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INTEGRAL TYPE CONTRACTIVE CONDITIONS FOR INTUITIONISTIC FUZZY MAPPINGS WITH APPLICATIONS

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Abstract. In this article, integral type contractive conditions for intuitionistic fuzzy set-valued mappings are defined. Using these contractions, some coincidence points and fuzzy fixed point theorems are proved. An example is thereafter provided to support the validity of the results. Moreover, these results are applied to establish some existence theorems for solutions of a few Cauchy problems of Caputo type fractional differential equations .

1. Introduction

Throughout this paper, we shall use the following notations for a metric space (X, d). These basic concepts are recorded from [1, 7, 8, 21, 28]. Often, we shall let $(X, d) = X$, when there is need for simplicity. Thus, for a metric space X,

$$2^X = \{A : A \text{ is a subset of } X\},$$
$$C(2^X) = \{A \in 2^X : \text{A is nonempty and compact}\}$$
$$CB(2^X) = \{A \in 2^X : A \text{ is a nonempty closed and bounded subset of } X\}.$$

For $A, B \in CB(2^X)$,

$$d(x, A) = \inf_{y \in A} d(x, y), \qquad d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

Hence, the Hausdorff metric H on $CB(2^X)$ is defined as :

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\right\}.$$

Denote by ψ, the class of functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ which satisfy the following conditions:

(i) φ is nonnegative, Lebesgue integrable, and
(ii) $\int_0^\tau \varphi(t)dt \geq 0$, for each $\tau > 0$.

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A fuzzy set A in X is a set of ordered pairs $\{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \longrightarrow [0,1]$ is called the membership function of A and $\mu_A(x)$ is the grade of membership of $x \in X$ in A. We denote the family of all fuzzy sets in X by I^X. Taking $\mu_A(x) = A(x)$, the α-level set of a fuzzy set A in X denoted by $[A]_\alpha$, is defined as:

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \quad \text{if} \quad \alpha \in (0,1]$$
$$[A]_0 = \{x \in X : \overline{\{x : A(x) > 0\}};$$

where $\overline{B}$ denotes the closure of nonfuzzy set B.

$$\hat{A} = \left\{ x \in X : A(x) = \max_{y \in X} A(y) \right\}.$$

For $A, B \in I^X$, $A \subset B$ implies $A(x) \leq B(x)$ for each $x \in X$. Let $\alpha \in [0,1]$ such that $[A]_\alpha, [B_\alpha] \in C(2^X)$. Then, we define

$$p_\alpha(A,B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x,y),$$
$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha),$$
$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

A fuzzy mapping T is a fuzzy set-valued mapping of X into I^X. In other words, a fuzzy mapping T is a fuzzy subset of $X \times Y$ with membership function $T(x)(y)$. The function value $T(x)(y)$ is the grade of membership of y in $T(x)$. An $x^* \in X$ is called a fuzzy fixed point of T if there exists an $\alpha \in (0,1]$ such that $x^* \in [Tx^*]_\alpha$. Similarly, x^* is known as *Heilpern fixed point* of T if $\{x^*\} \subset T(x^*)$. For some of these variant of classical fixed point, see [1, 7, 9, 21] and the reference therein.

Definition 1.1. [8] *Let X be a universal set. An intuitionistic fuzzy set (IFS) A in X is a set of ordered triples of the form :*

$$A = \{\langle x, \mu_A(x), v_A(x) | x \in X \rangle\};$$

where the function $\mu_A : X \longrightarrow [0,1]$ and $v_A : X \longrightarrow [0,1]$ define the degree of membership and nonmembership of the element $x \in X$ to the set A, respectively, such that for every $x \in X$, $0 \leq \mu_A(x) + v_A(x) \leq 1$.

Definition 1.2. [8] *Let A be an intuitionistic fuzzy set and $x \in X$. The α-level set of A is denoted by $[A]_\alpha$ and is defined as:*

$$[A]_\alpha = \{x \in X : \mu_A(x) \geq \alpha \quad \text{and} \quad v_A(x) \leq 1 - \alpha\}, \text{if} \quad \alpha \in [0,1].$$

Remark. *(See [4]) If A is an intuitionistic fuzzy set, then $\hat{A}$ is defined as*

$$\hat{A} = \left\{ x \in X : \mu_A(x) = \max_{y \in X} \mu_A(y) \quad \text{and} \quad \nu_A(x) = \min_{y \in X} \nu_A(y) \right\}.$$

Definition 1.3. [31]

Let $L = \{(\alpha, \beta) : \alpha + \beta \leq 1, \alpha, \beta \in (0,1] \times [0,1)\}$ and A is an IFS on X. The (α, β)-cut set of A is defined as:

$$A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \quad \text{and} \quad v_A(x) \leq \beta\}.$$

Definition 1.4. [32] *Let X be a nonempty set. A mapping $T : X \longrightarrow (IFS)^X$ is called an intuitionistic fuzzy mapping. A point $x^* \in X$ is called an intuitionistic fuzzy fixed point of T if there exists $(\alpha, \beta) \in (0,1] \times [0,1)$ such that $x^* \in [Tx^*]_{(\alpha,\beta)}$.*

Definition 1.5. [4] *A point $x^* \in X$ is said to be a fixed point of an intuitionistic fuzzy mapping $T : X \longrightarrow (IFX)^X$, if*

$$\mu_{(Sx^*)}(x^*) \geq \mu_{(Sx^*)}(x) \quad \text{and} \quad \nu_{(Sx^*)}(x^*) \leq \nu_{(Sx^*)}(x) \quad \text{for all } x \in X.$$

In what follows, we state some results that will be used in the main discussion.

Lemma 1.6. [28] *Let A and B be nonempty closed and bounded subsets of a metric space X. If $x \in A$, then*

$$d(x, B) \leq H(A, B).$$

Lemma 1.7. [28] *Let A and B be nonempty closed and bounded subsets of a metric space X. Then, for any $\epsilon > 0$, and $x \in A$, there exists $y \in B$ such that*

$$d(x, y) \leq H(A, B) + \epsilon.$$

Lemma 1.8. [28] *Let (X, d) be a metric space and A, B be nonempty closed and bounded subsets of X. Then, for $x \in A$, there exists $y \in B$ such that*

$$d(x, y) \leq H(A, B).$$

Lemma 1.9. [4] *Let (X, d) be a metric space, $x^* \in X$ and $T : X \longrightarrow (IFS)^X$ is an intuitionistic fuzzy mapping such that $\hat{T}(x) \in C(2^X)$ for all $x \in X$. Then $x^* \in \hat{T}x^*$ if and only if*

$$\mu_{Sx^*}(x^*) \geq \mu_{Sx^*}(x) \quad \text{and} \quad \nu_{Sx^*}(x^*) \leq \nu_{Sx^*}(x) \quad \text{for all} \quad x \in X.$$

Lemma 1.10. [24] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence of real numbers and $\varphi \in \psi$. Then*

$$\lim_{n\longrightarrow\infty} \int_0^{x_n} \varphi(t)dt = 0,$$

if and only if $\lim_{n\longrightarrow\infty} x_n = 0$.

Azam et al [4], used a contractive condition involving rational expression as follows:

Theorem 1.11. *Let $S, T : X \longrightarrow (IFS)^X$, and for $x \in X$ there exists $(\alpha, \beta)_{Sx}, (\alpha, \beta)_{Tx} \in (0, 1] \times [0, 1)$ such that $[Sx]_{(\alpha,\beta)_{Sx}}, [Tx]_{(\alpha,\beta)_{Tx}}$ are nonempty closed and bounded subsets of X. If*

$$\begin{aligned} H\left([Sx]_{(\alpha,\beta)_{Sx}}, [Ty]_{(\alpha,\beta)Ty}\right) \leq\ & ad(x, y) + bd(x, [Sx]_{(\alpha,\beta)S(x)}) \\ & + cd(y, [Ty]_{(\alpha,\beta)T(y)}) \\ & + \frac{ed(x, [Sx]_{(\alpha,\beta)S(x)})d(y, [Ty]_{(\alpha,\beta)T(y)})}{1 + d(x, y)}, \end{aligned}$$

and

$$c + \frac{ed(x, [Sx]_{(\alpha,\beta)_{S(x)}})}{1 + d(x, y)} < 1, \quad b + \frac{ed(y, [Ty]_{(\alpha,\beta)_{T(y)}})}{1 + d(x, y)} < 1,$$

where a, b, c, e, are nonnegative real numbers with $a + b + c + e < 1$, then there exists $z \in X$ such that $z \in [Sz]_{(\alpha,\beta)Sz} \cap [Tz]_{(\alpha,\beta)Tz}$.

2. Main Results

First in this section, we present an integral version of the above contractive condition used in [4, Theorem 3.11] and some associated consequences.

Theorem 2.1. *Let X be a complete metric space, $S, T : X \longrightarrow (IFS)^X$, $\varphi \in \psi$ and for $x \in X$, there exists $(\alpha,\beta)_{S(x)}, (\alpha,\beta)_{T(x)} \in (0,1] \times [0,1)$ such that $[Sx]_{(\alpha,\beta)_{S(x)}}, [Tx]_{(\alpha,\beta)_{T(x)}} \in CB(2^X)$ for all $x \in X$. If*

$$\begin{aligned}\int_0^{H\left([Sx]_{(\alpha,\beta)_{S(x)}},[Ty]_{(\alpha,\beta)_{T(y)}}\right)} \varphi(t)dt \quad \leq \quad & a\int_0^{d(x,y)} \varphi(t)dt \\ & +b\int_0^{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})} \varphi(t)dt \\ & +c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \\ & +e\int_0^{\left(\frac{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})d(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)}\right)} \varphi(t)dt,\end{aligned} \tag{2.1}$$

where $a, b, c, e,$ are nonnegative real numbers with $a + b + c + e < 1$, then there exists $z \in X$ such that $z \in [Sz]_{(\alpha,\beta)S(z)} \cap [Tz]_{(\alpha,\beta)T(z)}$.

Proof. We consider the following three possible cases:

(i) $a + b + e = 0$;
(ii) $a + c + e = 0$;
(iii) $a + b + e \neq 0$, $a + c + e \neq 0$.

Case (i): $a + b + e = 0$. Let $x \in X$ be arbitrary. Then, for $x \in X$, there exists $(\alpha,\beta)_{S(x)} \in (0,1] \times [0,1)$ such that $[Sx]_{(\alpha,\beta)_{S(x)}}$ is a nonempty closed and bounded subset of X. Let $y \in [Sx]_{(\alpha,\beta)_{S(x)}}$ and $u \in [Ty]_{(\alpha,\beta)_{T(y)}}$. Then by Lemma 1.6, we have

$$d(y,[Ty]_{(\alpha,\beta)_{T(y)}}) \leq H\left([Sx]_{(\alpha,\beta)_{S(x)}},[Ty]_{(\alpha,\beta)_{T(y)}}\right). \tag{2.2}$$

From ineqs. (2.1) and (2.2), we obtain

$$\begin{aligned}\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \quad \leq \quad & \int_0^{H([Sx]_{(\alpha,\beta)_{S(x)}},[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \\ \leq \quad & a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})} \varphi(t)dt \\ & +c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt + e\int_0^{\frac{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})d(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)}} \varphi(t)dt\end{aligned} \tag{2.3}$$

Using $a + b + e = 0$, ineq. (2.3) becomes

$$\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \leq c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt,$$

which implies $y \in [Ty]_{(\alpha,\beta)_{T(y)}}$.

Similarly,

$$\int_0^{d(y,[Sy]_{(\alpha,\beta)_{S(y)}})} \varphi(t)dt \leq \int_0^{H\left([Sy]_{(\alpha,\beta)_{S(y)}},[Ty]_{(\alpha,\beta)_{T(y)}}\right)} \varphi(t)dt \tag{2.4}$$

From (2.1) and (2.4), we have

$$\int_0^{d(y,[Sy]_{(\alpha,\beta)_{S(y)}})} \varphi(t)dt \leq 0. \tag{2.5}$$

Consequently, $y \in [Sy]_{(\alpha,\beta)_{S(y)}} \cap [Ty]_{(\alpha,\beta)_{T(y)}}$.

Case (ii): $a + c + e = 0$. For $x \in X$, by hypothesis, there exists $(\alpha, \beta)_{S(x)} \in (0, 1] \times [0, 1)$ such that $[Sx]_{(\alpha,\beta)_{S(x)}}$ is a nonempty closed and bounded subset of X. Let $y \in [Sx]_{(\alpha,\beta)_{S(x)}}$ and $u \in [Ty]_{(\alpha,\beta)_{T(y)}}$. Then, by Lemma 1.6,

$$d(u, [Su]_{(\alpha,\beta)_{S(u)}}) \leq H\left([Su]_{(\alpha,\beta)_{S(u)}}, [Ty]_{(\alpha,\beta)_{T(y)}}\right). \tag{2.6}$$

From (2.1) and (2.6), we get

$$\begin{aligned}\int_0^{d(u,[Su]_{(\alpha,\beta)_{S(u)}})} \varphi(t)dt \quad &\leq \quad a\int_0^{d(u,y)} \varphi(t)dt + b\int_0^{d(u,[Su]_{(\alpha,\beta)_{S(u)}})} \varphi(t)dt \\ &+c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt + e\int_0^{\frac{d(u,[Su]_{(\alpha,\beta)_{S(u)}})d(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(u,y)}} \varphi(t)dt\end{aligned}$$

Using $a + c + e = 0$, the above ineq. reduces to

$$\int_0^{d(u,[Su]_{(\alpha,\beta)_{S(u)}})} \varphi(t)dt \leq b\int_0^{d(u,[Su]_{(\alpha,\beta)_{S(u)}})} \varphi(t)dt,$$

this implies $u \in [Su]_{(\alpha,\beta)_{S(u)}}$. Similarly, one can show that $u \in [Tu]_{(\alpha,\beta)_{T(u)}}$. Hence, $u \in [Su]_{(\alpha,\beta)_{S(u)}} \cap [Tu]_{(\alpha,\beta)_{T(u)}}$.

Case (iii): $a + b + e \neq 0$, $a + c + e \neq 0$.
Let $\max\left\{\left(\frac{a+b}{1-c-e}\right), \left(\frac{a+c}{1-b-e}\right)\right\} = \xi$.
Observe that $\xi = 0$ implies $a = b = c = 0$ and proof follows trivially. Assume that $\xi \neq 0$. Since $a + b + c + e < 1$, then clearly $0 < \xi < 1$.
Let $x_0 \in X$. Then by hypothesis, there exists $(\alpha, \beta)_{S(x_0)} \in (0, 1] \times [0, 1)$ such that $[Sx_0]_{(\alpha,\beta)_{S(x_0)}}$ is a nonempty closed and bounded subset of X. Let $x_1 \in [Sx_0]_{(\alpha,\beta)_{S(x_0)}}$. Then for this x_1, there exists $(\alpha, \beta)_{T(x_1)} \in (0, 1] \times [0, 1)$ such that $[Tx_1]_{(\alpha,\beta)_{T(x_1)}}$ is a nonempty closed and bounded subset of X. Hence, by Lemma 1.6, there exists $x_2 \in [Tx_1]_{(\alpha,\beta)_{T(x_1)}}$ and $x_3 \in [Sx_2]_{(\alpha,\beta)_{S(x_2)}}$ such that

$$d(x_1, x_2) \leq H\left([Sx_0]_{(\alpha,\beta)_{S(x_0)}},[Tx_1]_{(\alpha,\beta)_{T(x_1)}}\right) + \xi(1 - c - e) \tag{2.7}$$

$$d(x_2, x_3) \leq H\left([Sx_2]_{(\alpha,\beta)_{S(x_2)}}, [Tx_1]_{(\alpha,\beta)_{Tx(x_1)}}\right) + \xi^2(1 - b - e) \tag{2.8}$$

Continuing in this fashion, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X can be generated as:

$$x_{2k+1} = [Sx_{2k}]_{(\alpha,\beta)_{Sx_{2k}}}, k = 0, 1, 2, \cdots$$

$$x_{2k+2} = [Tx_{2k+1}]_{(\alpha,\beta)_{Tx_{2k+1}}}, k = 0, 1, 2, \cdots$$

such that

$$d(x_{2k+1}, x_{2k+2}) \leq H\left([Sx_{2k}]_{(\alpha,\beta)_{Sx_{2k}}}, [Tx_{2k+1}]_{(\alpha,\beta)_{Tx_{2k+1}}}\right) + \xi^{2k+1}(1-c-e) \quad (2.9)$$

$$d(x_{2k+2}, x_{2k+3}) \leq H\left([Sx_{2k+2}]_{(\alpha,\beta)_{Sx_{2k+2}}}, [Tx_{2k+1}]_{(\alpha,\beta)_{Tx_{2k+1}}}\right) + \xi^{2k+2}(1-b-e) \quad (2.10)$$

Now, from (2.1) and (2.9), we have

$$\begin{aligned}
\int_0^{d(x_1,x_2)} \varphi(t)dt &\leq a\int_0^{d(x_0,x_1)} \varphi(t)dt + b\int_0^{d(x_0,[Sx_0]_{(\beta,\beta)_{Sx_0}})} \varphi(t)dt \\
&\quad + c\int_0^{d(x_1,[Tx_1]_{(\alpha,\beta)_{Tx_1}})} \varphi(t)dt \\
&\quad + e\int_0^{\frac{d(x_0,[Sx_0]_{(\beta,\beta)_{Sx_0}})d(x_1,[Tx_1]_{(\alpha,\beta)_{Tx_1}})}{1+d(x_0,x_1)}} \varphi(t)dt + \xi(1-c-e) \\
&\leq a\int_0^{d(x_0,x_1)} \varphi(t)dt + b\int_0^{d(x_0,x_1)} \varphi(t)dt \\
&\quad + c\int_0^{d(x_1,x_2)} \varphi(t)dt \\
&\quad + e\int_0^{\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}} \varphi(t)dt + \xi(1-c-e) \\
&\leq (a+b)\int_0^{d(x_0,x_1)} \varphi(t)dt + (c+e)\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi(1-c-e).
\end{aligned}$$

This yields

$$\begin{aligned}
\int_0^{d(x_1,x_2)} \varphi(t)dt &\leq \left(\frac{a+b}{1-c-e}\right)\int_0^{d(x_0,x_1)} \varphi(t)dt + \xi \qquad (2.11) \\
&\leq \xi\int_0^{d(x_0,x_1)} \varphi(t)dt + \xi.
\end{aligned}$$

Again, from (2.1) and (2.10), we have

$$\begin{aligned}
\int_0^{d(x_2,x_3)} \varphi(t)dt &\leq a\int_0^{d(x_1,x_2)} \varphi(t)dt + b\int_0^{d(x_2,[Sx_2]_{(\alpha,\beta)_{Sx_2}})} \varphi(t)dt \\
&\quad + c\int_0^{d(x_1,[Tx_1]_{(\alpha,\beta)_{Tx_1}})} \varphi(t)dt \\
&\quad + e\int_0^{\frac{d(x_2,[Sx_2]_{(\alpha,\beta)_{Sx_2}})d(x_1,[Tx_1]_{(\alpha,\beta)_{Tx_1}})}{1+d(x_1,x_2)}} \varphi(t)dt + \xi^2(1-b-e) \\
&\leq a\int_0^{d(x_1,x_2)} \varphi(t)dt + b\int_0^{d(x_2,x_3)} \varphi(t)dt \\
&\quad + c\int_0^{d(x_1,x_2)} \varphi(t)dt \\
&\quad + e\int_0^{\frac{d(x_2,x_3)d(x_1,x_2)}{1+d(x_1,x_2)}} \varphi(t)dt + \xi^2(1-b-e) \\
&\leq (a+c)\int_0^{d(x_1,x_2)} \varphi(t)dt + (b+e)\int_0^{d(x_2,x_3)} \varphi(t)dt + \xi^2(1-b-e).
\end{aligned}$$

Simplifying the above inequality, we have

$$\begin{aligned}
\int_0^{d(x_2,x_3)} \varphi(t)dt &\leq \left(\frac{a+c}{1-b-e}\right)\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi^2 \qquad (2.12) \\
&\leq \xi\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi^2.
\end{aligned}$$

Combining (2.11) and (2.12), we obtain

$$\begin{aligned}
\int_0^{d(x_2,x_3)} \varphi(t)dt &\leq \xi\int_0^{d(x_1,x_2)} \varphi(t)dt + \xi^2 \\
&\leq \xi\left[\xi\int_0^{d(x_0,x_1)} \varphi(t)dt + \xi\right] + \xi^2 \\
&\leq \xi^2\int_0^{d(x_0,x_1)} \varphi(t)dt + 2\xi^2.
\end{aligned}$$

Continuing this iteration repeatedly, we have

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \xi^n\int_0^{d(x_0,x_1)} \varphi(t)dt + n\xi^n, \quad n \in \mathbb{N}.$$

Hence, for $n > m \geq 1$, we get

$$\begin{aligned}
\int_0^{d(x_m,x_n)} \varphi(t)dt &\leq \int_0^{d(x_m,x_{m+1})} \varphi(t)dt + \int_0^{d(x_{m+1},x_{m+2})} \varphi(t)dt + \cdots + \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \\
&\leq \xi^m \int_0^{d(x_0)} \varphi(t)dt + m\xi^m + \xi^{m+1} \int_0^{d(x_0,x_1)} \varphi(t)dt \\
&\quad +(m+1)\xi^{m+1} + \cdots + \xi^{n-1} \int_0^{d(x_0,x_1)} \varphi(t)dt + (n-1)\xi^{n-1} \\
&\leq \left(\xi^m + \xi^{m+1} + \cdots + \xi^{n-1}\right) \int_0^{d(x_0,x_1)} \varphi(t)dt \\
&\quad + \left(m\xi^m + (m+1)\xi^{m+1} + \cdots + (n-1)\xi^{n-1}\right) \\
&\leq \sum_{i=m}^{n-1} \xi^i \int_0^{d(x_0,x_1)} \varphi(t)dt + \sum_{m}^{n-1} i\xi^i \\
&\leq \frac{\xi^m}{1-\xi} \int_0^{d(x_0,x_1)} \varphi(t)dt + \sum_{i=m}^{n-1} i\xi^i.
\end{aligned}$$

Observe that $(u_n)^{\frac{1}{n}} = n^{\frac{1}{n}}\xi < 1$ as $n \longrightarrow \infty$. Hence, by Cauchy's root test, the series $\sum_{i=m}^{n-1} i\xi^i$ is convergent. It follows that $d(x_m, x_n) \longrightarrow 0$ as $n, m \longrightarrow \infty$. This shows that $\{x_n\}$ is a Cauchy sequence of points of X. By completeness of X, there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. By Lemma 1.6, we get

$$\begin{aligned}
d(z, [Sz]_{(\alpha,\beta)_{S(z)}}) &\leq d(z, x_{2n}) + d(x_{2n}, [Sz]_{(\alpha,\beta)_{S(z)}}) \qquad (2.13) \\
&\leq d(z, x_{2n}) + H\left([Sz]_{(\alpha,\beta)_{S(z)}}, [Tx_{2n-1}]_{(\alpha,\beta)_{T(x_{2n-1})}}\right)
\end{aligned}$$

From (2.1) and (2.13), we have

$$\begin{aligned}
\int_0^{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})} \varphi(t)dt &\leq \int_0^{d(z,x_{2n})} \varphi(t)dt + a\int_0^{d(z,x_{2n-1})} \varphi(t)dt \\
&\quad +b\int_0^{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})} \varphi(t)dt + c\int_0^{d(x_{2n-1},[Tx_{2n-1}]_{(\alpha,\beta)_{T(x_{2n-1})}})} \varphi(t)dt \\
&\quad +e\int_0^{\frac{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})d(d(x_{2n-1},[Tx_{2n-1}]_{T(x_{2n-1})}))}{1+d(z,x_{2n-1})}} \varphi(t)dt \\
&\leq \int_0^{d(z,x_{2n-1})} \varphi(t)dt + a\int_0^{d(z,x_{2n-1})} \varphi(t)dt \\
&\quad +b\int_0^{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})} \varphi(t)dt + c\int_0^{d(x_{2n-1},x_{2n})} \varphi(t)dt \\
&\quad +e\int_0^{\frac{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})d(x_{2n-1},x_{2n})}{1+d(z,x_{2n-1})}} \varphi(t)dt
\end{aligned}$$

As $n \longrightarrow \infty$,the above expression reduces to

$$\int_0^{d(z,[Sz]_{(\alpha,\beta)_{S(z)}})} \varphi(t)dt \leq b\int_0^{[Sz]_{(\alpha,\beta)_{S(z)}}} \varphi(t)dt.$$

Hence, $z \in [Sz]_{(\alpha,\beta)_{S(z)}}$.

On similar steps, one can show that $z \in [Tz]_{(\alpha,\beta)_{T(z)}}$. Consequently, $z \in [Sz]_{(\alpha,\beta)_{S(z)}} \cap [Tz]_{(\alpha,\beta)_{T(z)}}$. □

Remark. *Notice that for $\varphi(t) \equiv 1$, Theorem 2.1 becomes* [4, Theorem 3.11], *and moreover, our result does not require the following condition of* [4]*:*

$$c + \frac{ed(x,[Sx]_{(\alpha,\beta)_{S(x)}})}{1+d(x,y)} < 1, \quad b + \frac{ed(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)} < 1.$$

Since intuitionistic fuzzy sets is a generalization of fuzzy sets, the following corollary is immediate from Theorem 2.1.

Corollary 2.2. *Let $S,T : X \longrightarrow I^X$, $\varphi(t) \in \psi$ and for $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}$ and $[Tx]_{\alpha_T(x)}$ are nonempty closed and bounded subsets of X. If for all $x,y \in X$,*

$$\begin{aligned}\int_0^{H([Sx]_{\alpha_S(x)},[Ty]_{\alpha_T(y)})} \varphi(t)dt \quad \leq \quad & a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{d(x,[Sx]_{\alpha_S(x)})} \varphi(t)dt \\ & +c\int_0^{d(y,[Ty]_{\alpha_T(y)})} \varphi(t)dt \\ & +e\int_0^{\frac{d(x,[Sx]_{\alpha_S(x)})d(y,[Ty]_{\alpha_T(y)})}{1+d(x,y)}} \varphi(t)dt,\end{aligned}$$

where a,b,c,e are nonnegative reals with $a+b+c+e<1$, then there exists $z \in X$ such that $z \in [Sz]_{\alpha_S(z)} \cap [Tz]_{\alpha_T(z)}$.

In what follows, we provide an example to support the validity of Theorem 2.1.

Example 2.3. *Let $X=[0,1]$, $d(x,y)=|x-y|$, whenever, $x,y \in X$ and $S,T : X \longrightarrow (IFS)^X$ be intuitionistic fuzzy mappings. Let $(\alpha,\beta)_{Sx}, (\alpha,\beta)_{Ty} \in (0,1] \times [0,1)$. Consider the following possible cases:*

Case(i) If $x=0$,

$$\mu_{S0}(t) = \begin{cases} 1, & \text{if } t=0 \\ \frac{1}{5}, & \text{if } 0<t\leq \frac{1}{250} \\ 0, & \text{if } \ t>\frac{1}{250}. \end{cases}$$

$$v_{S0}(t) = \begin{cases} 0, & \text{if } t=0 \\ \frac{1}{7}, & \text{if } 0<t\leq \frac{1}{252} \\ 1, & \text{if } \ t>\frac{1}{252}. \end{cases}$$

$$\mu_{T0}(t) = \begin{cases} 1, & \text{if } t=0 \\ \frac{1}{8}, & \text{if } 0<t\leq \frac{1}{190} \\ 0, & \text{if } \ t>\frac{1}{190}. \end{cases}$$

$$v_{T0}(t) = \begin{cases} 0, & \text{if } t=0 \\ \frac{1}{3}, & \text{if } 0<t\leq \frac{1}{150} \\ 1, & \text{if } \ t>\frac{1}{150}. \end{cases}$$

Case(ii) If $x \neq 0$,

$$\mu_{Sx}(t) = \begin{cases} \varrho, & \text{if } 0 \leq t \leq \frac{x}{30} \\ \frac{\varrho}{6}, & \text{if } \frac{x}{30} \leq t \leq \frac{x}{24} \\ \frac{\varrho}{12}, & \text{if } \frac{x}{24} < t < x \\ 0, & \text{if } \quad x \leq t < \infty. \end{cases}$$

$$v_{Sx}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{x}{27} \\ \frac{\beta}{6}, & \text{if } \frac{x}{27} \leq t \leq \frac{x}{24} \\ \frac{\beta}{5}, & \text{if } \frac{x}{20} < t < x \\ \beta, & \text{if } \quad x \leq t < \infty. \end{cases}$$

$$\mu_{Tx}(t) = \begin{cases} \varrho, & \text{if } 0 \leq t \leq \frac{x}{32} \\ \frac{\varrho}{8}, & \text{if } \frac{x}{32} \leq t \leq \frac{x}{24} \\ \frac{\varrho}{14}, & \text{if } \frac{x}{24} < t < x \\ 0, & \text{if } \quad x \leq t < \infty. \end{cases}$$

$$v_{Tx}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{x}{28} \\ \frac{\beta}{8}, & \text{if } \frac{x}{28} \leq t \leq \frac{x}{24} \\ \frac{\beta}{7}, & \text{if } \frac{x}{14} < t < x \\ \beta, & \text{if } \quad x \leq t < \infty. \end{cases}$$

We see that for

$$a = \frac{1}{10}, b = \frac{1}{5}, c = \frac{1}{4}, e = \frac{1}{6} \text{ and } \varphi(t) = \begin{cases} \frac{\sin(nt)}{t}, & \text{if } \quad t > 0, \quad \text{and } n \in \mathbb{N} \\ n, & \text{if } \quad t = 0, \text{and } n \in \mathbb{N} \end{cases}$$

all the conditions of Theorem 2.1 are satisfied to obtain $z \in [Sz]_{(\alpha,\beta)Sz} \cap [Tz]_{(\alpha,\beta)Tz}$.

Again, in the following, we furnish another example to authenticate the utility of Theorem 2.1.

Example 2.4. *Let* $X = \{5, 6, 7\}, \{5\}, \{6\}, \{7\}$ *be crisp sets. Define* $d : X \times X \longrightarrow \mathbb{R}$ *by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \frac{9}{20}, & \text{if } x \neq y \quad \text{and} \quad x, y \in X - \{6\} \\ 1, & \text{if } x \neq y \quad \text{and} \quad x, y \in X - \{5\} \\ \frac{11}{20}, & \text{if } \quad x \neq y \quad \text{and} \quad x, y \in X - \{7\}. \end{cases}$$

Consider two intuitionistic fuzzy mappings $S, T : X \longrightarrow (IFS)^X$ *defined as follows:*

$$\mu_{S(5)}(t) = \mu_{S(6)}(t) = \mu_{S(7)}(t) = \begin{cases} \frac{4}{5}, & \text{if } t = 5, \\ \frac{6}{91}, & \text{if } t = 6, \\ 0, & \text{if } \quad t = 7, \end{cases}$$

$$\nu_{S(5)}(t) = \nu_{S(6)}(t) = \nu_{S(7)}(t) = \begin{cases} 0, & \text{if } t = 5, \\ \frac{4}{13}, & \text{if } t = 6, \\ 1, & \text{if } \quad t = 7, \end{cases}$$

and

$$\mu_{T(5)}(t) = \mu_{T(7)} = \begin{cases} \frac{4}{5}, & \text{if } t = 5, \\ \frac{2}{7}, & \text{if } t = 6, \\ 0, & \text{if } \quad t = 7, \end{cases}$$

$$\mu_{T(6)}(t) = \begin{cases} 0, & \text{if } t = 5, \\ \frac{1}{18}, & \text{if } t = 6, \\ \frac{4}{5}, & \text{if } \quad t = 7, \end{cases}$$

$$\nu_{T(5)}(t) = \nu_{T(7)}(t) = \begin{cases} 0, & \text{if } t = 5, \\ \frac{1}{10}, & \text{if } t = 6, \\ \frac{3}{16} & \text{if } \quad t = 7, \end{cases}$$

$$\nu_{T(6)}(t) = \begin{cases} \frac{1}{30}, & \text{if } t = 5, \\ \frac{6}{91}, & \text{if } t = 6, \\ 0, & \text{if } \quad t = 7. \end{cases}$$

Then, for $x \in X$, by taking $\alpha_{S(x)} = \frac{4}{5} = \alpha_{T(x)}$, and $\beta_{S(x)} = 0 = \beta_{T(x)}$, we have

$$\begin{aligned} [Sx]_{(\alpha,\beta)_{S(x)}} &= \left\{ t \in X : \mu_{S(x)}(t) \geq \alpha_{S(x)} \quad \text{and} \quad \nu_{S(x)} \leq \beta_{S(x)} \right\} \\ &= \{5\}, \quad \text{for all } x \in X, \end{aligned}$$

and

$$\begin{aligned} [Tx]_{(\alpha,\beta)_{T(x)}} &= \left\{ \mu_{T(x)}(t) \geq \alpha_{T(x)} \text{ and } \nu_{T(x)} \leq \beta_{T(x)} \right\} \\ &= \begin{cases} \{5\}, & \text{if } x \neq 6, \\ \{7\}, & \text{if } x = 6. \end{cases} \end{aligned}$$

Therefore,

$$H\left([Sx]_{(\alpha,\beta)_{S(x)}}, [Ty]_{(\alpha,\beta)_{T(y)}}\right) = \begin{cases} H(\{5\},\{5\}) = 0 & \text{if } y \neq 6 \\ H(\{5\},\{7\}) = \frac{9}{20} & \text{if } y = 6. \end{cases}$$

Take $\varphi(t) = \frac{\ln(1+t)}{1+t}$, for all $t \in [0, \infty)$. Clearly, $\varphi \in \psi$. In fact,

$$\int_0^{d(x,y)} \varphi(t)dt = \frac{1}{2}\left[\ln^2(1+t)\right]_0^{d(x,y)}.$$

Now, for $y \neq 6$, we have

$$\begin{aligned} \int_0^{H\left([Sx]_{(\alpha,\beta)_{S(x)}}, [Ty]_{(\alpha,\beta)_{T(y)}}\right)} \varphi(t)dt = 0 \quad \leq \quad & a\int_0^{d(x,y)} \varphi(t)dt \\ & + b\int_0^{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})} \varphi(t)dt + c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \\ & + e\int_0^{\frac{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})d(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)}} \varphi(t)dt. \end{aligned}$$

Notice that for $y = 6$, $d(y, [Ty]_{(\alpha,\beta)_{T(y)}}) = d(6, 7) = 1$ *and hence*

$$\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt = \frac{1}{2}\left[\ln^2(1+t)\right]_0^1 \approx 0.2402.$$

Therefore, for $y = 6$, $a = b = e = 0$ *and* $c = \frac{1}{2}$, *we obtain*

$$\begin{aligned}
\int_0^{H\left([Sx]_{(\alpha,\beta)_{S(x)}},[Ty]_{(\alpha,\beta)_{T(y)}}\right)} \varphi(t)dt &= \int_0^{\frac{9}{20}} \frac{\ln(1+t)}{1+t}dt \\
&= \frac{1}{2}\left[\ln^2(1+t)\right]_0^{\frac{9}{20}} \approx 0.0690 \\
&\leq 0.1201 = \frac{1}{2}\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \\
&= \frac{1}{4}\left[\ln^2(1+t)\right]_0^{\frac{9}{20}} \\
&\leq a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})} \varphi(t)dt \\
&\leq +c\int_0^{d(y,[Ty]_{(\alpha,\beta)_{T(y)}})} \varphi(t)dt \\
&\leq +e\int_0^{\frac{d(x,[Sx]_{(\alpha,\beta)_{S(x)}})d(y,[Ty]_{(\alpha,\beta)_{T(y)}})}{1+d(x,y)}} \varphi(t)dt.
\end{aligned}$$

Consequently, all the conditions of Theorem 2.1 are satisfied to obtain $5 \in [S5]_{(\frac{4}{5},0)_{S(5)}} \cap [T5]_{(\frac{4}{5},0)_{T(5)}}$.

Next, using Theorem 2.1, we establish a result on the existence of a common fixed point for a pair of intuitionistic fuzzy mappings as follows.

Theorem 2.5. *Let* $S, T : X \longrightarrow (IFS)^X$ *be mappings such that* $\hat{S}x, \hat{T}x \in CB(2^X)$. *If for all* $x, y \in X$, *and* $\varphi \in \psi$,

$$\begin{aligned}
\int_0^{H\left(\hat{S}x,\hat{T}y\right)} \varphi(t)dt \leq\ & a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{d(x,\hat{S}x)} \varphi(t)dt \\
& +c\int_0^{d(y,\hat{T}y)} \varphi(t)dt + e\int_0^{\frac{d(x,\hat{S}x)d(y,\hat{T}y)}{1+d(x,y)}} \varphi(t)dt,
\end{aligned}$$

where a, b, c, e *are nonnegative real numbers with* $a+b+c+e < 1$, *then there exists* $z^* \in X$ *such that*

$$\mu_{Sz^*}(z^*) \geq \mu_{Sz^*}(z) \quad \nu_{Tz^*}(z^*) \leq \nu_{Tz^*}(z) \quad \textit{for all} \quad z \in X.$$

Proof. Assume

$$\max_{t\in X} \mu_{Sx}(t) = \alpha_1 \quad \min_{t\in X} \nu_{Tx}(t) = \beta_1$$

and

$$\max_{t\in X} \mu_{Tx}(t) = \alpha_2 \quad \min_{t\in X} \nu_{Tx}(t) = \beta_2.$$

Therefore, for all $x, y \in X$,

$$\hat{S}x = [Sx]_{(\alpha_1,\beta_1)} \quad \text{and} \quad \hat{T}y = [Ty]_{(\alpha_2,\beta_2)},$$

and

$$\begin{aligned}\int_0^{H\left([Sx]_{(\alpha_1,\beta_1)},[Ty]_{(\alpha_2,\beta_2)}\right)}\varphi(t)dt &\leq a\int_0^{d(x,y)}\varphi(t)dt+b\int_0^{d(x,[Sx]_{(\alpha_1,\beta_1)})}\varphi(t)dt\\ &\quad +c\int_0^{d(y,[Ty]_{(\alpha_2,\beta_2)})}\varphi(t)dt\\ &\quad +e\int_0^{\frac{d(x,[Sx]_{(\alpha_1,\beta_1)})d(y,[Ty]_{(\alpha_2,\beta_2)})}{1+d(x,y)}}\varphi(t)dt.\end{aligned}$$

Therefore, by Theorem 2.1, there exists $z^*\in X$ such that

$$z^*\in[Sz^*]_{(\alpha_1,\beta_1)}\cap[Tz^*]_{(\alpha_2,\beta_2)}=\hat{S}z^*\cap\hat{T}z^*.$$

It follows from Lemma 1.9 that

$$\mu_{Sz^*}(z^*)\geq\mu_{Sz^*}(z)\quad\text{and}\quad\nu_{Sz^*}(z^*)\leq\nu_{Sz^*}(z)\quad\text{for all}\quad z\in X,$$

Similarly,

$$\mu_{Tz^*}(z^*)\geq\mu_{Tz^*}(z)\qquad\text{and}\quad\nu_{Tz^*}(z^*)\leq\nu_{Tz^*}(z)\quad\text{for all}\quad z\in X.$$

□

In the next result, as an application of the techniques of proof of Theorem 2.1, one can also prove the existence of a common Heilpern fixed point for a pair of intuitionistic fuzzy mapping under a complete metric space X. First, recall that in [4], the set $N(X)$ is defined as:

$$N(X)=\{A\in(IFS)^X:[A]_{(\alpha,\beta)}\in CB(2^X)\}.$$

Theorem 2.6. *Let $S,T:X\longrightarrow N(X)$, $\varphi\in\psi$ and for all $x\in X$,*

$$\begin{aligned}\int_0^{d_\infty(S(x),T(y))}\varphi(t)dt &\leq a\int_0^{d(x,y)}\varphi(t)dt+b\int_0^{p(x,S(x))}\varphi(t)dt\\ &\quad +c\int_0^{p(x,T(y))}\varphi(t)dt\\ &\quad +e\int_0^{\left(\frac{p(x,S(x))p(y,T(y))}{1+d(x,y)}\right)}\varphi(t)dt,\end{aligned}$$

where a,b,c,e are nonnegative reals with $a+b+c+e<1$; then there exists $z\in X$ such that $\{z\}\subset\{Sz\}$ and $\{z\}\subset\{Tz\}$.

Proof. Let $x\in X$, then by hypothesis, $[Sx]_{(1,0)_{S(x)}}$ and $[Tx]_{(1,0)_{T(x)}}$ are nonempty closed and bounded subsets of X. Therefore, for all $x,y\in X$,

$$D_{(1,0)}(S(x),T(y))\leq d_\infty(S(x),T(y)).$$

Since

$$[Sx]_{(1,0)}\subset[Sx]_{(\alpha,\beta)}\in CB(2^X),\quad\text{for each}\quad(\alpha,\beta)\in(0,1]\times[0,1),$$

then

$$d(x,[Sx]_{(\alpha,\beta)})\leq d(x,[Sx]_{(1,0)}),\quad\text{for each}\quad(\alpha,\beta)\in(0,1]\times[0,1).$$

Therefore,

$$p(x,S(x))\leq d(x,[Sx]_{(1,0)}).$$

Similarly,

$$p(y,T(y)) \leq d(y,[Ty]_{(1,0)}).$$

Consequently,

$$\begin{aligned}
\int_0^{H\left([Sx]_{(1,0)},[Ty]_{(1,0)}\right)} \varphi(t)dt &= \int_0^{D_{(1,0)}(S(x),T(y))} \varphi(t)dt \\
&\leq \int_0^{d_\infty(S(x),T(y))} \varphi(t)dt \\
&\leq a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{p(x,S(x))} \varphi(t)dt \\
&\quad +c\int_0^{p(y,T(y))} \varphi(t)dt \\
&\quad +e\int_0^{\frac{p(x,S(x))p(y,T(y))}{1+d(x,y)}} \varphi(t)dt \\
&\leq a\int_0^{d(x,y)} \varphi(t)dt + b\int_0^{d(x,[Sx]_{(1,0)})} \varphi(t)dt \\
&\quad +c\int_0^{d(y,[Ty]_{(1,0)})} \varphi(t)dt \\
&\quad +e\int_0^{\frac{d(x,[Sx]_{(1,0)})d(y,[Ty]_{(1,0)})}{1+d(x,y)}} \varphi(t)dt.
\end{aligned}$$

It follows by Theorem 2.1 that there exists $z \in X$ such that $\{z\} \subset Sz$ and $\{z\} \subset Tz$. □

The next result follows from Theorem 2.1. For use in the next section, we present it as a theorem .

Theorem 2.7. *Let (X,d) be a complete metric space and $T : X \longrightarrow (IFS)^X$ be an intuitionistic set-valued mapping. Let $\varphi \in \psi$, and for $x \in X$, there exists $\eta \in (0,1)$ and $(\alpha,\beta)_{T(x)} \in (0,1] \times [0,1)$ such that $[Tx]_{(\alpha,\beta)_{T(x)}}$ is a nonempty closed and bounded subset of X. If for all $x,y \in X$, $x \neq y$,*

$$\int_0^{H\left([Tx]_{(\alpha,\beta)_{T(x)}},[Ty]_{(\alpha,\beta)_{T(y)}}\right)} \varphi(t)dt \leq \eta\int_0^{d(x,y)} \varphi(t)dt, \tag{2.14}$$

then there exists $z \in X$ such that $z \in [Tz]_{(\alpha,\beta)_{T(z)}}$.

Proof. Put $b = c = e = 0$ and $S = T$ in Theorem 2.1, the proof is complete. □

3. Applications

In this section, a few applications of Theorem 2.7 are presented. We apply the Theorem to establish existence results for some Cauchy problems of Caputo type differential equations.

For the present purpose, we give some specific definitions from fractional calculus as follows.

Definition 3.1. [25] *Let $\Omega = [a,b](-\infty < a < b < \infty)$ be a finite interval of the real line $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^v_{a^+}f$ and $I^v_{b^-}f$ of f of order $v \in \mathbb{C}$ are respectively defined by*

$$\left(I^v_{a^+}f\right)(x) = \frac{1}{\Gamma(v)}\int_a^x (x-t)^{v-1}f(t)dt \quad (x > a, Re(v) > 0), \tag{3.1}$$

and

$$\left(I^v_{b^-}f\right)(x) = \frac{1}{\Gamma(v)}\int_x^b (t-x)f(t)dt \quad (x < b, Re(v) > 0), \tag{3.2}$$

where

$$\Gamma(v) = \int_0^\infty t^{v-1}e^{-t}dt, \quad t^{v-1} = e^{(z-1)\log(t)}$$

is the Gamma function.

The integrals (3.1) and (3.2) are called left-sided and right-sided integrals , respectively.

Definition 3.2. [25] *The Riemann-Liouville fractional derivatives $D^v_{a^+}f$ and $D^v_{b^-}f$ of order $v \in \mathbb{C}$ are respectively defined by*

$$(D^v_{a^+}f)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-v}_{a^+}f\right)(x)$$

and

$$(D^v_{b^-}f)(x) = \left(-\frac{d}{dx}\right)^n \left(I^{n-v}_{b^-}f\right)(x).$$

Lemma 3.3. [25] *If $Rev > 0$ and $f(x) \in L^p(a,b)(1 \le p \le \infty)$, then the following equalities*

$$\left(D^v_{a^+}I^v_{a^+}f\right)(x) = f(x) \quad \text{and} \quad (D^v_{b^-}I^v_{b^-}f)(x) = f(x)$$

hold almost everwhere on $[a,b]$.

Definition 3.4. [25] *Let $n = [Re(v)] + 1$ for $v \notin \mathbb{N}_0$; $n = v$ for $v \in \mathbb{N}_0$. Define by $|.|C^n([a,b])$, the space of absolutely continuous functions defined on $[a,b]$. The Caputo fractional derivatives of f of order v, respectively denoted $({}^CD^v_{a^+}f)(x)$ and $({}^CD^v_{b^-}f)(x)$ are defined by*

$$\left({}^CD^v_{a^+}f\right)(x) = \frac{1}{\Gamma(n-v)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{v-n+1}}dt = \left(I^{n-v}_{a^+}D^nf\right)(x)$$

and

$$\left({}^CD^v_{b^-}f\right)(x) = \frac{(-1)^n}{\Gamma(n-v)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{v-n+1}}dt = (-1)^n\left(I^{n-v}_{b^-}D^nf\right)(x),$$

where $D = \frac{d}{dx}$.

Remark. *If $v = n \in \mathbb{N}_0$, then $({}^CD^v_{b^-}f)(x) = \frac{d^n}{dx^n}f(x)$.*

Consider the Cauchy type problem of Caputo fractional differential equation (CFDE) given by

$$\begin{aligned} {}^{C}D^{v}_{0+}x(t) &= g(t, x(t)) \\ x_0 = \beta, &\quad t \in [0, \delta], \delta < 1; \end{aligned} \tag{3.3}$$

where $g \in C([0, \delta] \times \mathbb{R}, \mathbb{R})$, $0 < v < 1$.

The Volterra fractional integral equation equivalent of (3.3) is given by

$$x(t) = \beta + \frac{1}{\Gamma(v)} \int_0^t (t - \eta)^{v-1} g(\eta, x(\eta)) d\eta, \quad t \in [0, \delta]. \tag{3.4}$$

Since f is continuous, every solution of (3.3) is also a solution of (3.4).

Now, using Theorem 2.7, we establish an existence theorem for the CFDE (3.3) as follows.

Theorem 3.5. *Let $g : \mathbb{R} \times [0, \delta] \longrightarrow \mathbb{R}$ be a continuous function. Assume there exist $\lambda, k \in (0, 1)$ and $\varphi \in \psi$ such that for all $x, y \in C([0, \delta], \mathbb{R})$,*

(i)
$$|g(t, x) - g(t, y)| \leq k|x(t) - y(t)|$$

and

(ii)
$$\int_0^{N|x(t)-y(t)|} \varphi(t)dt \leq \lambda \int_0^{|x(t)-y(t)|} \varphi(t)dt; \quad t \in [0, \delta]$$

where $N = \frac{k\delta^v}{v\Gamma(v)} < 1$ and $N \leq \lambda$. Then the CFDE (3.3) has atleast one solution in $C([0, \delta], \mathbb{R})$.

Proof. Let $X = C([0, \delta], \mathbb{R})$ and $d : X \times X \longrightarrow \mathbb{R}$ be defined by

$$d(x, y) = \max_{0 \leq t \leq \delta} |x(t) - y(t)|.$$

Then (X, d) is a complete metric space. Let $A, B : X \longrightarrow (0, 1]$ be any two mappings. For $x \in X$, take

$$h_x(t) = \beta + \frac{1}{\Gamma(v)} \int_0^t (t - \eta)^{v-1} g(\eta, x(\eta)) d\eta. \tag{3.5}$$

Define an intuitionistic fuzzy mapping $T : X \longrightarrow (IFS)^X$ by

$$\mu_{Tx}(r) = \begin{cases} A(x), & \text{if } r(t) = h_x(t) \\ 0, & \text{otherwise.} \end{cases} \qquad \nu_{Tx}(r) = \begin{cases} 0, & \text{if } r(t) = h_x(t) \\ B(x), & \text{otherwise.} \end{cases}$$

Taking $\alpha_{T(x)} = A(x)$ and $\beta_{T(x)} = 0$, we have

$$\begin{aligned} [Tx]_{(\alpha,\beta)_{Tx}} &= \{r \in X : \alpha_{T(x)} = A(x) \quad \text{and } \beta_{T(x)} = 0\} \\ &= \{h_x(t)\}. \end{aligned}$$

Therefore,

$$H\left([Tx]_{(\alpha,\beta)_{Tx}}, [Ty]_{(\alpha,\beta)_{Ty}}\right) = \max_{t \in [0,\delta]} |h_x(t) - h_y(t)|.$$

Hence,

$$\begin{aligned}
\int_0^{H\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)} \varphi(t)dt &= \int_0^{\max_{t\in[0,\delta]}|h_x(t)-h_y(t)|} \varphi(t) \\
&= \int_0^{\max_{t\in[0,\delta]}\left|\frac{1}{\Gamma(v)}\int_0^t (t-\eta)^{v-1}g(\eta,x(\eta))d\eta-\frac{1}{\Gamma(v)}\int_0^t (t-\eta)^{v-1}g(\eta,y(\eta))d\eta\right|} \varphi(t)dt \\
&\leq \int_0^{\frac{1}{\Gamma(v)}\max_{t\in[0,\delta]}\int_0^t (t-\eta)^{v-1}d\eta|g(\eta,x(\eta))-g(\eta,y(\eta))|} \varphi(t)dt \\
&\leq \int_0^{\frac{k}{\Gamma(v)}\max_{t\in[0,\delta]}\int_0^t (t-\eta)^{v-1}|d\eta||x(\eta)-y(\eta)|} \varphi(t)dt.
\end{aligned}$$

Letting $u = t - \eta$, yields

$$\frac{1}{\Gamma(v)}\int_0^t (t-\eta)^{v-1}d\eta = \frac{t^v}{v\Gamma(v)}.$$

This implies

$$\frac{k}{\Gamma(v)}\max_{t\in[0,\delta]}\int_0^t (t-\eta)^{v-1}d\eta = \frac{k\delta^v}{v\Gamma(v)} = N.$$

Therefore,

$$\begin{aligned}
\int_0^{H\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)} \varphi(t)dt &\leq \int_0^{Nd(x,y)} \varphi(t)dt \\
&\leq \lambda\int_0^{d(x,y)} \varphi(t)dt.
\end{aligned}$$

Consequently, all the conditions of Theorem 2.7 hold. Therefore, the CFDE (3.3) has at least one solution in X. □

Next, we give an example to support the validity of Theorem 3.5.

Example 3.6. *Consider the Cauchy problem with Caputo type fractional differential equation given by*

$$\begin{aligned} {}^{C}D_{0^+}^{\frac{1}{2}}x(t) &= t, \quad t \in \left[0, \frac{1}{4}\right] \\ x(0) &= 0. \end{aligned} \tag{3.6}$$

From (3.6), $g(t, x(t)) = t$ *and*

$$x(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\eta}{(t-\eta)^{\frac{1}{2}}} d\eta.$$

Since $0 < k, \delta < 1$, *we can take* $k = \frac{1}{2}$ *and* $\delta = \frac{1}{4}$. *Then*

$$N = \frac{k\delta^v}{v\Gamma(v)} = \frac{\frac{1}{2}(\frac{1}{4})^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{1}{2})} \approx 0.3.$$

Define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ *by* $\varphi(t) = \sin t$. *By direct calculation, it is easy to see that*

$$|g(t, x) - g(t, y)| \leq k|x(t) - y(t)|,$$

and

$$\int_0^{N|x(t)-y(t)|} \varphi(t)dt \leq \lambda \int_0^{|x(t)-y(t)|} \varphi(t)dt, \qquad \forall N \leq \lambda.$$

Hence, by Theorem 3.5, the CFDE (3.6) has at least one solution in X.

Again,using Theorem 2.7, we establish an existence result for the following Cauchy type problem of Caputo fractional differential equation (CFDE) of order $v \in (0, 1)$, given by

$${}^{C}D_{0^+}^{v}x(t) = g\left(t, x(t), \int_0^{\delta} f(t, s, x(s))ds\right), \quad x(0) = \beta, \tag{3.7}$$

where

$$t \in [0, \delta] = \Omega, \delta < 1, g \in C\left([0, \delta] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right), f \in C\left([0, \delta] \times [0, \delta] \times \mathbb{R}^n, \mathbb{R}^n\right).$$

By Lemma 3.3, the CFDE (3.7) can be reformulated as

$$x(t) = \beta + \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} g\left(\eta, x(\eta), \int_0^{\delta} f(\eta, s, x(s))ds\right) d\eta. \tag{3.8}$$

Since g is assumed to be a continuous function, every solution of the CFDE (3.7) is also a solution of the Volterra fractional integral equation (3.8).

Theorem 3.7. *Consider the CFDE* (3.7) *and its integral equivalent* (3.8). *Assume that* $\varphi \in \psi$ *and there exist* $p, q, r \in (0, 1)$ *such that*

(i) $\| g(t, u_1, v_1) - g(t, u_2, v_2) \| \leq p \| u_1 - u_2 \| + q \| v_1 - v_2 \|$
(ii) $\| f(t, s, w_1) - f(t, s, w_2) \| \leq r \| w_1 - w_2 \|$
(iii) $\int_0^{(ME_t(v,\theta))d(x,y)} \varphi(t)dt \leq \lambda \int_0^{d(x,y)} \varphi(t)dt,$

given that $0 < (ME_t(v,\theta)) < 1$ *and* $(ME_t(v,\theta)) \leq \lambda\,(ME_t(v,\theta))$, *for* $t, s \in \Omega = [0,\delta]$, $u_i, v_i, w_i \in \mathbb{R}^n$, $\quad x, y \in C\left([0,\delta] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right)$;
where

$$M = (p + \delta qr) \quad \textit{and} \quad E_t(v,\theta) = \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} e^{\theta\eta} d\eta, \theta > 0.$$

Then the CFDE (3.7) *has a solution in* $C\left([0,\delta] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right)$

Proof. Let $X = C\left([0,\delta] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right)$. For our purpose, we define the weighted Chebyshev metric $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x,y) = \max_{t\in\Omega} \parallel x(t) - y(t) \parallel e^{-\theta t}, \quad \theta > 0, \Omega = [0,\delta].$$

Then (X,d) is a complete metric space. Let $L, M : X \longrightarrow (0,1]$ be any two mappings. For $x \in X$, take

$$w_x(t) = \beta + \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} g\left(\eta, x(\eta), \int_0^\delta f(\eta, s, x(s)) ds\right) d\eta.$$

Let an intuitionistic fuzzy mapping $T : X \longrightarrow (IFS)^X$ be defined by

$$\mu_{Tx}(r) = \begin{cases} L(x), & \text{if } r(t) = w_x(t) \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_{Tx} = \begin{cases} 0, & \text{if } r(t) = w_x(t) \\ M(x), & \text{otherwise.} \end{cases}$$

By taking $\alpha_{T(x)} = L(x)$ and $\beta_{T(x)} = 0$, we obtain

$$\begin{aligned} [Tx]_{(\alpha,\beta)_{T(x)}} &= \{r \in X : \alpha_{T(x)} = L(x) \text{ and } \beta_{T(x)} = 0.\} \\ &= \{w_x(t)\}. \end{aligned}$$

Hence,

$$H\left([Tx]_{(\alpha,\beta)_{T(x)}}, [Ty]_{(\alpha,\beta)_{T(y)}}\right) = \max_{t\in\Omega} \parallel w_x(t) - w_y(t) \parallel .$$

First, notice that by conditions $(i) - (ii)$, for $x, y \in X$, we have

$$\begin{aligned} H\left([Tx]_{(\alpha,\beta)_{Tx}}, [Ty]_{(\alpha,\beta)_{Ty}}\right) &\leq \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} d\eta e^{\theta s} e^{-\theta s} \\ &\quad \times \max_{s\in\Omega} \left\{ \left[p \left\| x(s) - y(s) \right\| + q \left\| \int_0^\delta f(\eta, s, x(s)) - \int_0^\delta f(\eta, s, y(s)) \right\| \right] ds \right\} \\ &\leq \frac{1}{\Gamma(v)} \int_0^t (t-\eta)^{v-1} e^{\theta\eta} d\eta \\ &\quad \times \max_{s\in\Omega} \left\{ p \left\| x(s) - y(s) \right\| e^{-\theta s} + q \int_0^\delta \left\| f(\eta, s, x(s)) - f(\eta, s, y(s)) \right\| e^{-\theta s} ds \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{H\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)}\varphi(t)dt &= \int_0^{\max_{t\in\Omega}\|x(t)-y(t)\|}\varphi(t)dt\\ &\leq \int_0^{\frac{1}{\Gamma(v)}\int_0^t(t-\eta)^{v-1}e^{\theta\eta}d\eta\left\{pd(x,y)+qr\max_{s\in\Omega}\int_0^\delta\|x(s)-y(s)\|e^{\theta s}ds\right\}}\varphi(t)dt\\ &\leq \int_0^{\frac{1}{\Gamma(v)}\int_0^t(t-\eta)^{v-1}e^{\theta\eta}d\eta\{pd(x,y)+\delta qrd(x,y)\}}\varphi(t)dt\\ &\leq \int_0^{\frac{1}{\Gamma(v)}\int_0^t(t-\eta)^{v-1}e^{\theta\eta}d\eta(p+\delta qr)d(x,y)}\varphi(t)dt.\end{aligned}$$

If we make the change of variable $u = t - \eta$, we see that

$$\frac{1}{\Gamma(v)}\int_0^t(t-\eta)^{v-1}e^{\theta\eta}d\eta = \frac{e^{\theta t}}{\Gamma(v)}\int_0^t u^{v-1}e^{-\theta u}du \tag{3.9}$$

Recall that the incomplete Gamma function $\gamma^*(v,t)$ is given by $\gamma^*(v,t) = \frac{1}{\Gamma(v)}t^v\int_0^t\zeta^{v-1}e^{-\zeta}d\zeta$ (see[26, p. 48]). Therefore, (3.9) may be written as

$$\frac{1}{\Gamma(v)}\int_0^t(t-\eta)^{v-1}e^{\theta\eta}d\eta = t^ve^{\theta t}\gamma^*(v,\theta t). \tag{3.10}$$

Let (3.10) be denoted by $E_t(v,\theta)$ (see [26, p. 48]). Consequently, the above calculation reduces to

$$\begin{aligned}\int_0^{H\left([Tx]_{(\alpha,\beta)_{Tx}},[Ty]_{(\alpha,\beta)_{Ty}}\right)}\varphi(t)dt &\leq \int_0^{(E_t(v,\theta)M)d(x,y)}\varphi(t)dt\varphi(t)dt\\ &\leq \lambda\int_0^{d(x,y)}\varphi(t)dt.\end{aligned}$$

□

Hence, all the assumptions of Theorem 2.7 are satisfied. It follows that there exists $x \in [Tx]_{(\alpha,\beta)_{Tx}}$ which corresponds to a solution of the Cauchy type CFDE (3.7).

Next, we provide an example to support the validity of Theorem 3.7.

Example 3.8. *Consider the Cauchy type CFDE*

$$\begin{aligned}{}^{C}D^{\frac{1}{2}}_{0^+}x(t) &= \frac{1}{1+t}\sin x(t) + \frac{2}{2+t}\sin y(t), \quad t\in[0,1]\\ x(0) &= 0.\end{aligned} \tag{3.11}$$

From (3.11)*, we have*

$$g(t,u,v) = \frac{1}{1+t}\sin u(t) + \frac{2}{2+t}\sin v(t).$$

Therefore, condition (i) *of Theorem 3.7 becomes*

$$\|g(t,u_1,v_1) - g(t,u_2,v_2)\| \leq \frac{1}{2}\|u_1-v_1\| + \frac{2}{3}\|u_2-v_2\|.$$

Hence, $p = \frac{1}{2}, q = \frac{2}{3}$. *Since* $r, \delta < 1$, *we can take* $r = \frac{1}{4}$ *and* $\delta = \frac{1}{2}$ *so that* $M = (p + \delta qr) = \frac{7}{12}$.
It is also clear that condition (ii) *hold. Now, from* (3.11),

$$x(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(1+\eta)(t-\eta)^{\frac{1}{2}}} \sin x(\eta) d\eta + \frac{1}{\sqrt{\pi}} \int_0^t \frac{2}{(2+\eta)(t-\eta)^{\frac{1}{2}}} \sin y(\eta) d\eta.$$

By taking

$$\varphi(t) = \begin{cases} \frac{\sin(nt)}{t}, & t > 0, \quad and \quad n \in \mathbb{N} \\ n, & t = 0 \quad and \quad n \in \mathbb{N}. \end{cases}$$

We see that $\varphi \in \psi$ *and* $\int_0^\delta \varphi(t)dt = \frac{\pi}{2}$. *By direct calculation, it can be verified that*

$$\int_0^{(ME_t(v,\theta))d(x,y)} \varphi(t)dt \leq \lambda \int_0^{d(x,y)} \varphi(t)dt,$$

for all $(ME_t(v,\theta)) \leq \lambda$. *Consequently, all the conditions of Theorem 3.7 are satisfied. It follows that the CFDE* (3.8) *has a solution in* $C([0,\delta] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Conclusion

This paper presents some coincidence and common fixed point of intuitionistic fuzzy set-valued mappings defined on a metric space using integral type contractive conditions. In Theorem 2.1, the contractive condition is used to obtain a coincidence point of two intuitionistic fuzzy mappings. Theorem 2.6 is a direct consequence of Theorem 2.1. The main result (Theorem 2.1) is an extension of the contractive condition used in [4, Theorem 3.11] and the reference therein. Further, as applications of Theorem 2.7, a few existence results for some Cauchy type problems of Caputo fractional differential equations are provided. Hence, we hope that these results will contribute positively, in some ways, to the field of fuzzy analysis as well as metric fixed point theory .

Competing Interests

The authors declare that they have no competing interests.

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