Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 10 Issue 1 (2019), Pages 49-61.

EXPLICIT SOLUTION TO THE OPERATOR EQUATION $AXD + FX^*B = C$ OVER HILBERT C^*-MODULES

ZAHRA NIAZI MOGHANI, MAHNAZ KHANEHGIR*, MEHDI MOHAMMADZADEH KARIZAKI

Abstract. In this paper, by using operator matrices, we investigate the explicit solution of the operator equation

$$AXD + FX^*B = C$$

in the general setting of the adjointable operators between Hilbert C^*-modules. This solution is expressed in terms of the Moore-Penrose inverses of the operators A, B, D and F. The obtained results extend and generalize some known operator equations studied previously by a number of mathematicians.

1. Introduction and preliminaries

Matrix equations are very useful in engineering problems, information theory, linear system theory, linear estimation theory, numerical analysis, and other topics. The matrix equation $AXD + FX^TB = C$ has been widely used in system theory, such as eigenstructure assignment [7, 12, 16, 18, 19]. Other relevant applications include Lyapunov and Sylvester equations [2, 17]. The generalized Sylvester equation $AV + BW = EVJ + R$ with unknown matrices V and W, has numerous applications in linear systems theory [5], and fault detection [8]. The equation $TXS^* - SX^*T^* = A$ was studied by Yuan for finite matrices [22]. When T equals an identity matrix or identity operator, this equation is reduced to $XS^* - SX^* = A$, which was studied by Braden [1] for finite matrices, later, by Djordjevic [4] for the Hilbert space operators. Cvetkovic-Ilic [3] investigated the solvability of the operator equation $AX + X^*C = B$ for g-invertible Hilbert space operators.

Hilbert C^*-modules are natural generalizations of finite-dimensional spaces, Hilbert spaces and C^*-algebras. They were first introduced by Kaplansky [10] in 1953, and since then, they proved to be effective tool in the theory of C^*-algebra, especially in the study of KK-groups, induced representations, locally compact quantum groups and non-commutative geometry [6, 9, 11]. Compared to the Hilbert space case, there exist some differences when we deal with operators in a general Hilbert C^*-module; for instance, a closed topologically complemented submodule of a Hilbert

2000 *Mathematics Subject Classification.* 47A62, 15A24, 46L08.
Key words and phrases. Hilbert C^*-module; Moore-Penrose inverse; Operator equation; Operator matrix.

*Corresponding author.
Submitted November 2, 2018. Published February 19, 2019.
Communicated by Roman Drnovsek.

C^*-module may not be orthogonally complemented. By [11, Theorem 3.2], we know that the former deficit can be mended if we restrict our attention to those adjointable operators whose ranges are closed. It is therefore meaningful to put forward a generalized version of the previous results about the equation

$$AXD + FX^*B = C. \tag{1.1}$$

for matrices and bounded linear operators between Hilbert spaces in the context of Hilbert C^*-modules. For instance, Xu *et al.* [21] and Mohammadzadeh Karizaki *et al.* [13, 14, 15] studied the operator equation $TXS^* - SX^*T^* = A$ under closed range assumption for adjointable Hilbert C^*-modules.

In this paper, applying operator matrices we provide a new approach to the study of the operator equation $AXD + FX^*B = C$ in the general setting of the adjointable operators between Hilbert C^*-modules. Based on the Moore-Penrose inverses [20] of the associated operators, we propose the necessary and sufficient conditions for the existence of a solution to this equation, and obtain the general expression of the solution in the solvable case.

Throughout the paper, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert modules over the same C^*-algebra $\mathcal{A}$. We denote the set of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ and by $\mathcal{L}(\mathcal{X})$, when $\mathcal{X} = \mathcal{Y}$. For $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, let $\mathrm{ran(T)}$ and $\ker(T)$ be the range of T and the null space of T, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

For the sake of convenience of the reader, here we collect a few facts which will be needed further on.

Theorem 1.1. [11, Theorem 3.2] *Suppose that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\mathrm{ran(T^*)}$.
- $\mathrm{ran(T)}$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$.
- *The map $T^* \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ has closed range.*

Xu *et al.* [20] showed that an adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse $T^\dagger$ of T is the unique element in $\mathcal{L}(\mathcal{Y},\mathcal{X})$ which satisfies the following conditions:

$$TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

From these conditions we obtain that $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have

$$\mathrm{ran(T)} = \mathrm{ran(TT^\dagger)}, \qquad \mathrm{ran(T^\dagger)} = \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)},$$

$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*).$$

Remark. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by*

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$.

Theorem 1.2. [15, Theorem 2.5.] Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.
(b) $A = A^*$.
If (a) or (b) is satisfied, then any solution to

$$TXS^* + SX^*T^* = A, \ X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $Z^* = -Z$.

Remark. [15, Remark 2.1.] Let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation

$$TXS = A, \qquad X \in \mathcal{L}(\mathcal{Y}), \tag{1.2}$$

has a solution if and only if

$$TT^\dagger AS^\dagger S = A.$$

In which case, any solution of (1.2) has the form

$$X = T^\dagger AS^\dagger.$$

We terminate this section with analogues versions of [15, Theorem 2.2.] and [14, Theorem 2.2.]. We will not reproduce here the corresponding proofs which can be easily obtained by slight modifications of the proofs given in these papers.

Theorem 1.3. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules. Suppose $S \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) There exists a solution $X \in \mathcal{L}(\mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.
(b) $A = A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.
If (a) or (b) is satisfied, then any solution to $TXS^* + SX^*T^* = A$ has the form

$$X = \frac{1}{2}T^\dagger ATT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z + Z^*)T = 0$ and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary.

Theorem 1.4. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $A, S \in \mathcal{L}(\mathcal{X})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that S is self-adjoint, both S and T have closed ranges, and $AS^\dagger S = A$ and $T^\dagger S^\dagger S = T^\dagger$. Then the following statements are equivalent:*
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to the operator equation $TXS + SX^*T^* = A$.
(b) $A = A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.
If (a) or (b) is satisfied, then any solution to $TXS + SX^*T^* = A$ has the form

$$X = T^\dagger AS^\dagger - \frac{1}{2}T^\dagger ATT^\dagger S^\dagger + T^\dagger ZTT^\dagger S^\dagger + V - T^\dagger TVSS^\dagger,$$

where $Z \in \mathcal{L}(\mathcal{X})$ satisfies $T^*(Z + Z^*)T = 0$ and $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is arbitrary.

2. Main results

In this section, in the general setting of Hilbert C^*-module operators, we provide some new approaches for solving operator equation (1.1) Furthermore, the necessary and sufficient conditions for the existence of a solution of Eq. (1.1), were given and the set of solutions was completely described.

Theorem 2.1. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $A, F \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be invertible and $C \in \mathcal{L}(\mathcal{Y})$. If*

$$A(B^*)^{-1}C^*(F^*)^{-1}D + F(D^*)^{-1}C^*(A^*)^{-1}B = 2C, \tag{2.1}$$

then the operator equation (1.1) has a solution $X \in \mathcal{L}(\mathcal{X})$. In the case any solution of the Eq. (1.1) is represented by

$$X = \frac{1}{4}A^{-1}CD^{-1} + \frac{1}{4}(B^*)^{-1}C^*(F^*)^{-1} + (B^*)^{-1}K^*(F^*)^{-1} - A^{-1}KD^{-1}, \tag{2.2}$$

where $K \in \mathcal{L}(\mathcal{Y})$ satisfies $A(B^*)^{-1}K^*(F^*)^{-1}D = F(D^*)^{-1}K^*(A^*)^{-1}B$.

Proof. Suppose that (2.1) is satisfied. Taking $K = 0$ in (2.2), we have that the operator X defined by (2.2) is the solution of the equation (2.1).

We will now show that if the equation (1.1) has a solution $X \in \mathcal{L}(\mathcal{X})$, then it must be of the form (2.2). Let $T = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{Y}\oplus\mathcal{Y}$, $Y = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{X}\oplus\mathcal{X}$, $S = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X}\oplus\mathcal{X} \to \mathcal{Y}\oplus\mathcal{Y}$ and $N = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} : \mathcal{Y}\oplus\mathcal{Y} \to \mathcal{Y}\oplus\mathcal{Y}$. Obviously, T and S are invertible.
By $TYS^* + SY^*T^* = N$, we have

$$\begin{aligned} TYS^* + SY^*T^* &= \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} \\ &+ \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix}\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}, \end{aligned}$$

or equivalently

$$\begin{bmatrix} 0 & AXD + FX^*B \\ B^*XF^* + D^*X^*A^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}.$$

Therefore

$$AXD + FX^*B = C.$$

By Theorem 1.2, it follows that Y has the following representation

$$Y = \frac{1}{2}T^{-1}N(S^*)^{-1} - T^{-1}Z(S^*)^{-1}, \tag{2.3}$$

where $Z \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y})$ satisfies $Z^* = -Z$. Let $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. From (2.3), we have

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A^{-1} & 0 \\ 0 & (B^*)^{-1} \end{bmatrix} \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$$
$$- \begin{bmatrix} A^{-1} & 0 \\ 0 & (B^*)^{-1} \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

It enforces that

$$X = \frac{1}{2} A^{-1} C D^{-1} - A^{-1} Z_2 D^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} - (B^*)^{-1} Z_3 (F^*)^{-1},$$

and

$$-A^{-1} Z_1 (F^*)^{-1} = -(B^*)^{-1} Z_4 D^{-1} = 0.$$

From invertibility of the operators A, B, D and F it follows that $Z_1 = 0$ and $Z_4 = 0$. On the other hand, from $Z^* = -Z$, we get $Z_3^* = -Z_2$. For $K = \frac{1}{2} Z_2$, we have

$$X = \frac{1}{2} A^{-1} C D^{-1} - 2A^{-1} K D^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} + 2(B^*)^{-1} K^* (F^*)^{-1}, \quad (2.4)$$

which implies that

$$\frac{1}{2} A^{-1} C D^{-1} = \frac{1}{2} (B^*)^{-1} C^* (F^*)^{-1} + 2(B^*)^{-1} K^* (F^*)^{-1} + 2A^{-1} K D^{-1}. \quad (2.5)$$

Replacing the second part of (2.4) in FX^*B and the third part of it in AXD, it can be derived that

$$A(B^*)^{-1} K^* (F^*)^{-1} D - F(D^*)^{-1} K^* (A^*)^{-1} B =$$
$$\frac{1}{2}(AXD + FX^*B) - \frac{1}{4}[A(B^*)^{-1} C^* (F^*)^{-1} D + F(D^*)^{-1} C^* (A^*)^{-1} B].$$

From this relation and the condition (2.1), it can be concluded that

$$A(B^*)^{-1} K^* (F^*)^{-1} D = F(D^*)^{-1} K^* (A^*)^{-1} B.$$

On the other hand, applying the third part of (2.4) and also using (2.5), we have

$$\begin{aligned} X &= A^{-1} C D^{-1} - A^{-1} F X^* B D^{-1} \\ &= A^{-1} C D^{-1} - A^{-1} F (\frac{1}{2} F^{-1} C B^{-1} + 2F^{-1} K B^{-1}) B D^{-1} \\ &= \frac{1}{2} A^{-1} C D^{-1} - 2A^{-1} K D^{-1} \\ &= \frac{1}{4} A^{-1} C D^{-1} + \frac{1}{4} A^{-1} C D^{-1} - 2A^{-1} K D^{-1} \\ &= \frac{1}{4} A^{-1} C D^{-1} + \frac{1}{4} (B^*)^{-1} C^* (F^*)^{-1} + (B^*)^{-1} K^* (F^*)^{-1} \\ &\quad + A^{-1} K D^{-1} - 2A^{-1} K D^{-1} \\ &= \frac{1}{4} A^{-1} C D^{-1} + \frac{1}{4} (B^*)^{-1} C^* (F^*)^{-1} + (B^*)^{-1} K^* (F^*)^{-1} - A^{-1} K D^{-1}. \end{aligned}$$

□

As a consequence of the above theorem we have the following result.

Corollary 2.2. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be invertible and $C \in \mathcal{L}(\mathcal{Y})$. If*

$$A(B^*)^{-1}C^* + C^*(A^*)^{-1}B = 2C,$$

then the operator equation $AX + X^*B = C$ has a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. In the case any solution to

$$AX + X^*B = C, \qquad X \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),$$

is represented by

$$X = \frac{1}{4}A^{-1}C + \frac{1}{4}(B^*)^{-1}C^* + (B^*)^{-1}K^* - A^{-1}K,$$

where $K \in \mathcal{L}(\mathcal{Y})$ satisfies $(B^*)^{-1}K^*B^{-1} = A^{-1}K^*(A^*)^{-1}$.

Now, we will consider the more general case when operators F and D have closed ranges.

Theorem 2.3. *Suppose $A, F \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that A, B are invertible and F, D have closed ranges, and $C \in \mathcal{L}(\mathcal{Y})$ such that*

$$A(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger D = -F(D^*)^\dagger C^* FF^\dagger (A^*)^{-1}B,$$
$$A(B^*)^{-1}C^*(F^*)^\dagger D + F(D^*)^\dagger C^*(A^*)^{-1}B = C.$$

Then the following statements are equivalent:

(a) *There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to Eq. (1.1).*
(b) $(1 - FF^\dagger)C(1 - D^\dagger D) = 0$.

If (a) or (b) is satisfied, then any solution to Eq. (1.1) has the form

$$\begin{aligned} X = {} & \frac{1}{2}A^{-1}CD^\dagger + \frac{1}{2}(B^*)^{-1}C^*(F^*)^\dagger - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger - \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger \\ & -\frac{1}{2}A^{-1}FF^\dagger KD^\dagger + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger \\ & +\frac{1}{2}(B^*)^{-1}V^*(1 - F^\dagger F) - \frac{1}{2}A^{-1}W^*(1 - DD^\dagger), \end{aligned} \tag{2.6}$$

where $K \in \mathcal{L}(\mathcal{Y})$ and $V, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfy

$$\begin{aligned} A(B^*)^{-1}V^*(1 - F^\dagger F)D &= F(1 - DD^\dagger)W(A^*)^{-1}B, \\ A(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger D &= F(D^*)^\dagger K^* FF^\dagger (A^*)^{-1}B. \end{aligned}$$

Proof. $(a) \Rightarrow (b)$: Suppose that Eq. (1.1) has a solution $X \in \mathcal{L}(\mathcal{X})$. Then

$$\begin{aligned} (1 - FF^\dagger)C(1 - D^\dagger D) &= (1 - FF^\dagger)(AXD + FX^*B)(1 - D^\dagger D) \\ &= (1 - FF^\dagger)AXD(1 - D^\dagger D) + (1 - FF^\dagger)FX^*B(1 - D^\dagger D) \\ &= 0. \end{aligned}$$

$(b) \Rightarrow (a)$: Let $T = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $\hat{X} = \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{X} \oplus \mathcal{X}$, $S = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$ and $N = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$. Further, we consider the operator equation $T\hat{X}S^* + S(\hat{X})^*T^* = N$, where it can be written in the following form

$$
\begin{aligned}
&T\hat{X}S^* + S(\hat{X})^*T^* \\
&= \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} \\
&= \begin{bmatrix} 0 & FX^*B + AXD \\ D^*X^*A^* + B^*XF^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}.
\end{aligned}
$$

From this, we find that $AXD + FX^*B = C$. On the other hand, by condition (b) we have

$$
\begin{aligned}
(1 - TT^\dagger)N(1 - TT^\dagger) &= \begin{bmatrix} 0 & (1 - FF^\dagger)C(1 - D^*(D^*)^\dagger) \\ (1 - D^*(D^*)^\dagger)C^*(1 - FF^\dagger) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

According to Theorem 1.3 the operator equation $T\hat{X}S^* + S(\hat{X})^*T^* = N$, has the solution

$$\hat{X} = \frac{1}{2}T^\dagger NTT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger N(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y})$ satisfies $T^*(Z + Z^*)T = 0$ and $Y \in \mathcal{L}(\mathcal{Y} \oplus \mathcal{Y}, \mathcal{X} \oplus \mathcal{X})$ is arbitrary. Let us assume that $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$. Then from the above solution we obtain

$$
\begin{aligned}
\begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 0 & F^\dagger CD^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger C^*FF^\dagger(A^*)^{-1} & 0 \end{bmatrix} \\
&+ \begin{bmatrix} F^\dagger Z_1 FF^\dagger(A^*)^{-1} & F^\dagger Z_2 D^*(D^*)^\dagger B^{-1} \\ (D^*)^\dagger Z_3 FF^\dagger(A^*)^{-1} & (D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & F^\dagger C(1 - D^*(D^*)^\dagger)B^{-1} \\ (D^*)^\dagger C^*(1 - FF^\dagger)(A^*)^{-1} & 0 \end{bmatrix} \\
&+ \begin{bmatrix} (1 - F^\dagger F)Y_1(A^*)^{-1} & (1 - F^\dagger F)Y_2 B^{-1} \\ (1 - (D^*)^\dagger D^*)Y_3(A^*)^{-1} & (1 - (D^*)^\dagger D^*)Y_4 B^{-1} \end{bmatrix}.
\end{aligned}
$$

It in turns implies that

$$F^\dagger Z_1 FF^\dagger(A^*)^{-1} + (1 - F^\dagger F)Y_1(A^*)^{-1} = 0. \tag{2.7}$$

$$(D^*)^\dagger Z_4 D^*(D^*)^\dagger B^{-1} + (1 - (D^*)^\dagger D^*)Y_4 B^{-1} = 0. \tag{2.8}$$

$$
\begin{aligned}
X^* = \frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^{-1} &+ F^\dagger Z_2 D^*(D^*)^\dagger B^{-1} \\
&+ F^\dagger C(1 - D^*(D^*)^\dagger)B^{-1} + (1 - F^\dagger F)Y_2 B^{-1}.
\end{aligned} \tag{2.9}
$$

$$X^* = \frac{1}{2}(D^*)^\dagger C^*FF^\dagger(A^*)^{-1} + (D^*)^\dagger Z_3 FF^\dagger(A^*)^{-1} \tag{2.10}$$

$$+(D^*)^\dagger C^*(1-FF^\dagger)(A^*)^{-1}+(1-(D^*)^\dagger D^*)Y_3(A^*)^{-1}.$$

By multiplication F on the left and A^* on the right to (2.7), we get $FF^\dagger Z_1 FF^\dagger = 0$. Also, by multiplication D^* on the left and B on the right to (2.8), we obtain that $D^\dagger D Z_4 D^\dagger D = 0$. Further, utilizing the relation $T^*(Z+Z^*)T = 0$, we observe that

$$\begin{aligned} F^*Z_1F &= -F^*Z_1^*F, \\ F^*Z_2D^* &= -F^*Z_3^*D^*, \\ DZ_3F &= -DZ_2^*F, \\ DZ_4D^* &= -DZ_4^*D^*. \end{aligned}$$

Then the relation $(Z+Z^*)T \in \ker(T^*) = \ker(T^\dagger)$ implies that $T^\dagger(Z+Z^*)T = 0$ and so it yields that

$$F^\dagger Z_1 F = -F^\dagger Z_1^* F, \qquad F^\dagger Z_2 D^* = -F^\dagger Z_3^* D^*, \tag{2.11}$$

$$(D^*)^\dagger Z_3 F = -(D^*)^\dagger Z_2^* F,\ (D^*)^\dagger Z_4 D^* = -(D^*)^\dagger Z_4^* D^*.$$

By taking adjoint from (2.9) and (2.10) and setting the two equal to each other, we get

$$\begin{aligned} & \frac{1}{2}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger + (B^*)^{-1}D^\dagger DZ_2^*(F^*)^\dagger \\ & +(B^*)^{-1}(1-D^\dagger D)C^*(F^*)^\dagger + (B^*)^{-1}Y_2^*(1-F^\dagger F) \\ = \ & \frac{1}{2}A^{-1}(F^*)^\dagger F^*CD^\dagger + A^{-1}(F^*)^\dagger F^*Z_3^*D^\dagger + A^{-1}CD^\dagger \\ & -A^{-1}(F^*)^\dagger F^*CD^\dagger + A^{-1}Y_3^*(1-(D^*)^\dagger D^*). \end{aligned}$$

Take $K = Z_2$, $V = Y_2$ and $W = Y_3$ in the above relation. Using (2.11), it can be deduced that

$$\begin{aligned} A^{-1}CD^\dagger &= \frac{1}{2}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger + (B^*)^{-1}D^\dagger DK^*(F^*)^\dagger \\ & +(B^*)^{-1}(1-D^\dagger D)C^*(F^*)^\dagger + (B^*)^{-1}V^*(1-F^\dagger F) \\ & +\frac{1}{2}A^{-1}(F^*)^\dagger F^*CD^\dagger + A^{-1}(F^*)^\dagger F^*KD^\dagger \\ & -A^{-1}W^*(1-(D^*)^\dagger D^*). \end{aligned} \tag{2.12}$$

On the other hand, let X be any solution of Eq. (1.1), then $AXD = C - FX^*B$. Put (2.9) in the right side of this equation we get

$$\begin{aligned} AXD &= C - \frac{1}{2}FF^\dagger CD^*(D^*)^\dagger - FF^\dagger KD^*(D^*)^\dagger \\ & -FF^\dagger C(1-D^*(D^*)^\dagger) - F(1-FF^\dagger)V \\ &= C - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger C(1-D^\dagger D). \end{aligned}$$

Set $M = C - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D - FF^\dagger C(1-D^\dagger D)$. In view of the condition (b), it can be concluded that

$$\begin{aligned} AA^{-1}MD^\dagger D &= CD^\dagger D - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D \\ &= C - FF^\dagger C(1-D^\dagger D) - \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D \\ &= M. \end{aligned}$$

So, taking into account Remark 1, (2.9) and (2.12) we observe that

$$\begin{aligned}
X &= A^{-1}(C - FX^*B)D^\dagger \\
&= A^{-1}CD^\dagger - A^{-1}F\Big(\frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^{-1} + F^\dagger KD^*(D^*)^\dagger B^{-1} \\
&\quad + F^\dagger C(1 - D^*(D^*)^\dagger)B^{-1} + (1 - F^\dagger F)VB^{-1}\Big)BD^\dagger \\
&= A^{-1}CD^\dagger - \frac{1}{2}A^{-1}FF^\dagger CD^\dagger - A^{-1}FF^\dagger KD^\dagger \\
&= \frac{1}{2}A^{-1}CD^\dagger + \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger \\
&\quad + \frac{1}{2}(B^*)^{-1}(1 - D^\dagger D)C^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}V^*(1 - F^\dagger F) \\
&\quad - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger + \frac{1}{2}A^{-1}FF^\dagger KD^\dagger + \frac{1}{2}A^{-1}FF^\dagger CD^\dagger \\
&\quad - \frac{1}{2}A^{-1}W^*(1 - DD^\dagger) - \frac{1}{2}A^{-1}FF^\dagger CD^\dagger - A^{-1}FF^\dagger KD^\dagger \\
&= \frac{1}{2}A^{-1}CD^\dagger + \frac{1}{2}(B^*)^{-1}C^*(F^*)^\dagger - \frac{1}{4}A^{-1}FF^\dagger CD^\dagger \\
&\quad - \frac{1}{4}(B^*)^{-1}D^\dagger DC^*(F^*)^\dagger - \frac{1}{2}A^{-1}FF^\dagger KD^\dagger \\
&\quad + \frac{1}{2}(B^*)^{-1}D^\dagger DK^*(F^*)^\dagger + \frac{1}{2}(B^*)^{-1}V^*(1 - F^\dagger F) - \frac{1}{2}A^{-1}W^*(1 - DD^\dagger).
\end{aligned}$$

□

In the sequel, we shall establish the explicit solution to Eq. (1.1) when operators A, B, D and F have closed ranges.

Theorem 2.4. *Let $A, B, C \in \mathcal{L}(\mathcal{Y})$, $F \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, such that A, B are self-adjoint, A, B, D, F have closed ranges. Moreover, suppose that $CB^\dagger B = C$, $AA^\dagger C = C$, $F^\dagger A^\dagger A = F^\dagger$, $BB^\dagger D^\dagger = D^\dagger$, $AB^\dagger C^*(F^*)^\dagger D + F(D^*)^\dagger C^* A^\dagger B = C$, and $AB^\dagger D^\dagger DC^*(F^*)^\dagger D = -F(D^*)^\dagger C^* FF^\dagger A^\dagger B$.*
Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ to Eq. (1.1).
(b) $(1 - FF^\dagger)C(1 - D^\dagger D) = 0$.
If (a) or (b) is satisfied, then any solution to Eq. (1.1) has the form

$$\begin{aligned}
X = \frac{1}{2}A^\dagger CD^\dagger + \frac{1}{2}B^\dagger C^*(F^*)^\dagger - \frac{1}{4}A^\dagger FF^\dagger CD^\dagger \\
- \frac{1}{4}B^\dagger D^\dagger DC^*(F^*)^\dagger - \frac{1}{2}A^\dagger FF^\dagger KD^\dagger + \frac{1}{2}B^\dagger D^\dagger DK^*(F^*)^\dagger \\
- \frac{1}{2}U^* + \frac{1}{2}A^\dagger AU^*DD^\dagger + \frac{1}{2}W^* - \frac{1}{2}B^\dagger BW^*F^\dagger F,
\end{aligned} \tag{2.13}$$

where $K \in \mathcal{L}(\mathcal{Y})$ and $U, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfy

$$\begin{aligned}
AB^\dagger D^\dagger DK^*(F^*)^\dagger D &= F(D^*)^\dagger K^* FF^\dagger A^\dagger B, \\
AW^*D - AB^\dagger BW^*F^\dagger FD &= FUB - FDD^\dagger UAA^\dagger B.
\end{aligned}$$

Proof. $(a) \Rightarrow (b)$: The proof in this case is the same as the proof of Theorem 2.3.
$(b) \Rightarrow (a)$: Let $T = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$, $\hat{X} = \begin{bmatrix} 0 & X^* \\ X^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to$

$\mathcal{X}\oplus\mathcal{X}$, $S=\begin{bmatrix} A & 0\\ 0 & B^*\end{bmatrix}:\mathcal{Y}\oplus\mathcal{Y}\to\mathcal{Y}\oplus\mathcal{Y}$ and $N=\begin{bmatrix} 0 & C\\ C^* & 0\end{bmatrix}:\mathcal{Y}\oplus\mathcal{Y}\to\mathcal{Y}\oplus\mathcal{Y}$. By replacing these operator matrices in the operator equation $T\hat{X}S+S(\hat{X})^*T^*=N$, we get $AXD+FX^*B=C$. Further, from the condition (b) it can be derived that

$$(1-TT^\dagger)N(1-TT^\dagger) \quad = \quad \begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}.$$

According to Theorem 1.4 it follows that $\hat{X}$ has the following representation

$$\hat{X}=T^\dagger NS^\dagger-\frac{1}{2}T^\dagger NTT^\dagger S^\dagger+T^\dagger ZTT^\dagger S^\dagger+V-T^\dagger TVSS^\dagger,$$

where $Z\in\mathcal{L}(\mathcal{Y}\oplus\mathcal{Y})$ satisfies $T^*(Z+Z^*)T=0$ and $V\in\mathcal{L}(\mathcal{Y}\oplus\mathcal{Y},\mathcal{X}\oplus\mathcal{X})$ is arbitrary. Let us assume that $Z=\begin{bmatrix} Z_1 & Z_2\\ Z_3 & Z_4\end{bmatrix}$ and $V=\begin{bmatrix} V_1 & V_2\\ V_3 & V_4\end{bmatrix}$. From the above solution we get

$$\begin{aligned}
\begin{bmatrix} 0 & X^*\\ X^* & 0\end{bmatrix} &= \begin{bmatrix} 0 & F^\dagger CB^\dagger\\ (D^*)^\dagger C^*A^\dagger & 0\end{bmatrix}\\
&- \frac{1}{2}\begin{bmatrix} 0 & F^\dagger CD^*(D^*)^\dagger B^\dagger\\ (D^*)^\dagger C^*FF^\dagger A^\dagger & 0\end{bmatrix}\\
&+ \begin{bmatrix} F^\dagger Z_1FF^\dagger A^\dagger & F^\dagger Z_2D^*(D^*)^\dagger B^\dagger\\ (D^*)^\dagger Z_3FF^\dagger A^\dagger & (D^*)^\dagger Z_4D^*(D^*)^\dagger B^\dagger\end{bmatrix}\\
&+ \begin{bmatrix} V_1 & V_2\\ V_3 & V_4\end{bmatrix}-\begin{bmatrix} F^\dagger FV_1AA^\dagger & F^\dagger FV_2BB^\dagger\\ (D^*)^\dagger D^*V_3AA^\dagger & (D^*)^\dagger D^*V_4BB^\dagger\end{bmatrix}.
\end{aligned}$$

From this we obtain

$$F^\dagger Z_1FF^\dagger A^\dagger+V_1-F^\dagger FV_1AA^\dagger=0, \tag{2.14}$$

$$(D^*)^\dagger Z_4D^*(D^*)^\dagger B^\dagger+V_4-(D^*)^\dagger D^*V_4BB^\dagger=0, \tag{2.15}$$

$$\begin{aligned}
X^*=F^\dagger CB^\dagger-\frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^\dagger+F^\dagger Z_2D^*(D^*)^\dagger B^\dagger\\
+V_2-F^\dagger FV_2BB^\dagger,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
X^*=(D^*)^\dagger C^*A^\dagger-\frac{1}{2}(D^*)^\dagger C^*FF^\dagger A^\dagger+(D^*)^\dagger Z_3FF^\dagger A^\dagger\\
+V_3-(D^*)^\dagger D^*V_3AA^\dagger.
\end{aligned} \tag{2.17}$$

Now, by multiplication F on the left and A on the right to (2.14) and our assumptions, we get

$$FF^\dagger Z_1FF^\dagger=0.$$

Also, by multiplication D^* on the left and B on the right to (2.15), we obtain

$$D^\dagger DZ_4D^\dagger D=0.$$

On the other hand, the relations $T^*(Z + Z^*)T = 0$ and $\ker(T^*) = \ker(T^\dagger)$ give us

$$(D^*)^\dagger Z_3 F = -(D^*)^\dagger Z_2^* F. \tag{2.18}$$

Regarding to (2.16) and (2.17) we observe that

$$\begin{aligned} & B^\dagger C^*(F^*)^\dagger - \frac{1}{2}B^\dagger D^\dagger DC^*(F^*)^\dagger + B^\dagger D^\dagger DZ_2^*(F^*)^\dagger \\ & +V_2^* - B^\dagger BV_2^* F^*(F^*)^\dagger \\ = \; & A^\dagger CD^\dagger - \frac{1}{2}A^\dagger(F^*)^\dagger F^* CD^\dagger + A^\dagger(F^*)^\dagger F^* Z_3^* D^\dagger \\ & +V_3^* - A^\dagger AV_3^* DD^\dagger. \end{aligned}$$

For $K = Z_2$, $W = V_2$, $U = V_3$ and also applying (2.18) we have

$$A^\dagger CD^\dagger = B^\dagger C^*(F^*)^\dagger - \frac{1}{2}B^\dagger D^\dagger DC^*(F^*)^\dagger + B^\dagger D^\dagger DK^*(F^*)^\dagger \tag{2.19}$$

$$\begin{aligned} & +W^* - B^\dagger BW^* F^*(F^*)^\dagger + \frac{1}{2}A^\dagger(F^*)^\dagger F^* CD^\dagger \\ & +A^\dagger(F^*)^\dagger F^* KD^\dagger - U^* + A^\dagger AU^* DD^\dagger. \end{aligned}$$

Next, let X be any solution of Eq. (1.1). Replace X^* in the right side of the equation $AXD = C - FX^*B$ with (2.16) and applying relations $CB^\dagger B = C$ and $(D^*)^\dagger B^\dagger B = (D^*)^\dagger$, it yields

$$\begin{aligned} AXD &= C - F\Big(F^\dagger CB^\dagger - \frac{1}{2}F^\dagger CD^*(D^*)^\dagger B^\dagger + F^\dagger Z_2 D^*(D^*)^\dagger B^\dagger \\ & \quad +V_2 - F^\dagger FV_2 BB^\dagger\Big)B \\ &= C - FF^\dagger CB^\dagger B + \frac{1}{2}FF^\dagger CD^*(D^*)^\dagger B^\dagger B - FF^\dagger KD^*(D^*)^\dagger B^\dagger B \\ &= C - FF^\dagger C + \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D. \end{aligned}$$

Set $M = C - FF^\dagger C + \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D$. Then applying relations $AA^\dagger C = C$, $F^\dagger A^\dagger A = F^\dagger$ and regarding the condition (b) we deduce

$$\begin{aligned} AA^\dagger MD^\dagger D &= AA^\dagger CD^\dagger D - AA^\dagger FF^\dagger CD^\dagger D \\ & \quad +\frac{1}{2}AA^\dagger FF^\dagger CD^\dagger DD^\dagger D - AA^\dagger FF^\dagger KD^\dagger DD^\dagger D \\ &= CD^\dagger D - (F^*)^\dagger F^* CD^\dagger D + \frac{1}{2}(F^*)^\dagger F^* CD^*(D^*)^\dagger - (F^*)^\dagger F^* KD^*(D^*)^\dagger \\ &= C - FF^\dagger C + FF^\dagger CD^\dagger D - FF^\dagger CD^\dagger D + \frac{1}{2}FF^\dagger CD^\dagger D - FF^\dagger KD^\dagger D \\ &= M. \end{aligned}$$

Finally, from Remark 1 and (2.19) we get

$$\begin{aligned}
X &= A^\dagger M D^\dagger \\
&= A^\dagger C D^\dagger - A^\dagger F F^\dagger C D^\dagger + \frac{1}{2} A^\dagger F F^\dagger C D^\dagger - A^\dagger F F^\dagger K D^\dagger \\
&= \frac{1}{2} A^\dagger C D^\dagger + \frac{1}{2} B^\dagger C^* (F^*)^\dagger - \frac{1}{4} B^\dagger D^\dagger D C^* (F^*)^\dagger \\
&\quad + \frac{1}{2} B^\dagger D^\dagger D K^* (F^*)^\dagger + \frac{1}{2} W^* - \frac{1}{2} B^\dagger B W^* F^* (F^*)^\dagger \\
&\quad + \frac{1}{4} A^\dagger (F^*)^\dagger F^* C D^\dagger + \frac{1}{2} A^\dagger (F^*)^\dagger F^* K D^\dagger - \frac{1}{2} U^* + \frac{1}{2} A^\dagger A U^* D D^\dagger \\
&\quad - A^\dagger F F^\dagger C D^\dagger + \frac{1}{2} A^\dagger F F^\dagger C D^\dagger - A^\dagger F F^\dagger K D^\dagger \\
&= \frac{1}{2} A^\dagger C D^\dagger + \frac{1}{2} B^\dagger C^* (F^*)^\dagger - \frac{1}{4} A^\dagger F F^\dagger C D^\dagger - \frac{1}{4} B^\dagger D^\dagger D C^* (F^*)^\dagger \\
&\quad - \frac{1}{2} A^\dagger F F^\dagger K D^\dagger + \frac{1}{2} B^\dagger D^\dagger D K^* (F^*)^\dagger \\
&\quad - \frac{1}{2} U^* + \frac{1}{2} A^\dagger A U^* D D^\dagger + \frac{1}{2} W^* - \frac{1}{2} B^\dagger B W^* F^\dagger F.
\end{aligned}$$

□

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] H. Braden, *The equations $A^T X \pm X^T A = B$*, SIAM J. Matrix Anal. Appl. **20** (1998), 295–302.

[2] C. Chiang, E. King-Wah Chu, Lin Wen-Wei, *On the $\star$-Sylvester equation $AX \pm X^\star B^\star = C$*, Appl. Math. Comput. **218** (2012), 8393–8407.

[3] D.S. Cvetković-Ilić, *The solutions of some operator equations,* J. Korean Math. Soc. **45 5** (2008), 1417–1425.

[4] D.S. Djordjevic, *Explicit solution of the operator equation $A^*X + X^*A = B$*, J. Comput. Appl. Math. **200** (2007), 701–704.

[5] G.R. Duan, *The solution to the matrix equation AV+BW=EVJ+R,* Appl. Math. Lett. **17** (2004), 1197–1202.

[6] X. Fang, J. Yu, H. Yao, *Solutions to operator equations on Hilbert C^*-modules,* Linear Alg. Appl. **431** (2009), 2142–2153.

[7] L.R. Fletcher, J. Kuatsky, N.K. Nichols, *Eigenstructure assignment in descriptor systems*, IEEE Trans. Auto. Control **31 12** (1986), 1138–1141.

[8] P.M. Frank, *Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy a survey and some new results,* Automatica **26** (1990), 459–474.

[9] K.K. Jensen, K. Thomsen, *Elements of KK-Theory,* Birkhäuser, Cambridge, Mass, USA, 1991.

[10] I. Kaplansky, *Modules over operator algebras,* Amer J. Math. **75** (1953), 839–858.

[11] E.C. Lance, *Hilbert C^*-Modules, LMS Lecture Note Series 210,* Cambridge Univ. Press, 1995.

[12] X.Z. Ma, J.H. Han, F.X. Piao, *A study of solution existence for matrix equation $AX+X^TC = B$*, J. Shenyang Inst. Aero. Eng. **20** (2003), 64–66.

[13] M. Mohammadzadeh Karizaki, D. Djordjevic, *Solutions to some solvable modular operator equations*, Funct. Anal. Approx. Comput. **8 1** (2016), 7–13.

[14] M. Mohammadzadeh Karizaki, M. Hassani, D. Djordjevic, *The solutions to the operator equation $TXS^* - SX^*T^* = A$ in Hilbert C^*-modules*, Intern. J. Nonlinear Anal. Appl. **7 2** (2016), 127–132.

[15] M. Mohammadzadeh Karizaki, M. Hassani, S.S. Dragomir, *Explicit solution to modular operator equation $TXS^* - SX^*T^* = A$*, Kragujevac J. Math. **40 2** (2016), 280–289.

[16] F. Piao, Q. Zhanga, Z. Wang, *The solution to matrix equation $AX + X^TC = B$*, J. Franklin. I. **344** (2007), 1056–1062.

[17] F. de Teran, F. Dopico, *Consistency and efficient solution of the Sylvester equation for *-congruence: $AX + X^\star B = C$*, Electron. J. Linear Alg. **22** (2011), 849–863.

[18] M. Wang, X. Cheng, M. Wei, *Iterative algorithms for solving the matrix equation $AXB + CX^TD = E$,* Appl. Math. Comput. **187 2** (2007), 622–629.

[19] L. Xie, Y. Liu, H. Yang, *Gradient based and least squares based iterative algorithms for matrix equations $AXB + CX^TD = F$,* Appl. Math. Comput. **217** (2010), 2191–2199.

[20] Q. Xu, L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C*-modules*, Linear Alg. Appl. **428** (2008), 992–1000.

[21] Q. Xu, L. Sheng, Y. Gu, *The solutions to some operator equations*, Linear Alg. Appl. **429** (2008), 1997–2024.

[22] Y. Yuan, *Solvability for a class of matrix equation and its applications*, J. Nanjing Univ. (Math. Biquart.) **18** (2001), 221–227.

Zahra Niazi Moghani
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: zahra_nm79@yahoo.com

Mahnaz Khanehgir
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: khanehgir@mshdiau.ac.ir

Mehdi Mohammadzadeh Karizaki
Department of Computer Engineering, University of Torbat Heydarieh, Torbat Heydarieh, Iran
E-mail address: m.mohammadzadeh@torbath.ac.ir