1})} \right\} = \frac{p(x_n,x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_n,x_{n+1})}
\]
for some \( n \in \mathbb{N}_0 \), then we obtain
\[
p(x_{n+1},x_{n+2}) \leq \lambda p(x_n,x_{n+1}).
\]

Case 4 : If
\[
\min\left\{ p(x_n,x_{n+1}), p(x_{n+1},x_{n+2}), \frac{p(x_n,x_{n+1})[1 + p(x_{n+1},x_{n+2})]}{1 + p(x_n,x_{n+1})} \right\} = \frac{\min\{p^2(x_n,x_{n+1}), p^2(x_{n+1},x_{n+2})\}}{p(x_n,x_{n+1})}
\]
for some \( n \in \mathbb{N}_0 \), then we obtain
\[
p(x_{n+1},x_{n+2}) \leq \lambda p(x_n,x_{n+1}).
\]

Consequently, the following inequality is obtained from all these cases
\[
p(x_{n+1},x_{n+2}) \leq \sqrt[2]{\lambda} p(x_n,x_{n+1}).
\]

By induction we have
\[
p(x_{n+1},x_{n+2}) \leq \sqrt[2]{\lambda} p(x_n,x_{n+1}) \leq \lambda p(x_{n-1},x_n) \leq \cdots \leq \sqrt[2]{\lambda}^{n+1} p(x_0,x_1) \quad (2.2)
\]
for any \( n \in \mathbb{N} \).

We will show that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((X,p)\). Now, let \( m, n \in \mathbb{N}_0 \) be such that \( m > n \). Then by (2.2), we have
\[
p(x_n,x_m) - p(x_n,x_n) \leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{m-1},x_m)
\]

\[
- [p(x_{n+1},x_{n+1}) + p(x_{n+2},x_{n+2}) + \cdots + p(x_{m-1},x_{m-1})]
\]

\[
\leq \sqrt[2]{\lambda}^{m-n} p(x_0,x_1) + \sqrt[2]{\lambda}^{m-1} p(x_0,x_1) + \cdots + \sqrt[2]{\lambda}^{m-1} p(x_0,x_1)
\]

\[
\leq \sum_{i=n}^{m-1} \sqrt[2]{\lambda}^i p(x_0,x_1) \to 0 \quad \text{as} \quad n \to \infty.
\]

Then, \( \{x_n\}_{n \in \mathbb{N}} \) satisfies (*) of Lemma 1.6. Hence \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((X,p)\). As \( x_n = F^n x_0 \) for all \( n \), and \((X,p)\) is \( F \)-orbitally complete, there is \( u \in X \) such that \( x_n \to u \) with respect to \( \tau_{d_p} \). Using the orbitally continuity of \( F \), we conclude that \( x_n \to Fu \) with respect to \( \tau_{d_p} \). Therefore \( u = Fu \) which deduces the proof. \( \square \)

The following corollary is a generalization to partial metric spaces of the theorem of Dhage [5].

**Corollary 2.2.** Let \( F \) be an orbitally continuous self map of a \( F \)-orbitally complete partial metric space \((X,p)\). If there is \( \alpha \geq 0 \) and \( \lambda \in [0,1) \) is such that
\[
\min\{p(x,y),p(Fx,Fy),p(x,Fx),p(y,Fy) - \alpha \min\{p(x,Fy),p(y,Fx)\}\}
\]

\[
\leq \lambda \max\{p(x,y),p(x,Fx)\} \quad (2.3)
\]
for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of $F$.

The following result is an immediate consequence of corollary 2.2.

**Remark.** If there is $\alpha \geq 0$ and $\delta + t \in [0, 1)$ is such that 

$$\min\{p(Fx, Fy), p(x, Fx), p(y, Fy) - \alpha \min\{p(x, Fy), p(y, Fx)\} \leq tp(x, y) + \delta p(x, Fx)$$

(2.4) for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of $F$. Actually, since $tp(x, y) + \delta p(x, Fx) \leq (t + \delta)\max\{p(x, y), p(x, Fx)\}$, the inequality (2.4) implies (2.3) with $\lambda = \delta + t$.

The following corollary is a generalization to partial metric spaces of the theorem of Pathak [18].

**Corollary 2.3.** Let $F$ be an orbitally continuous self map of a $F$-orbitally complete partial metric space $(X, p)$. If there is $h \in (0, 1)$ is such that 

$$\min\{p(x, Fx), p(Fx, Fy), p^2(x, y), p(x, Fx)p(y, Fy) \leq \frac{1}{2}h[p(x, y) + p(x, Fx)]\max\{p(x, y), p(x, Fx)\}$$

(2.5) for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of $F$.

The following result is an immediate consequence of Corollary 2.3.

**Remark.** If there is $h < 1$ is such that 

$$\min\{p(x, Fx)p(Fx, Fy), p^2(x, y), p(x, Fx)p(y, Fy) \leq \frac{1}{2}h[p(x, y) + p(x, Fx)]\max\{p(x, y), p(x, Fx)\}$$

(2.6) for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of $F$. To see that (2.6) implies (2.3) we have that 

$$\min\{p^2(Fx, Fy), p^2(x, y), p^2(x, Fx), p^2(y, Fy) \leq \min\{p(Fx, Fy)p(x, Fx), p^2(x, y), p(x, Fx)p(y, Fy)\},$$

$$\frac{1}{2}h[p(x, y) + p(x, Fx)]\max\{p(x, y), p(x, Fx)\}$$

$$\leq \frac{1}{2}h.2\max\{p(x, y), p(x, Fx)\} \times \max\{p(x, y), p(x, Fx)\}$$

$$= h \max\{p^2(x, y), p^2(x, Fx)\}$$

with $\lambda = \sqrt{h}$ and $\alpha = 0$.

**Theorem 2.4.** Let $F$ be an orbitally continuous self map of a $F$-orbitally complete partial metric space $(X, p)$. If there is $\alpha > 0$ and $\lambda \in (0, 1)$ is such that 

$$\min\{p(Fx, Fy), p(y, Fy)\} - \alpha \min\{d^p_m(x, Fx), d^p_m(y, Fx)\} \leq \lambda \max\{p(x, y), p(x, Fx), p(y, Fy)\}$$

(2.7) for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges with respect to $\tau_{d^p}$ to a fixed point of $F$. 

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Proof. Let \( x_0 \) be an arbitrary point of \( X \) and choose \( x_n \in X \) such that
\[
x_{n+1} = Fx_n.
\]
If for some \( n_0 \in \{1, 2, \ldots, n\} \) we have \( x_{n_0} = x_{n_0+1} \) then
\[
x_{n_0+1} = x_{n_0} = Fx_{n_0}
\]
that is \( x_{n_0} \) is a fixed point of \( F \) which completes the proof. We suppose that \( x_n \neq x_{n+1} \) for all \( n \in \mathbb{N}_0 \). From (2.7) for \( x = x_n \) and \( y = x_{n+1} \) we obtain
\[
\min\{p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})\} = \alpha \min\{d^p_m(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}
\]
where
\[
\alpha \min\{d^p_m(x_n, x_{n+2}), p(x_{n+1}, x_{n+2})\} = 0.
\]
Then we obtain
\[
p(x_{n+1}, x_{n+2}) \leq \lambda \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}.
\]
We will consider the following two cases:

Case 1 : If
\[
\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})
\]
for some \( n \in \mathbb{N}_0 \), then we obtain that
\[
p(x_{n+1}, x_{n+2}) \leq \lambda p(x_{n+1}, x_{n+2}) < p(x_{n+1}, x_{n+2})
\]
a contraction.
Case 2 : If
\[
\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})
\]
for some \( n \in \mathbb{N}_0 \), then we obtain
\[
p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}).
\]
Consequently, the following inequality is obtained from all these cases
\[
p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}).
\]
By induction we have
\[
p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}) \leq \lambda^2 p(x_{n-1}, x_n) \leq \cdots \leq \lambda^{n+1} p(x_0, x_1) \tag{2.8}
\]
for any \( n \in \mathbb{N} \). We will show that \( \{x_n\} \in \mathbb{N} \) is a Cauchy sequence in \((X, p)\). Now, let \( m, n \in \mathbb{N}_0 \) be such that \( m > n \). Then by (2.8) and \( p_4 \), we have
\[
p(x_n, x_m) - p(x_n, x_m) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m)
\]
\[
- [p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_{n+2}) + \cdots + p(x_{m-1}, x_{m-1})]
\]
\[
\leq \lambda^n p(x_0, x_1) + \lambda^{n+1} p(x_0, x_1) + \cdots + \lambda^{m-1} p(x_0, x_1)
\]
\[
\leq \sum_{i=n}^{m-1} \lambda^i p(x_0, x_1) \to 0 \text{ as } n \to \infty.
\]
Then, \( \{x_n\}_{n \in \mathbb{N}} \) satisfies (*) of Lemma 1.6. Hence \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((X, p)\). As \( x_n = F^n x_0 \) for all \( n \), and \((X, p)\) is \( F \)-orbitally complete, there is \( u \in X \) such that \( x_n \to u \) with respect to \( \tau_{d^p} \). Using the orbitally continuity of \( F \), we conclude that \( x_n \to Fu \) with respect to \( \tau_{d^p} \). Therefore \( u = Fu \) which deduces the proof.
Theorem 2.5. Let $F$ be an orbitally continuous self map of a $F$-orbitally complete partial metric space $(X, p)$. If there is $\alpha > 0$ and $\lambda \in (0, 1)$ is such that

$$\min\{p^2(Fx, Fy), p(Fx, Fy)p(x, y), p^2(y, Fy)\} - \alpha \min\{d_m^p(x, Fx)d_m^p(y, Fy), d_m^p(x, Fy)d_m^p(y, Fx)\} \leq \lambda p(x, Fx)p(y, Fy)$$

(2.9)

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges with respect to $\tau_{d_p}$ to a fixed point of $F$.

Proof. Let $x_0$ be an arbitrary point of $X$ and choose $x_n \in X$ such that

$$x_{n+1} = Fx_n.$$ 

If for some $n_0 \in \{1, 2, \ldots n\}$ we have $x_{n_0} = x_{n_0+1}$ then

$$x_{n_0+1} = x_{n_0} = Fx_{n_0}$$

that is $x_{n_0}$ is a fixed point of $F$ which completes the proof. We suppose that $x_{n} \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. From (2.9) for $x = x_n$ and $y = x_{n+1}$ we obtain

$$\min\{p^2(Fx_n, Fx_{n+1}), p(Fx_n, Fx_{n+1})p(x_n, x_{n+1}), p^2(x_{n+1}, Fx_{n+1})\}$$

$$- \alpha \min\{d_m^p(x_n, Fx_n)d_m^p(x_{n+1}, Fx_{n+1}), d_m^p(x_n, Fx_{n+1})d_m^p(x_{n+1}, Fx_n)\}$$

$$\leq \lambda p(x_n, Fx_n)p(x_{n+1}, Fx_{n+1})$$

which imply that

$$\min\{p^2(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})p(x_n, x_{n+1}), p^2(x_{n+2}, x_{n+2})\}$$

$$- \alpha \min\{d_m^p(x_n, x_{n+1})d_m^p(x_{n+1}, x_{n+2}), d_m^p(x_n, x_{n+2})d_m^p(x_{n+1}, x_{n+1})\}$$

$$\leq \lambda p(x_n, x_{n+1})p(x_{n+1}, x_{n+2})$$

where

$$\alpha \min\{d_m^p(x_n, x_{n+1})d_m^p(x_{n+1}, x_{n+2}), d_m^p(x_n, x_{n+2})d_m^p(x_{n+1}, x_{n+1})\} = 0.$$ 

Then we obtain

$$\min\{p^2(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})p(x_n, x_{n+1}), p^2(x_{n+1}, x_{n+2})\}$$

$$\leq \lambda p(x_n, x_{n+1})p(x_{n+1}, x_{n+2}).$$

We will consider the following two cases:

Case 1: If

$$\min\{p^2(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})p(x_n, x_{n+1})\} = p(x_{n+1}, x_{n+2})p(x_n, x_{n+1})$$

for some $n \in \mathbb{N}_0$, then we obtain that

$$p(x_{n+1}, x_{n+2})p(x_n, x_{n+1}) \leq \lambda p(x_n, x_{n+1})p(x_{n+1}, x_{n+2})$$

a contraction.

Case 2: If

$$\min\{p^2(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})p(x_n, x_{n+1})\} = p^2(x_{n+1}, x_{n+2})$$

for some $n \in \mathbb{N}_0$, then we obtain

$$p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}).$$

Consequently, the following inequality is obtained from all these cases

$$p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}).$$
By induction we have
\[ p(x_{n+1}, x_{n+2}) \leq \lambda p(x_n, x_{n+1}) \leq \lambda^2 p(x_{n-1}, x_n) \leq \cdots \leq \lambda^{n+1} p(x_0, x_1) \] (2.10)
for any \( n \in \mathbb{N} \). We will show that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((X, p)\). Now, let \( m, n \in \mathbb{N}_{n_0} \) be such that \( m > n \). Then by (2.10) and \( p_4 \), we have
\[
p(x_n, x_m) - p(x_n, x_n) \leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - [p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_{n+2}) + \cdots + p(x_{m-1}, x_{m-1})]
\leq \lambda^n p(x_0, x_1) + \lambda^{n+1} p(x_0, x_1) + \cdots + \lambda^{m-1} p(x_0, x_1)
\leq \sum_{i=n}^{m-1} \lambda^i p(x_0, x_1) \to 0 \text{ as } n \to \infty.
\]
Then, \( \{x_n\}_{n \in \mathbb{N}} \) satisfies (*) of Lemma 1.6. Hence \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \((X, p)\). As \( x_n = F^n x_0 \) for all \( n \), and \((X, p)\) is \( F\)-orbitally complete, there is \( u \in X \) such that \( x_n \to u \) with respect to \( \tau_{d_p} \). Using the orbitally continuity of \( F \), we conclude that \( x_n \to Fu \) with respect to \( \tau_{d_p} \). Therefore \( u = Fu \) which deduces the proof. 

The following corollary is a generalization to partial metric spaces of the theorem of Pachpatte [17].

**Corollary 2.6.** Let \( F \) be an orbitally continuous self map of a \( F\)-orbitally complete partial metric space \((X, p)\). If there is \( \alpha > 0 \) and \( \lambda \in (0, 1) \) is such that
\[
\min\{p^2(Fx, Fy), p(Fx, Fy)p(x, y), p^2(y, Fy)\} - \alpha \min\{p(x, Fx)p(y, Fy), p(x, Fy)p(y, Fx)\} \\
\leq \lambda(p(x, Fx)p(y, Fy)) \] (2.11)
for all \( x, y \in X \), then for each \( x_0 \in X \) the sequence \( \{F^n x_0\}_{n \in \mathbb{N}} \) converges to a fixed point of \( F \).

The following is example where main theorems can be applied but not corollaries for partial metric spaces.

**Example 2.7.** Let \( X = [0, \infty) \) and define \( p : X \times X \to \mathbb{R}^+ \) by \((X, p)\) be a partial metric space where \( p(x, y) = \max\{x, y\} \) for all \( x, y \in X \). Define \( F : X \to X \) by
\[ Fx = \begin{cases} 0, & x < 1 \\ \frac{x}{x+1}, & x \geq 2. \end{cases} \]
Since \((X, p)\) is a complete partial metric space, then it is \( F\)-orbitally complete. Furthermore, \( F \) is continuous with respect to \( \tau_{d_p} \), so it is orbitally continuous.

Now we show that \( F \) satisfies the contraction condition
\[
\min\{p(x, y)p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, \frac{p(y, Fy)[1 + p(x, Fx)]}{1 + p(x, y)}, \min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}\} - \alpha \min\{d^p_m(x, Fx), d^p_m(y, Fx)\} \leq \lambda \max\{p(x, y), p(x, Fx)\} \] (2.12)
for any \( \alpha > 0 \) and \( \lambda = \frac{1}{2} \). Hence the conditions of Theorems 2.1 are satisfied. We obtain some cases for \( x, y \in X \):
Let’s first examine some cases for $x < 1$

**Case 1 :** $x = y$. Then

$$
\min\{p(x, y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, \frac{p(y, Fy)[1 + p(x, Fx)]}{1 + p(x, y)}, \min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}\}
- \alpha \min\{d_m^m(x, Fy), d_m^m(y, Fx)\}
= \min\{0, x, x, \frac{x(1 + x)}{1 + x}, \frac{x(1 + x)}{1 + x}, \frac{\min\{0, x^2\}}{x}\} - \alpha x \leq \lambda x.
$$

**Case 2 :** $x \neq y$. Thus, without loss of generality, we assume that $x > y$. If $Fx = y$, we obtain

$$
\min\{p(x, y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, \frac{p(y, Fy)[1 + p(x, Fx)]}{1 + p(x, y)}, \min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}\}
- \alpha \min\{d_m^m(x, Fy), d_m^m(y, Fx)\}
= \min\{0, x, x, \frac{x(1 + y)}{1 + x}, \frac{y(1 + x)}{1 + x}, \frac{\min\{0, x^2, y^2\}}{x}\} - \alpha \min\{x - 0, 0\} \leq \lambda x.
$$

If $Fx < y$, we obtain

$$
\min\{p(x, y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, \frac{p(y, Fy)[1 + p(x, Fx)]}{1 + p(x, y)}, \min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}\}
- \alpha \min\{d_m^m(x, Fy), d_m^m(y, Fx)\}
= \min\{0, x, x, \frac{x(1 + y)}{1 + x}, \frac{y(1 + x)}{1 + x}, \frac{\min\{0, x^2, y^2\}}{x}\} - \alpha \min\{x - 0, y - 0\} \leq \lambda x.
$$

Let’s examine some cases for $x \geq 2$.

**Case 1 :** $x = y$. Then

$$
\min\{p(x, y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, \frac{p(y, Fy)[1 + p(x, Fx)]}{1 + p(x, y)}, \min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}\}
- \alpha \min\{d_m^m(x, Fy), d_m^m(y, Fx)\}
= \min\{\frac{x - 1}{2}, x, x, \frac{x(1 + x)}{1 + x}, \frac{x(1 + x)}{1 + x}, \frac{\min\{(\frac{x-1}{2})^2, x^2, x^2\}}{x}\} - \alpha \min\{x - \frac{x - 1}{2}, x - \frac{x - 1}{2}\} \leq \lambda x.
$$
Case 2: $x \neq y$. Thus, with out loss of generality, we assume that $x > y$. If $Fx \geq y$, we obtain
\[
\min\{p(x,y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, p(y, Fy)[1 + p(x, Fx)]\frac{\min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}}{p(x, y)} - \alpha \min\{d_p^m(x, Fy), d_p^m(y, Fx)\}
\]
\[
= \min\left\{ \frac{x - 1}{2}, x, y, \frac{x(1 + y)}{1 + x}, \frac{y(1 + x)}{1 + x}, \frac{\min\left\{ \frac{(x - 1)^2}{2}, x^2, y^2 \right\}}{x} \right\} - \alpha \min\{x - y - 1, x - y - 1\} \leq \lambda x.
\]
If $Fx < y$, we obtain
\[
\min\{p(x,y), p(Fx, Fy), p(x, Fx), p(y, Fy), \frac{p(x, Fx)[1 + p(y, Fy)]}{1 + p(x, y)}, p(y, Fy)[1 + p(x, Fx)]\frac{\min\{p^2(Fx, Fy), p^2(x, Fx), p^2(y, Fy)\}}{p(x, y)} - \alpha \min\{d_p^m(x, Fy), d_p^m(y, Fx)\}
\]
\[
= \min\left\{ \frac{x - 1}{2}, x, y, \frac{x(1 + y)}{1 + x}, \frac{y(1 + x)}{1 + x}, \frac{\min\left\{ \frac{(x - 1)^2}{2}, x^2, y^2 \right\}}{x} \right\} - \alpha \min\{x - y - 1, y - x - 1\} \leq \lambda x.
\]
Thus, $F$ satisfies the conditions of Theorem 2.1 with $\lambda = \frac{1}{2}$. Notice that $u = 0$ is the fixed point of $F$. Also example satisfies the conditions of Theorem 2.4 and Theorem 2.5 with $\lambda = \frac{1}{2}$ by making similar process.

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References


Seher Sultan Sepet
Institute of Science and Technology Kahramanmaraş Sütçü İmam University
Kahramanmaraş, 46040, Turkey;
E-mail address: sultanseher20@gmail.com

Cafer Aydın
Department of Mathematics Kahramanmaraş Sütçü İmam University
Kahramanmaraş, 46040, Turkey
E-mail address: caydin61@gmail.com