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SIMULTANEOUS STATISTICAL APPROXIMATION OF ANALYTIC FUNCTIONS IN ANNULUS BY $k-$POSITIVE LINEAR OPERATORS

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Abstract. In this work, we investigate the results concerning on simultaneous statistical approximation of analytic functions and their derivatives in the annulus of complex plane with the topology of compact convergence.

1. Introduction and Background

Let $\Re = \{z \in \mathbf{C} : r < |z - z_0| < R\}$ be the annulus of centered at z_0 and $A(\Re)$ denote the space of all analytic functions in $\Re$ with the topology of compact convergence. This means that, by a convergence in this space the uniform convergence in any compact of $\Re$. For $r < r' < R' < R$, we will consider the seminorms $\|f\|_{A(\Re),r',R'} := \|f\|_{A(\Re)} := \max_{r'<|z-z_0|<R'} |f(z)|$ that convert $A(\Re)$ into a Frechet-type space.

It is easy to see that any linear operator acting from $A(\Re)$ to $A(\Re)$ can be represented in the form

$$T(f;z) = \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} f_m T_{k,m} \right) (z - z_0)^k \tag{1.1}$$

where f_m is the Laurent coefficient of f and for each $T_{k,m}$ is Laurent coefficient of $T\left((z - z_0)^k\right)$

We will study the sequence of linear operators

$$T_n(f;z) = \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)} \right) (z - z_0)^k, \tag{1.2}$$

where $\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}$ is the Laurent coefficient of $T_n(f)$ and $n \in \mathbf{N}$, acting on functions $f \in A(\Re)$ and having the properties of "$k-$positivity" in the sense of the paper [7]. It was also first introduced by Gadjiev. Recall that by the definition (see [7]) a linear operator T, acting from $A(\Re)$ to $A(\Re)$, is called $k-$positive if it preserves the class of functions with non-negative Laurent coefficients. Note

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that the different aproximation and statistical approximation properties of linear $k-$positive operators were studied by Ahadov [1], Duman [5], Doğru [4] Gadjiev [8], Gadjiev and Ghorbanalized [10, 11], Gadjiev and Orhan [12], Gadjiev and Aliev [14], Aliev [2], Özarslan [3], İspir [15] and İspir and Atakut [16]. We require the following definition of statistically convergence which was first introduced by Fast [6] for a sequence of real numbers:

Definition 1.1. [6] *A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0, \tag{1.3}$$

where vertical bars indicate the number of elements in the enclosed set. In this case we write $st - \lim x = L$.

Note that in the paper [12] it was the first proved the theorems of the Krovkin type on the statistical approximation by positive linear operators. In our investigation, first result related on simultaneous statistical approximation of analytic functions by $k-$positive linear operators was obtained by Gadjiev [9].

In this study, we used annulus instead of unit disc which was studied by Gadjiev [9] and similar results were obtained.

2. Main Results

In this section, by using statistical convergence we use some approximation theorems by means of $k-$positive linear operators defined on the space of all analytic functions on the annulus. Let $f_n(z)$ be a sequence of analytic functions in $\Re$. Then we can write

$$f_n(z) = \sum_{k=-\infty}^{\infty} f_{n,k} (z - z_0)^k, \tag{2.1}$$

where the coefficients $f_{n,k},\ k \in \mathbf{Z}$ are defined by the formula

$$f_{n,k} = \frac{1}{2\pi i} \int_\Gamma f_n(z) (z - z_0)^{-k-1} dz \tag{2.2}$$

and Γ is any circle centered at the point z_0 with a radius greater than r and less than R.

The following theorem is a statistical analogue of the lemma on uniform convergence of sequence of analytic function in $A(\Re)$, proved in [13].

Theorem 2.1 ([13],Theorem 7)**.** *The sequence $f_n(z)$ statistically tends to zero in $A(\Re)$ necessary and sufficient the coefficient of expansion $f_n(z) = \sum\limits_{k=-\infty}^{\infty} f_{n,k}(z - z_0)^k$ satisfy the conditions*

$$|f_{n,k}| \leq \varepsilon_n \begin{cases} (1+\delta_n)^k R^{-k} & ; \quad k \geq 0 \\ (1+\delta_n)^{-k} r^{-k} & ; \quad k < 0 \end{cases} \tag{2.3}$$

for any $n \in \mathbf{N}$ where δ_n tends to zero and ε_n-statistically tends to zero as $n \to \infty$.

Corollary 2.2. *$f_n^{(p)}(z)$ statistically tends to zero in $A(\Re)$ for any $p = 0, 1, 2, ...$; if and only if the coefficients of $f_n^{(p)}$ for all $k \geq 0$,*

$$|f_{n,k+p}| \leq \frac{k!}{(k+p)!} \varepsilon_n (1+\delta_n)^{k+p} R^{-(k+p)} \tag{2.4}$$

and for all $k<0$,

$$|f_{n,k}| \leq \frac{(-k-1)!}{(-k+p-1)!}\varepsilon_n (1+\delta_n)^{-k} r^{-k}. \tag{2.5}$$

Proof. Firstly suppose that $f_n^{(p)}(z)$ statistically tends to zero. From the Laurent expansion of $f_n(z)$

$$f_n(z) = \sum_{k=-\infty}^{\infty} f_{n,k}(z-z_0)^k = \sum_{k=0}^{\infty} f_{n,k}(z-z_0)^k + \sum_{k=-\infty}^{-1} f_{n,k}(z-z_0)^k .$$

If the series on the right side of the above equation are defined as $\varphi_n(z)$ and $\psi_n(z)$ respectively, we obtain

$$f_n^{(p)}(z) = \varphi_n^{(p)}(z) + \psi_n^{(p)}(z) .$$

We can write the following obvious representation:

$$\varphi_n^{(p)}(z) = \sum_{k=0}^{\infty} (k+1)(k+2)...(k+p) f_{n,k+p}(z-z_0)^k ,$$
$$\psi_n^{(p)}(z) = \sum_{k=-\infty}^{-1} k(k-1)(k-2)...(k-p+1) f_{n,k}(z-z_0)^{k-p} ,$$

then

$$\begin{aligned} f_n^{(p)}(z) = &\sum_{k=0}^{\infty} (k+1)(k+2)...(k+p) f_{n,k+p}(z-z_0)^k + \\ &+ \sum_{k=-\infty}^{-1} k(k-1)...(k-p+1) f_{n,k}(z-z_0)^{k-p} . \end{aligned} \tag{2.6}$$

By Theorem 2.1, $st-lim_{n\to\infty}\left\|f_n^{(p)}\right\|_{A(\Re),r',R'} = 0$ if and only if

$$(k+1)(k+2)...(k+p)|f_{n,k+p}| \leq \varepsilon_n (1+\delta_n)^{k+p} R^{-(k+p)}, \quad (k\geq 0)$$
$$|k(k-1)...(k-p+1)|\,|f_{n,k}| \leq \varepsilon_n (1+\delta_n)^{-k} r^{-k}, \quad (k<0)$$

Therefore we obtain (2.4) and (2.5). Conversely, suppose that if (2.4) and (2.5) hold, then we have

$$\begin{aligned} \left\|f_n^{(p)}(z)\right\|_{A(\Re)} &\leq \sum_{k=0}^{\infty} \frac{(k+p)!}{k!}|f_{n,k+p}|(R')^k + \sum_{k=-\infty}^{-1} \frac{(-k+p-1)!}{(-k-1)!}|f_{n,k}|(r')^{k-p} \\ &\leqslant \sum_{k=0}^{\infty} \varepsilon_n (1+\delta_n)^{k+p} R^{-(k+p)}(R')^k + \sum_{k=-\infty}^{-1} \varepsilon_n (1+\delta_n)^{-k} r^{-k}(r')^k \\ &= \frac{\varepsilon_n (1+\delta_n)^p}{(R')^p}\sum_{k=0}^{\infty}(1+\delta_n)^k\left(\frac{R'}{R}\right)^k + \frac{\varepsilon_n}{(r')^p}\sum_{k=-\infty}^{-1}(1+\delta_n)^{-k}\left(\frac{r}{r'}\right)^{-k}, \end{aligned}$$

where $r<r'<R'<R$ and the desired conclusion follows by letting $n\to\infty$. □

Theorem 2.3. *For any* $p = 1, 2, 3, ...$

$$st - lim_{n\to\infty} \left\| f_n^{(p)} \right\|_{A(\Re),r',R'} = 0 \tag{2.7}$$

it necessary and sufficient to satisfy the condition (2.7) for $p = 0$.

Proof. Let the condition (2.7) holds for $p = 0$. Then by Theorem 2.1 there exist sequences ε_n and δ_n such that $\lim_{n\to\infty} \delta_n = 0,\ st-\lim_{n\to\infty} \varepsilon_n = 0$ and the inequality (2.3) holds. From the equality (2.6), we have for any natural p

$$\left\| f_n^{(p)}(z) \right\|_{A(\Re)} \leq \frac{\varepsilon_n (1+\delta_n)^p}{R^p} \sum_{k=0}^{\infty} (k+1)(k+2)...(k+p)(1+\delta_n)^k \left(\frac{R'}{R}\right)^k +$$
$$+ \frac{\varepsilon_n}{(r')^p} \sum_{k=-\infty}^{-1} (-k)(-k+1)...(-k+p-1)(1+\delta_n)^{-k} \left(\frac{r}{r'}\right)^{-k}.$$

Since

$$\tfrac{d^p}{dz^p}\left(\tfrac{1}{1-(z-z_0)}\right) = \sum_{k=0}^{\infty} (k+1)(k+2)...(k+p)(z-z_0)^k,$$

we can write

$$\frac{p!}{(1-(z-z_0))^{p+1}} = \sum_{k=0}^{\infty} (k+1)(k+2)...(k+p)(z-z_0)^k$$

,

where $|z - z_0| < 1$. Therefore, we get

$$\left\| f_n^{(p)}(z) \right\|_{A(\Re)} \leq \varepsilon_n \frac{p!(1+\delta_n)^p}{R^p\left(1-(1+\delta_n)\frac{R'}{R}\right)^{p+1}} + \varepsilon_n \frac{(1+\delta_n) rp!}{(r')^{p+1}\left(1-(1+\delta_n)\frac{r}{r'}\right)^{p+1}}$$

Since

$$st - lim_{n\to\infty}\alpha_n = \tfrac{p!}{R^p\left(1-\frac{R'}{R}\right)^{p+1}} \quad \text{and} \quad st - lim_{n\to\infty}\beta_n = \tfrac{rp!}{(r')^{p+1}\left(1-\frac{r}{r'}\right)^{p+1}},$$

where $\alpha_n := \frac{p!(1+\delta_n)^p}{R^p\left(1-(1+\delta_n)\frac{R'}{R}\right)^{p+1}}$ and $\beta_n := \frac{(1+\delta_n)rp!}{(r')^{p+1}\left(1-(1+\delta_n)\frac{r}{r'}\right)^{p+1}}$, there exists a constant $K := K(p, r, r', R', R)$ such that for any $n = 1, 2, 3, ...$

$$\alpha_n \leq K(p, r, r', R', R) \quad \text{and} \quad \beta_n \leq K(p, r, r', R', R).$$

So we have the inequality

$$\left\| f_n^{(p)}(z) \right\|_{A(\Re),r',R'} \leq \varepsilon_n\ K(p, r, r', R', R)$$

which implies

$$\left| \left\{ k \leq n : \left\| f_n^{(p)}(z) \right\|_{A(\Re),r',R'} > \varepsilon \right\} \right| \leq \left| \left\{ k \leq n : \varepsilon_n > \tfrac{\varepsilon}{K(p,r,r',R',R)} \right\} \right|.$$

The last inequality gives the proof as in the proof of Theorem 7 [13] □

Consider now a sequence of linear $k-$positive operator T_n given by the formula (1.2). Obviously, for any even natural p :

$$\begin{aligned}\frac{d^p}{dz^p}T_n\left(f\ ;z\right) &= \left(\sum_{k=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}\right)\left(z-z_0\right)^k\right)^{(p)} \\ &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}\right)\left(z-z_0\right)^k\right)^{(p)} + \\ &+ \left(\sum_{k=-\infty}^{-1}\left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}\right)\left(z-z_0\right)^k\right)^{(p)} \\ &= \sum_{k=0}^{\infty}\left(k+1\right)\left(k+2\right)...\left(k+p\right)\left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}\right)\left(z-z_0\right)^k + \\ &+ \sum_{k=-\infty}^{-1} k\left(k-1\right)...\left(k-p+1\right)\left(\sum_{m=-\infty}^{\infty} f_m T_{k,m}^{(n)}\right)\left(z-z_0\right)^{k-p}.\end{aligned}$$

Since T_n is a sequence of $k-$positive operators then, by the definition, $T_{k+p,m}^{(n)} \geq 0$, $T_{k,m}^{(n)} \geq 0$ for any $k,\ m$ and even p. For positive coefficients f_m we have

$$\begin{gathered}\left(k+1\right)\left(k+2\right)...\left(k+p\right)\textstyle\sum_{k=-\infty}^{\infty} f_m T_{k,m}^{(n)} \geq 0, \\ k\left(k-1\right)...\left(k-p+1\right)\textstyle\sum_{k=-\infty}^{\infty} f_m T_{k,m}^{(n)} \geq 0.\end{gathered}$$

T_n is $k-$positive operator if for any $f \in A\left(\Re\right)$, having non-negative Laurent coefficients, $T_n\left(f\ ;z\right) \in A\left(\Re\right)$ and also has a non-negative Laurent coefficients. Therefore we have the following Proposition 2.4.

Proposition 2.4. *If $T_n\left(f\ ;z\right)$ is a $k-$positive operators then for any even natural p, $\frac{d^p}{dz^p}\ T_n\left(f\ ;z\right)$ is also $k-$positive operator.*

Theorem 2.3 allows us to formulate any theorem on statistical convergence $T_n\left(f\ ;z\right)$ to $f\left(z\right)$ as $n \to \infty$ in $A\left(\Re\right)$ as a theorem on simultaneous convergence of $\frac{d^p}{dz^p}T_n\left(f\ ;z\right)$ to $f^{(p)}\left(z\right)$, p is a any even natural number.

For example, using the general result, proven in [11] (see, p.392, Theorem 2.1) we can formulate the following result.

Let the sequence $g = \{g_k\}_{k=-\infty}^{\infty}$ of positive numbers satisfy the conditions: $\forall k \in \mathbf{Z}$,

$$\triangle_k\left(g\right) = \inf_{p\in\mathbf{Z},p\neq k}\left|\sqrt{g_k}-\sqrt{g_p}\right| > 0, \lim_{k\to\mp\infty}\left(\triangle_k\left(g\right)\right)^{\frac{1}{k}} = 1, \lim_{k\to\mp\infty}\left(g_k\right)^{\frac{1}{k}} = 1. \quad (2.8)$$

Definition 2.5. [13] *By $A_g\left(\Re\right)$ we denote the set of analytic functions*

$$f\left(z\right) = \sum_{k=-\infty}^{\infty} f_k\left(z-z_0\right)^k \in A\left(\Re\right),$$

whose coefficients satisfy the following conditions:

$$\left|f_k\right| \leq \begin{cases} M\ g_k\ R^{-k} & ;if\ k \geq 0 \\ M\ g_k\ r^{-k} & ;if\ \ k < 0 \end{cases} \quad (2.9)$$

where M is a constant independent of k.

The following theorem shows that the system of three functions

$$g_{\vartheta}(z)=\sum_{k=-\infty}^{\infty} g_k^{\frac{\vartheta}{2}}\alpha_k (z-z_0)^k, \quad \vartheta=0,1,2, \tag{2.10}$$

where $\alpha_k := \begin{cases} R^{-k} & ; k\geq 0 \\ r^{-k} & ; k<0 \end{cases}$, is a Krovkin system in the space $A_g(\Re)$.

Theorem 2.6. *Let $T_n : A(\Re) \longrightarrow A(\Re)$ be a sequence of linear $k-$positive operators. Then for all even natural number p and each function $f\in A_g(\Re)$*

$$st-\lim_{n\to\infty}\left\|\tfrac{d^p}{dz^p}T_n f(z)-f^{(p)}(z)\right\|_{A(\Re),r',R'}=0$$

if and only if

$$st-\lim_{n\to\infty}\|T_n\ g_{\vartheta}(z)-g_{\vartheta}(z)\|_{A(\Re),r',R'}=0,\ \vartheta=0,1,2, \tag{2.11}$$

where $g_{\vartheta}(z)$ is defined as in (2.8), (2.10).

Proof. By the definition of the norm in $A(\Re)$, it follows that for any function $f\in A_g(\Re)$, the inequality

$$\begin{aligned}\left\|\frac{d^p}{dz^p}T_n\ f(z)-f^{(p)}(z)\right\| &\leq \sum_{k=0}^{\infty}\frac{(k+p)!}{k!}\left(\frac{R'}{R}\right)^k\sum_{m=-\infty}^{\infty}|f_m-f_{k+p}|\,T_{k+p,m}^{(n)}\\ &+\sum_{k=-\infty}^{\infty}\frac{(k+p)!}{k!}\left(\frac{R'}{R}\right)^k|f_{k+p}|\left|\sum_{m=-\infty}^{\infty}T_{k,m}^{(n)}-1\right|\\ &+\sum_{k=-\infty}^{\infty}\frac{(-k+p-1)!}{(-k-1)!}\left(\frac{r}{r'}\right)^{-k}\sum_{m=-\infty}^{\infty}|f_m-f_k|\,T_{k,m}^{(n)}\\ &+\sum_{k=-\infty}^{\infty}\frac{(-k+p-1)!}{(-k-1)!}\left(\frac{r}{r'}\right)^{-k}|f_k|\left|\sum_{m=-\infty}^{\infty}T_{k,m}^{(n)}-1\right|\\ &=I_n^{(1)}+I_n^{(2)}+I_n^{(3)}+I_n^{(4)}\end{aligned} \tag{2.12}$$

holds. To estimate the terms $I_n^{(1)}$ and $I_n^{(2)}$ of (2.12), it is easy to verify that for all m,k and p.

$$|f_m-f_{k+p}|\leq 8M\frac{g_{k+p}^3}{\Delta_{k+p}^2(g)}\left(\sqrt{g_m}-\sqrt{g_{k+p}}\right)^2 \tag{2.13}$$

$$|f_m-f_k|\leq 8M\frac{g_k^3}{\Delta_k^2(g)}\left(\sqrt{g_m}-\sqrt{g_k}\right)^2 \tag{2.14}$$

where $\Delta_k(g) = \min\{\sqrt{g_k - g_{k-1}}, \sqrt{g_{k+1} - g_k}\}$. Combining (2.12 - 2.14), we have

$$\begin{aligned}
I_n^{(1)} + I_n^{(2)} &= \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \left(\frac{R'}{R}\right)^k 8M \frac{g_{k+p}^3}{\Delta_{k+p}^2(g)} \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} \left(\sqrt{g_m} - \sqrt{g_{k+p}}\right)^2 \\
&+ \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \left(\frac{R'}{R}\right)^k |f_{k+p}| \left| \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} - 1 \right|. \\
&\leqslant 8M \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \left(\frac{R'}{R}\right)^k \left\{ \frac{g_{k+p}^3}{\Delta_{k+p}^2(g)} \left| \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} g_m - g_{k+p} \right| \right. \\
&+ 2 \frac{g_{k+p}^{\frac{7}{2}}}{\Delta_{k+p}^2(g)} \left| \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} \sqrt{g_m} - \sqrt{g_{k+p}} \right| \\
&\left. + \left(\frac{g_{k+p}^4}{\Delta_{k+p}^2(g)} + \frac{1}{8} g_{k+p} \right) \left| \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} - 1 \right| \right\}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
I_n^{(3)} + I_n^{(4)} &= \sum_{k=-\infty}^{-1} \frac{(-k+p-1)!}{(-k-1)!} \left(\frac{r}{r'}\right)^{-k} 8M \frac{g_k^3}{\Delta_k^2(g)} \sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} \left(\sqrt{g_m} - \sqrt{g_k}\right)^2 + \\
&+ \sum_{k=0}^{\infty} \frac{(-k+p-1)!}{(-k-1)!} \left(\frac{r}{r'}\right)^{-k} |f_k| \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right|. \\
&\leq 8M \sum_{k=0}^{\infty} \frac{(-k+p-1)!}{(-k-1)!} \left(\frac{r}{r'}\right)^{-k} \left\{ \frac{g_k^3}{\Delta_k^2(g)} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} g_m - g_k \right| \right. \\
&+ 2 \frac{g_k^{\frac{7}{2}}}{\Delta_k^2(g)} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} \sqrt{g_m} - \sqrt{g_k} \right| \\
&\left. + \left(\frac{g_k^4}{\Delta_k^2(g)} + \frac{1}{8} g_k \right) \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \right\}.
\end{aligned}$$

Setting $C := C(k,p) = 8M \max\left\{ \frac{g_{k+p}^3}{\Delta_{k+p}^2(g)}, \frac{2g_{k+p}^{\frac{7}{2}}}{\Delta_{k+p}^2(g)}, \left(\frac{g_{k+p}^4}{\Delta_{k+p}^2(g)} + \frac{1}{8} g_{k+p} \right), \frac{g_k^3}{\Delta_k^2(g)}, \frac{2g_k^{\frac{7}{2}}}{\Delta_k^2(g)}, \right.$
$\left. , \left(\frac{g_k^4}{\Delta_k^2(g)} + \frac{1}{8} g_k \right) \right\}$.

The above inequalities yield

$$\left\|\left(T_n\ f(z)\right)^{(p)} - f^{(p)}(z)\right\| \leq \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \left(\frac{R'}{R}\right)^k C \left[\left|\sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} g_m - g_{k+p}\right| + \left|\sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} \sqrt{g_m} - \sqrt{g_{k+p}}\right| + \left|\sum_{m=-\infty}^{\infty} T_{k+p,m}^{(n)} - 1\right|\right]$$
$$+ \sum_{k=-\infty}^{-1} \frac{(-k+p-1)!}{(-k-1)!} \left(\frac{r}{r'}\right)^{-k} C \left[\left|\sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} g_m - g_k\right| + \left|\sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} \sqrt{g_m} - \sqrt{g_k}\right| + \left|\sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1\right|\right].$$

By applying condition (2.11), there are six sequences $\{t_{n,k,k+p,\vartheta}\}$, $\vartheta = 0,1,2$ of positive numbers such that

$$st - \lim_{n\to\infty} t_{n,k,k+p,\vartheta} = 0 \tag{2.15}$$

for any fixed $k \in \mathbf{Z},\ p \in \mathbf{N}$ and

$$\left|\sum_{m=-\infty}^{\infty} T_{k+p,\ m}^{(n)} g_{\ m}^{\frac{\vartheta}{2}} - g_{k\ +p}^{\frac{\vartheta}{2}}\right| \leq t_{n,\ k\ +\ p,\ \vartheta},$$

$$\left|\sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} g_m^{\frac{\vartheta}{2}} - g_k^{\frac{\vartheta}{2}}\right| \leq t_{n,k,\vartheta},$$

where $\vartheta = 0,1,2$. We have

$$\left\|\frac{d^p}{dz^p} T_n\ f(z) - f^{(p)}(z)\right\| \leq \sum_{k=0}^{\infty} \frac{(k+p)!}{k!} \left(\frac{R'}{R}\right)^k C \sum_{\vartheta=0}^{2} t_{n,\ k+p,\ \vartheta}$$
$$+ \sum_{k=-\infty}^{-1} \frac{(-k+p-1)!}{(-k-1)!} \left(\frac{r}{r'}\right)^{-k} C \sum_{\vartheta=0}^{2} t_{n,\ k,\ \vartheta}.$$

We now show that the sequence

$$\xi_{n,\ k+p} := \left(\sum_{\vartheta=0}^{2} t_{n,\ k+p,\ \vartheta}\right),\ \eta_{n,\ k} := \left(\sum_{\vartheta=0}^{2} t_{n,\ k,\ \vartheta}\right)$$

satisfy

$$st - \lim_{n\to\infty} (\xi_{n,\ k+p}) = 0 \ and\ st - \lim_{n\to\infty} (\eta_{n,\ k}) = 0.$$

For a given ε', consider the sets

$$U = \left\{n : \textstyle\sum_{\vartheta=0}^{2} t_{n,\ k+p,\ \vartheta} \geq \frac{\varepsilon'}{C}\right\}, \quad U_0 = \left\{n : t_{n,k+p,0} \geq \frac{\varepsilon'}{6C}\right\}$$

$$U_1 = \left\{n : t_{n,k+p,1} \geq \frac{\varepsilon'}{6C}\right\}, \qquad U_2 = \left\{n : t_{n,k+p,2} \geq \frac{\varepsilon'}{6C}\right\}$$

$$V = \left\{n : \textstyle\sum_{\vartheta=0}^{2} t_{n,\ k,\ \vartheta} \geq \frac{\varepsilon'}{C}\right\}, \quad V_0 = \left\{n : t_{n,k,\ 0} \geq \frac{\varepsilon'}{6C}\right\}$$

$$V_1 = \left\{n : t_{n,k,\ 1} \geq \frac{\varepsilon'}{6C}\right\}, \quad V_2 = \left\{n : t_{n,k,\ 2} \geq \frac{\varepsilon'}{6C}\right\}.$$

Obviously, we have $U \subseteq U_0 \cup U_1 \cup U_2$ and $V \subseteq V_0 \cup V_1 \cup V_2$. Hence by virtue of (2.15), we obtain $st - \lim_{n\to\infty} (\xi_{n,\ k+p}) = 0$ and $st - \lim_{n\to\infty} (\eta_{n,\ k}) = 0$. By Lemma 2.1 in [11], the proof is completed. □

Theorem 2.7. *Let T_n be a sequence of linear $k-$positive operators from $A(\Re)$ into itself and $g(z) = \sum\limits_{k=-\infty}^{\infty} g_k (z-z_0)^k$ be an analytic function in $\Re$. If for any $r < r' < R' < R$ and $p = 0, 1, 2,$*

$$st - \lim_{n\to\infty} \left\| T_n (z-z_0)^p g^{(p)}(z) - (z-z_0)^p g^{(p)}(z) \right\|_{A(\Re),\ r',\ R'} = 0, \tag{2.16}$$

then for any function $f \in A_g(\Re)$ and even number p

$$st - \lim_{n\to\infty} \left\| \frac{d^p}{dz^p} T_n f(z) - f^{(p)}(z) \right\|_{A(\Re),\ r',\ R'} = 0. \tag{2.17}$$

Proof. Obviously, we have

$$g(z) = \sum_{k=-\infty}^{\infty} g_k (z-z_0)^k = \sum_{k=0}^{\infty} g_k (z-z_0)^k + \sum_{k=-\infty}^{-1} g_k (z-z_0)^k,$$

$$g'(z) = \sum_{k=0}^{\infty} k g_k (z-z_0)^{k-1} + \sum_{k=-\infty}^{-1} k g_k (z-z_0)^{k-1}.$$

Multiplying both side with $(z-z_0)$, we obtain

$$(z-z_0) g'(z) = \sum_{k=-\infty}^{\infty} k g_k (z-z_0)^k.$$

In the same way, we get

$$(z-z_0)^2 g''(z) = \sum_{k=-\infty}^{\infty} k(k-1) g_k (z-z_0)^k$$

and therefore using (1.1), the condition (2.16) give as $n \to \infty$

$$\left\| \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)}\ g_m\ (z-z_0)^k - \sum_{k=-\infty}^{\infty} g_k\ (z-z_0)^k \right\|_{A(\Re)} \overrightarrow{st}\ 0,$$

$$\left\| \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} m\ g_m\ (z-z_0)^k - \sum_{k=-\infty}^{\infty} k\ g_k\ (z-z_0)^k \right\|_{A(\Re)} \overrightarrow{st}\ 0,$$

$$\left\| \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} m(m-1)\ g_m\ (z-z_0)^k - \sum_{k=-\infty}^{\infty} k(k-1) g_k\ (z-z_0)^k \right\| \overrightarrow{st}\ 0,$$

where $\overrightarrow{st}$ denote that the statistical limit. By Theorem 2.1 this means that there exist sequences δ_n and ε_n such that $st - \lim_{n\to\infty} \delta_n = 0$ and $st - \lim_{n\to\infty} \varepsilon_n = 0$ and the following inequalities hold

$$\left| \sum_{k=-\infty}^{\infty} T_{k,m}^{(n)} g_m\ - g_k \right| < \varepsilon_n\ \alpha_{n,k}\ ,$$

$$\left| \sum_{k=-\infty}^{\infty} T_{k,m}^{(n)} m g_m \ - k g_k \right| < \varepsilon_n \ \alpha_{n,k}$$

and

$$\left| \sum_{k=-\infty}^{\infty} T_{k,m}^{(n)} m \left(m-1\right) g_m \ - k \left(k-1\right) g_k \right| < \varepsilon_n \ \alpha_{n,k} \ ,$$

where $\alpha_{n,k} \ := \begin{cases} \left(1+\delta_n\right)^k R^{-k} & ; k \geq 0 \\ \left(1+\delta_n\right)^{-k} r^{-k} & k < 0 \end{cases}$. From these inequalities it follows that

$$\sum_{k=-\infty}^{\infty} \left(m-k\right)^2 g_m \ T_{k,m}^{(n)} < \varepsilon_n \ \alpha_{n,k} \ \left(\left(k-1\right)^2 + 1\right). \tag{2.18}$$

Let now $f\left(z\right)$ be any function in $A\left(\Re\right)$ with Laurent coefficients and $\rho := \begin{cases} R' & ; k \geq 0 \\ r' & ; k < 0 \end{cases}$ satisfying

$$\begin{aligned}
\|T_n f - f\| &\leq \sum_{k=-\infty}^{\infty} \rho^k \sum_{m=-\infty}^{\infty} |f_m \ - f_k \ | T_{k,m}^{(n)} + \sum_{k=-\infty}^{\infty} \rho^k \left|f_k\right| \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\
&\leq M \sum_{k=-\infty}^{\infty} \rho^k \sum_{m=-\infty}^{\infty} \left(g_m + g_{_k}\right) T_{k,m}^{(n)} + M \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\
&= M \left[\sum_{k=-\infty}^{\infty} \rho^k g_{_k} \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} + \sum_{k=-\infty}^{\infty} \rho^k \sum_{m=-\infty}^{\infty} g_m T_{k,m}^{(n)} \right] \\
&+ M \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\
&\leq M \left[\sum_{k=-\infty}^{\infty} \rho^k g_{_k} \sum_{m=-\infty}^{\infty} \left(m-k\right)^2 g_m T_{k,m}^{(n)} \right. \\
&\left. + \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \sum_{m=-\infty}^{\infty} \left(m-k\right)^2 g_m T_{k,m}^{(n)} \right] + M \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\
&\leq 2M \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \sum_{m=-\infty}^{\infty} \left(m-k\right)^2 g_m T_{k,m}^{(n)} \\
&+ \ 2M \sum_{k=-\infty}^{\infty} \rho^k g_{_k} \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\
&= 2M \left\{ S_n' + S_n'' \right\}.
\end{aligned}$$

where $S_n' = \sum\limits_{k=-\infty}^{\infty} \rho^k g_k \sum\limits_{m=-\infty}^{\infty} (m-k)^2 g_m T_{k,m}^{(n)}$, $S_n'' = \sum\limits_{k=-\infty}^{\infty} \rho^k g_k \left| \sum\limits_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right|$.
Using (2.18), we have

$$\begin{aligned} S_n' &= \sum_{k=-\infty}^{\infty} \rho^k g_k \sum_{m=-\infty}^{\infty} (m-k)^2 g_m T_{k,m}^{(n)} \\ &< \sum_{k=-\infty}^{\infty} \rho^k g_k \varepsilon_n \alpha_{n,k} \left((k-1)^2 + 1 \right) \\ &= \varepsilon_n \sum_{k=-\infty}^{\infty} \rho^k g_k \alpha_{n,k} \left((k-1)^2 + 1 \right) \end{aligned}$$

and since the series converges, the right hand side tends to zero as $n \to \infty$ statistically. Now we estimate S_n'' using (2.18) and

$$\sum_{m \neq k} T_{k,m}^{(n)} \leq \sum_{m=-\infty}^{\infty} (m-k)^2 T_{k,m}^{(n)} \leq \varepsilon_n \alpha_{n,k} \left((k-1)^2 + 1 \right),$$

$$\left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} g_m - g_k \right| < \varepsilon_n \alpha_{n,k} \ or \ \left| g_k \left(T_{k,k} - 1 \right) + \sum_{m \neq k} T_{k,m}^{(n)} g_m \right| < \varepsilon_n \alpha_{n,k}.$$

We get from above the last inequality and (2.18) that

$$g_k \left| T_{k,k}^{(n)} - 1 \right| \leq \varepsilon_n \alpha_{n,k} + \sum_{m \neq k} T_{k,m}^{(n)} g_m (m-k)^2 \leq 2\varepsilon_n \alpha_{n,k} \left((k-1)^2 + 1 \right). \quad (2.19)$$

On the other side, (2.18) gives

$$\sum_{m \neq k} T_{k,m}^{(n)} \leq \varepsilon_n \alpha_{n,k} \left((k-1)^2 + 1 \right) \quad (2.20)$$

From the inequalities (2.19) and (2.20) we obtain

$$\begin{aligned} S_n'' &= \sum_{k=-\infty}^{\infty} \rho^k g_k \left| \sum_{m=-\infty}^{\infty} T_{k,m}^{(n)} - 1 \right| \\ &= \sum_{k=-\infty}^{\infty} \rho^k g_k \left| \left(T_{k,k}^{(n)} - 1 \right) + \sum_{m \neq k} T_{k,m}^{(n)} - 1 \right| \\ &\leq \sum_{k=-\infty}^{\infty} \rho^k g_k \left| T_{k,k}^{(n)} - 1 \right| + \sum_{k=-\infty}^{\infty} \rho^k g_k \left| \sum_{m \neq k} T_{k,m}^{(n)} - 1 \right| \\ &\leq 2\varepsilon_n \sum_{k=-\infty}^{\infty} \rho^k \alpha_{n,k} \left((k-1)^2 + 1 \right) + \sum_{k=-\infty}^{\infty} \rho^k g_k \sum_{m \neq k} T_{k,m}^{(n)} \\ &\leq 2\varepsilon_n \sum_{k=-\infty}^{\infty} \rho^k \alpha_{n,k} \left((k-1)^2 + 1 \right) + \varepsilon_n \sum_{k=-\infty}^{\infty} \rho^k g_k \alpha_{n,k} \left((k-1)^2 + 1 \right). \end{aligned}$$

Since both series in right hand side converge, then S_n'' statistically tends to zero. Therefore, we see that $\|T_n f - f\|$ statistically convergent to zero because both S_n' and S_n'' statistically tend to zero as $n \to \infty$. Using Theorem 2.1, we get (2.17) and the proof of Theorem 2.7 is completed. □

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