Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 10 Issue 1 (2019), Pages 10-18.

# RECTANGULAR $M_b$-METRIC SPACES AND FIXED POINT RESULTS

MOHAMMAD ASIM, A. RAUF KHAN AND MOHAMMAD IMDAD

Abstract. In this paper, we enlarge the class of rectangular $M$-metric spaces by considering the class of rectangular $M_b$-metric spaces and utilize the same to prove an analogue of Banach contraction principle in such spaces. Our main result extends and improves some relevant results of the existing literature. Finally, we adopt an example to highlight the utility of our main result.

## 1. Introduction

Banach contraction principle continues to inspire researchers to prove new results enriching the principle in several ways. One possible way is to improve this principle by enlarging the class of spaces. In 1993, Stefan Czerwik [6] extensively used the concept of $b$-metric space by replacing triangular inequality with a relatively more general condition which is also utilized to extend Banach contraction theorem. By now there already exists considerable literature in $b$-metric spaces and for the work of this kind one can consult Imdad et al. [8], Mustafa [11], Suzuki [13], Wong [14], Piri-Afshari [12] and similar others.

In 2000, Branciari [1] extended the idea of metric space by replacing the triangular inequality with a relatively more general inequality namely: quadrilateral inequality which involves four points instead of three and utilized this relatively larger class to prove an analogue of Banach contraction theorem. In 2008, George et al. [7] further enlarged the class of rectangular metric spaces by introducing the the class of rectangular $b$-metric spaces and proved an analogue of a Banach contraction principle in such spaces.

In 2014, Asadi et al. [3] enlarged the class of the partial metric spaces (see [10]) by introducing $M$-metric spaces. Soon, Mlaiki et al. [4] introduced $M_b$-metric spaces and utilized the same to prove fixed point results. Very recently, in an attempt to extend the "rectangular metric spaces" and "$M$-metric spaces" Özgür [5] introduced rectangular $M$-metric spaces.

2000 *Mathematics Subject Classification.* 35A07, 35Q53.
*Key words and phrases.* Bourgain spaces; KdV equation; local smoothing effect.

Submitted November 1, 2018. Published November 29, 2018.
Communicated by Wasfi Shatanawi.

Inspired by the concepts of $M_b$-metric space and rectangular $M$-metric space, we introduce rectangular $M_b$-metric space and utilize the same to prove fixed point result in such spaces. Finally, we furnish an example to establish the genuineness of our newly proved result.

## 2. Preliminaries

In this section, we begin with some notions and definitions which are needed in our subsequent discussions.

**Notation 1.** [3] The following notations will be utilized in the sequel.

(1) $m_{x,y} = \min\{m(x,x), m(y,y)\}$,
(2) $M_{x,y} = \max\{m(x,x), m(y,y)\}$,
(3) $m_{b_{x,y}} = \min\{m_b(x,x), m_b(y,y)\}$,
(4) $M_{b_{x,y}} = \max\{m_b(x,x), m_b(y,y)\}$,
(5) $m_{r_{x,y}} = \min\{m_r(x,x), m_r(y,y)\}$,
(6) $M_{r_{x,y}} = \max\{m_r(x,x), m_r(y,y)\}$.

**Definition 2.1.** [3] Let $X$ be a non-empty set. A mapping $m : X \times X \to \mathbb{R}_+$ is said to be a $M$-metric, if $m$ satisfies the following (for all $x, y, z \in X$):

(1) $m(x,x) = m(x,y) = m(y,y)$ if and only if $x = y$,
(2) $m_{x,y} \leq m(x,y)$,
(3) $m(x,y) = m(y,x)$,
(4) $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$.

Then the pair $(X, m)$ is said to be a $M$-metric space.

**Definition 2.2.** [4] Let $X$ be a non-empty set. A mapping $m_b : X \times X \to \mathbb{R}_+$ is said to be a $M_b$-metric with coefficient $s \geq 1$, if $m_b$ satisfies the following (for all $x, y, z \in X$):

(1) $m_b(x,x) = m_b(x,y) = m_b(y,y)$ if and only if $x = y$,
(2) $m_{b_{x,y}} \leq m_b(x,y)$,
(3) $m_b(x,y) = m_b(y,x)$,
(4) $(m_b(x,y) - m_{b_{x,y}}) \leq s[(m_b(x,z) - m_{b_{x,z}}) + (m_b(z,y) - m_{b_{z,y}})] - m_b(z,z)$.

Then the pair $(X, m_b)$ is said to be a $M_b$-metric space.

**Definition 2.3.** [5] Let $X$ be a non-empty set. A mapping $m_r : X \times X \to \mathbb{R}_+$ is said to be a rectangular $M$-metric, if $m_r$ satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$,):

(1) $m_r(x,x) = m_r(x,y) = m_r(y,y)$ if and only if $x = y$,
(2) $m_{r_{x,y}} \leq m_r(x,y)$,
(3) $m_r(x,y) = m_r(y,x)$,
(4) $(m_r(x,y) - m_{r_{x,y}}) \leq (m_r(x,u) - m_{r_{x,u}}) + (m_r(u,v) - m_{r_{u,v}}) + (m_r(v,y) - m_{r_{v,y}})$.

Then the pair $(X, m_r)$ is said to be a rectangular $M$-metric space.

Özgür et al. [5] proved the following:

**Theorem 2.1.** *Let $(X, m_r)$ be a rectangular $M$-metric space. Suppose, $f : X \to X$ satisfies the following conditions:*

*(i) for all $x, y \in X$, we have*

$$m_r(fx, fy) \leq \lambda m_r(x, y)$$

*where $\lambda \in [0, \frac{1}{s})$,*

*(ii) $(X, m_r)$ is complete.*

*Then $f$ has a unique fixed point $x$ such that $m_r(x, x) = 0$.*

## 3. Results

In this section, we introduce definition of a rectangular $M_b$-metric space. We also establish a fixed point theorem besides deducing natural corollaries. But first we introduce the following notation:

**Notation 4.**

(1) $m_{rb_{x,y}} = \min\{m_{rb}(x, x), m_{rb}(y, y)\}$,

(2) $M_{rb_{x,y}} = \max\{m_{rb}(x, x), m_{rb}(y, y)\}$.

**Definition 3.1.** Let $X$ be a non-empty set. A mapping $m_{rb} : X \times X \to \mathbb{R}_+$ is said to be a $M_b$-metric with coefficient $s \geq 1$, if $m_{rb}$ satisfies the following (for all $x, y, z \in X$ and all distinct $u, v \in X \setminus \{x, y\}$):

($1m_{rb}$) $m_{rb}(x, x) = m_r(x, y) = m_{rb}(y, y)$ if and only if $x = y$,

($2m_{rb}$) $m_{rb_{x,y}} \leq m_{rb}(x, y)$,

($3m_{rb}$) $m_{rb}(x, y) = m_{rb}(y, x)$,

($4m_{rb}$) $(m_{rb}(x, y) - m_{rb_{x,y}}) \leq s[(m_{rb}(x, u) - m_{rb_{x,u}}) + (m_{rb}(u, v) - m_{rb_{u,v}}) + (m_{rb}(v, y) - m_{rb_{v,y}})] - m_{rb}(u, u) - m_{rb}(v, v)$.

Then the pair $(X, m_{rb})$ is said to be a rectangular $M_b$-metric space.

The following example demonstrates that the results proved in this paper are genuinely new.

**Example 3.1.** Let $X = [0, \infty)$ and $p > 1$ any positive integer. Define $m_{rb} : X \times X \to \mathbb{R}_+$ by (for all $x, y \in X$):

$$m_{rb}(x, y) = \max\{x, y\}^p + |x - y|^p.$$

By routine calculation one can check that, $m_{rb}$ is a rectangular $M_b$-metric space with coefficient $s = 3^{p-1}$. Now, we show that $(X, m_{rb})$ is not rectangular $M$-metric space. By taking, $x = 1,\ u = 2,\ v = 3$ and $y = 4$, we have

$$m_{rb}(1, 4) - m_{rb_{1,4}} = \max\{1, 4\}^p + |1 - 4|^p - 1 = 4^p + 3^p - 1,$$

$$m_{rb}(1, 2) - m_{rb_{1,2}} = \max\{1, 2\}^p + |1 - 2|^p - 1 = 2^p + 1 - 1 = 2^p,$$

$$m_{rb}(2, 3) - m_{rb_{2,3}} = \max\{2, 3\}^p + |2 - 3|^p - 1 = 3^p + 1 - 2^p,$$

$$m_{rb}(3, 4) - m_{rb_{3,4}} = \max\{3, 4\}^p + |3 - 4|^p - 1 = 4^p + 1 - 3^3.$$

Therefore,

$$(m_{rb}(x, y) - m_{rb_{x,y}}) > (m_{rb}(x, u) - m_{rb_{x,u}}) + (m_{rb}(u, v) - m_{rb_{u,v}}) + (m_{rb}(v, y) - m_{rb_{v,y}}).$$

Hence, $(X, m_{rb})$ is not rectangular $M$-metric space.

In a rectangular $M_b$-metric space, the concepts of basic topological notions, such as: Cauchy sequence, convergent sequence and complete rectangular $M_b$-metric space can be easily adopted as under.

**Definition 3.2.** A sequence $\{x_n\}$ in $(X, m_{rb})$ is said to be $m_{rb}$-convergent to $x \in X$ if and only if

$$\lim_{n\to\infty}(m_{rb}(x_n, x) - m_{rb_{x_n,x}}) = 0.$$

**Definition 3.3.** A sequence $\{x_n\}$ in $(X, m_{rb})$ is said to be $m_{rb}$-Cauchy if and only if

$$\lim_{n,m\to\infty}(m_{rb}(x_n, x_m) - m_{rb_{x_n,x_m}}) \text{ and } \lim_{n,m\to\infty}(M_{rb_{x_n,x_m}} - m_{rb_{x_n,x_m}})$$

exists and finite.

**Definition 3.4.** A rectangular $M_b$-metric space $(X, m_{rb})$ is said to be a $m_{rb}$-complete if every $m_{rb}$-Cauchy in $X$ is $m_{rb}$-convergent to some point in X.

Firstly, we present the following lemma which is needed in the sequel.

**Lemma 3.1.** *Let $(X, m_{rb})$ be a rectangular $M_b$-metric space with coefficient $s \geq 1$ and $f : X \to X$ a self mapping on $X$. If there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$m_{rb}(fx, fy) \leq \lambda m_{rb}(x, y) \tag{3.1}$$

*and consider the sequence $\{x_n\}$ define by $x_{n+1} = fx_n$. If $x_n \to x$ as $n \to \infty$, then $fx_n \to fx$ as $n \to \infty$.*

*Proof.* If $m_{rb}(fx_n, fx) = 0$, then $m_{rb_{fx_n,fx}} \leq m_{rb}(fx_n, fx) = 0$, which implies that $m_{rb}(fx_n, fx) - m_{rb_{fx_n,fx}} \to 0$ as $n \to \infty$ and $fx_n \to fx$ as $n \to \infty$.

Otherwise, if $m_{rb}(fx_n, fx) > 0$, then by (3.1), we have $m_{rb}(fx_n, fx) \leq \lambda m_{rb}(x_n, x)$. Here, we distinguish two cases as under:

**Case 1.** If $m_{rb}(x, x) \leq m_{rb}(x_n, x_n)$, then using (3.1), one can easily show that $m_{rb}(x_n, x_n) \to 0$ as $n \to \infty$, which implies that $m_{rb}(x, x) = 0$ and since $m_{rb}(fx, fx) < m_{rb}(x, x) = 0$, we obtain that $m_{rb}(fx, fx) = 0$. By the definition of $m_{rb}$-convergent of a sequence $\{x_n\}$, which converges to $x$, we have

$$\lim_{n\to\infty}(m_{rb}(x_n, x) - m_{rb_{x_n,x}}) = 0.$$

Since $m_{rb_{x_n,x}} = \min\{m_{rb}(x_n, x_n), m_{rb}(x_n, x_n)\}$ so that $m_{rb_{x_n,x}} \to 0$ as $n \to \infty$ and henceforth $m_{rb}(x_n, x) \to 0$ as $n \to \infty$. Thus, $m_{rb}(fx_n, fx) < m_{rb}(x_n, x) \to 0$ as $n \to \infty$. Therefore, $m_{rb}(fx_n, fx) - m_{rbfx_n,fx} \to 0$ as $n \to \infty$ and thus $fx_n \to fx$ as $n \to \infty$.

**Case 2.** If $m_{rb}(x, x) \geq m_{rb}(x_n, x_n)$, then again $m_{rb}(x_n, x_n) \to 0$ as $n \to \infty$, which implies that $m_{rbx_n,x} \to 0$ as $n \to \infty$. Hence, $m_{rb}(x_n, x) \to 0$ as $n \to \infty$. As $m_{rb}(fx_n, fx) < m_{rb}(x_n, x) \to 0$ as $n \to \infty$, we have $m_{rb}(fx_n, fx) - m_{rbfx_n,fx} \to 0$ so that $fx_n \to fx$ as $n \to \infty$. □

Now, we state and prove our main result as follows:

**Theorem 3.2.** *Let $(X, m_{rb})$ be a rectangular $M_b$-metric space with coefficient $s \geq 1$. Suppose $f : X \to X$ satisfies the following conditions:*

*(i) for all $x, y \in X$, we have*

$$m_{rb}(fx, fy) \leq \lambda m_{rb}(x, y) \tag{3.2}$$

*where $\lambda \in [0, \frac{1}{s})$,*

*(ii) $(X, m_{rb})$ is $m_{rb}$-complete.*

*Then $f$ has a unique fixed point $x$ such that $m_{rb}(x, x) = 0$.*

*Proof.* Assume that $x_0 \in X$ and construct an iterative sequence $\{x_n\}$ by:

$$x_1 = fx_0,\ x_2 = f^2x_0,\ x_3 = f^3x_0, ..., x_n = f^nx_0, ...$$

Now, we assert that $\lim_{n\to\infty} m_{rb}(x_n, x_{n+1}) = 0$. On setting $x = x_n$ and $y = x_{n+1}$ in (3.2), we get

$$\begin{aligned} m_{rb}(x_n, x_{n+1}) &= m_{rb}(fx_{n-1}, fx_n) \\ &\leq \lambda m_{rb}(x_{n-2}, x_{n-1}) \\ &\leq \lambda^{n-1} m_{rb}(x_0, x_1), \end{aligned}$$

which on making $n \to \infty$, gives rise

$$\lim_{n\to\infty} m_{rb}(x_n, x_{n+1}) = 0.$$

Similarly, from condition (3.2), we get

$$m_{rb}(x_n, x_n) = m_{rb}(fx_{n-1}, fx_{n-1}) \leq \lambda m_{rb}(x_{n-1}, x_{n-1}) \leq ... \leq \lambda^{n-1} m_{rb}(x_0, x_0).$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} m_{rb}(x_n, x_n) = 0. \tag{3.3}$$

Firstly, we show that $x_n \neq x_m$ for any $n \neq m$. Let on contrary that, $x_n = x_m$ for some $n > m$, then we have $x_{n+1} = fx_n = fx_m = x_{m+1}$. On using (3.2) with $x = x_m$ and $y = x_{m+1}$, we have

$$m_{rb}(x_m, x_{m+1}) = m_{rb}(x_n, x_{n+1}) < m_{rb}(x_{n-1}, x_n) < ... < m_{rb}(x_m, x_{m+1}),$$

a contradiction. This in turn yields that $x_n \neq x_m$ for all $n \neq m$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in $(X, m_{rb})$. In doing so, we distinguish two cases as under:

**Case 1.** Firstly, let $p$ is odd, that is $p = 2m + 1$ for any $m \geq 1$. Now using $(4m_{rb})$ for any $n \in \mathbb{N}$, we have

$$\begin{aligned} m_{rb}(x_n, x_{n+p}) &\leq s[m_{rb}(x_n, x_{n+1}) + m_{rb}(x_{n+1}, x_{n+2}) + m_{rb}(x_{n+2}, x_{n+p})] \\ &\quad -m_{rb}(x_{n+1}, x_{n+1}) - m_{rb}(x_{n+2}, x_{n+2}) \\ &\leq s[\lambda^n m_{rb}(x_0, x_1) + \lambda^{n+1} m_{rb}(x_0, x_1)] + s m_{rb}(x_{n+2}, x_{n+2m+1}) \\ &\quad -\lambda^{n+1} m_{rb}(x_0, x_0) - \lambda^{n+2} m_{rb}(x_0, x_0) \\ &= s(\lambda^n + \lambda^{n+1}) m_{rb}(x_0, x_1) + s m_{rb}(x_{n+2}, x_{n+2m+1}) \\ &\quad -(\lambda^{n+1} + \lambda^{n+2}) m_{rb}(x_0, x_0) \\ &\leq s(\lambda^n + \lambda^{n+1}) m_{rb}(x_0, x_1) + s^2(\lambda^{n+2} + \lambda^{n+3}) m_{rb}(x_0, x_1) + ... + \\ &\quad s^m(\lambda^{n+2m-2} + \lambda^{n+2m-1}) m_{rb}(x_0, x_1) + s^m \lambda^{n+2m} m_{rb}(x_0, x_1) \\ &\quad -(\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + ...) m_{rb}(x_0, x_0) \\ &= \big(s\lambda^n(1 + s\lambda^2 + s^2\lambda^4 + ...) + s\lambda^{n+1}(1 + s\lambda^2 + s^2\lambda^4 + ...)\big) m_{rb}(x_0, x_1) \\ &= \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n m_{rb}(x_0, x_1) - \frac{\lambda^{n+1}}{1-\lambda} m_{rb}(x_0, x_0), \end{aligned}$$

yielding thereby

$$m_{rb}(x_n, x_{n+2m+1}) \leq \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n m_{rb}(x_0, x_1) - \frac{\lambda^{n+1}}{1-\lambda} m_{rb}(x_0, x_0). \tag{3.4}$$

Letting $n \to \infty$ in equation (3.4), we conclude that

$$\lim_{n,m\to\infty} m_{rb}(x_n, x_{n+2m+1}) = 0.$$

**Case 2.** Secondly, assume that $p$ is even, that is $p = 2m$ for any $m \geq 1$. Then

$$\begin{aligned}
m_{rb}(x_n, x_{n+p}) &\leq s[m_{rb}(x_n, x_{n+1}) + m_{rb}(x_{n+1}, x_{n+2}) + m_{rb}(x_{n+2}, x_{n+p})] \\
&\quad -m_{rb}(x_{n+1}, x_{n+1}) - m_{rb}(x_{n+2}, x_{n+2}) \\
&\leq s[\lambda^n m_{rb}(x_0, x_1) + \lambda^{n+1} m_{rb}(x_0, x_1)] + s m_{rb}(x_{n+2}, x_{n+2m}) \\
&\quad -\lambda^{n+1} m_{rb}(x_0, x_0) - \lambda^{n+2} m_{rb}(x_0, x_0) \\
&= s(\lambda^n + \lambda^{n+1}) m_{rb}(x_0, x_1) + s m_{rb}(x_{n+2}, x_{n+2m}) \\
&\quad -(\lambda^{n+1} + \lambda^{n+2}) m_{rb}(x_0, x_0) \\
&\leq s(\lambda^n + \lambda^{n+1}) m_{rb}(x_0, x_1) + s^2(\lambda^{n+2} + \lambda^{n+3}) m_{rb}(x_0, x_1) + ... + \\
&\quad s^{m-1}(\lambda^{n+2m-4} + \lambda^{n+2m-3}) m_{rb}(x_0, x_1) + s^{m-1}\lambda^{n+2m-2} m_{rb}(x_0, x_2) \\
&\quad +s^{m-1}\lambda^{n+2m-2} m_{rb}(x_0, x_2) - (\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + ...) m_{rb}(x_0, x_0) \\
&= \big(s\lambda^n(1 + s\lambda^2 + s^2\lambda^4 + ...) + s\lambda^{n+1}(1 + s\lambda^2 + s^2\lambda^4 + ...)\big) m_{rb}(x_0, x_1) \\
&= \frac{1+\lambda}{1-s\lambda^2} s\lambda^n r_\xi(x_0, x_1) + s^{m-1}\lambda^{n+2m-2} m_{rb}(x_0, x_2) - \frac{\lambda^{n+1}}{1-\lambda} m_{rb}(x_0, x_0),
\end{aligned}$$

so that

$$m_{rb}(x_n, x_{n+2m}) \leq \frac{1+\lambda}{1-s\lambda^2} s\lambda^n m_{rb}(x_0, x_1) + s^{m-1}\lambda^{n+2m-2} m_{rb}(x_0, x_2) - \frac{\lambda^{n+1}}{1-\lambda} m_{rb}(x_0, x_0). \tag{3.5}$$

Taking the limit as $n \to \infty$, in (3.5), we conclude that

$$\lim_{n,m\to\infty} m_{rb}(x_n, x_{n+2m}) = 0.$$

Therefore, in both the cases, we have

$$\lim_{n,m\to\infty} (m_{rb}(x_n, x_m) - m_{rb_{x_n,x_m}}) = 0.$$

On the other hand, without loss of generality we may assume that

$$M_{rb_{x_n,x_m}} = m_{rb}(x_n, x_n).$$

Hence, we obtain

$$\begin{aligned}
M_{rb_{x_n,x_m}} - m_{rb_{x_n,x_m}} &\leq M_{rb_{x_n,x_m}} \\
&= m_{rb}(x_n, x_n) \\
&\leq \lambda^n m_{rb}(x_0, x_0).
\end{aligned}$$

Taking the limit of the above inequality as $n \to \infty$, we deduce that

$$\lim_{n,m\to\infty} (M_{rb_{x_n,x_m}} - m_{rb_{x_n,x_m}}) = 0.$$

Therefore, the sequence $\{x_n\}$ is $m_{rb}$-Cauchy in $X$. Since $X$ is $m_{rb}$-complete, there exists $x \in X$ such that $x_n \to x$. Now, we show that $fx = x$. By Lemma 3.1, we have

$$\begin{aligned}
\lim_{n,m\to\infty}(m_{rb}(x_n,x)-m_{rb_{x_n,x}}) &= 0\\
&= \lim_{n,m\to\infty}(m_{rb}(x_{n+1},x)-m_{rb_{x_{n+1},x}})\\
&= \lim_{n,m\to\infty}(m_{rb}(fx_n,x)-m_{rb_{fx_n,x}})\\
&= \lim_{n,m\to\infty}(m_{rb}(fx,x)-m_{rb_{fx,x}}).
\end{aligned}$$

Hence, we find $m_{rb}(fx,x)=m_{rb_{x,fx}}$. Since $m_{rb_{x,fx}}=\min\{m_{rb}(x,x),m_{rb}(fx,fx)\}$. Therefore, $m_{rb_{x,fx}}=m_{rb}(x,x)$ or $m_{rb_{x,fx}}=m_{rb}(fx,fx)$ which implies that $fx=x$. Next, we show the uniqueness of the fixed point of $f$. Assume that $f$ has two fixed points $x,y\in X$, that is, $fx=x$ and $fy=y$. Thus

$$m_{rb}(x,y)=m_{rb}(fx,fy)\le\lambda m_{rb}(x,y)<m_{rb}(x,y),$$

which implies that $m_{rb}(x,y)=0$ and hence, $x=y$.
Finally, we show that if $x$ is a fixed point, then $m_{rb}(x,x)=0$. To accomplish this, let $x$ be a fixed point of $f$ then

$$\begin{aligned}
m_{rb}(x,x) &= m_{rb}(fx,fx)\\
&\le \lambda m_{rb}(x,x)\\
&< m_{rb}(x,x),
\end{aligned}$$

yielding thereby $m_{rb}(x,x)=0$. This concludes the proof. □

Now, we present an example which illustrates the utility of our newly proved result:

**Example 3.2.** Let $X=\{1,3,5,7\}$. Define $m_{rb}(x,y)=\left(\frac{x+y}{2}\right)^2$ with $s=3$, for all $x,y\in X$. Firstly, we show that $(X,m_{rb})$ is a rectangular $M_b$-metric space. It is easy to see that the conditions $(1m_{rb})$-$(3m_{rb})$ are satisfied. Now, to verify condition $(4m_{rb})$, we distinguish five cases as under:

**Case 1.** If $x=1,y=3,u=5$ and $v=7$, then we have

$$m_{rb}(x,y)-m_{rb_{x,y}}=(2)^2-1^2=3,$$
$$3[m_{rb}(x,u)-m_{rb_{x,u}}+m_{rb}(u,v)-m_{rb_{u,v}}+m_{rb}(v,y)-m_{rb_{v,y}}]=105,$$
$$m_{rb}(u,u)+m_{rb}(u,u)=(5)^2+(7)^2=74.$$

**Case 2.** If $x=1,y=5,u=3$ and $v=7$, then we have

$$m_{rb}(x,y)-m_{rb_{x,y}}=(3)^2-1^2=8,$$
$$3[m_{rb}(x,u)-m_{rb_{x,u}}+m_{rb}(u,v)-m_{rb_{u,v}}+m_{rb}(v,y)-m_{rb_{v,y}}]=90,$$
$$m_{rb}(u,u)+m_{rb}(u,u)=(3)^2+(7)^2=58.$$

**Case 3.** If $x=1,y=7,u=3$ and $v=5$, then we have

$$m_{rb}(x,y)-m_{rb_{x,y}}=(4)^2-1^2=15,$$
$$3[m_{rb}(x,u)-m_{rb_{x,u}}+m_{rb}(u,v)-m_{rb_{u,v}}+m_{rb}(v,y)-m_{rb_{v,y}}]=63,$$
$$m_{rb}(u,u)+m_{rb}(u,u)=(3)^2+(5)^2=34.$$

**Case 4.** If $x=3,y=5,u=7$ and $v=1$, then we have

$$m_{rb}(x,y)-m_{rb_{x,y}}=(4)^2-(3)^2=7,$$
$$3[m_{rb}(x,u)-m_{rb_{x,u}}+m_{rb}(u,v)-m_{rb_{u,v}}+m_{rb}(v,y)-m_{rb_{v,y}}]=117,$$

$$m_{rb}(u,u)+m_{rb}(u,u)=(7)^2+(1)^2=50.$$

**Case 5.** If $x=3, y=7, u=1$ and $v=5$, then we have

$$m_{rb}(x,y)-m_{rb_{x,y}}=(3)^2+(2)^2=16,$$

$$3[m_{rb}(x,u)-m_{rb_{x,u}}+m_{rb}(u,v)-m_{rb_{u,v}}+m_{rb}(v,y)-m_{rb_{v,y}}]=66,$$

$$m_{rb}(u,u)+m_{rb}(u,u)=(1)^2+(5)^2=26.$$

Then $(X,m_{rb})$ is a $m_{rb}$-complete rectangular $M_b$-metric space. Consider a mapping $f:X\to X$ defined by:

$$f=\begin{pmatrix}1&3&5&7\\1&1&1&3\end{pmatrix}.$$

In particular, if we take $x,y\in\{1,3,5\}$, then $fx=fy=1$ and hence one can easily check that condition (3.2) satisfy (for some $\lambda\in[0,\frac{1}{3})$)

$$m_{rb}(fx,fy)\le\lambda m_{rb}(x,y).$$

Now, by taking $x\in\{1,3,5\}$ and $y=7$, then (by (3.2)), we have

$$\left(\frac{1+3}{2}\right)^2\le\lambda\left(\frac{x+7}{2}\right)^2,$$

therefore

$$m_{rb}(fx,fy)\le\lambda m_{rb}(x,y).$$

Observe that, all the conditions of Theorem 3.2 are satisfied and $x=1$ is a unique fixed point of the involved map $f$.

The following corollary deduced form Theorem 3.2 remains a new result which is genuinely sharpened version of Theorem 2.1 due to Özgür et al. [5].

**Corollary 3.3.** Let $(X,m_{rb})$ be a rectangular $M_b$-metric space with coefficient $s=1$ and $f:X\to X$ satisfying the following condition:

(i) for all $x,y\in X$, we have

$$m_{rb}(fx,fy)\le\lambda m_{rb}(x,y)$$

where $\lambda\in[0,1)$,

(ii) $(X,m_{rb})$ is $m_{rb}$-complete.

Then $f$ has a unique fixed point $x$ such that $m_{rb}(x,x)=0$.

## References

[1] A. Branciari. A fixed point theorem of BanachCaccioppoli type on a class of generalized metric spaces, *Publ. Math.,* 57, 31-37, (2000).

[2] L. B. *Ćirić*. A generalization of Banach contraction principle, *Proc. Am. Math. Soc.*, 45, 267273, (1974).

[3] M. Asadi, E. Karapinar and P. Salimi. New extension of p-metric spaces with some fixed-point results on M-metric spaces, *J. Inequal. Appl.*, 18, (2014).

[4] N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer and T. Abdeljawed. Fixed Point Theorems in $M_b$-metric spaces, *J. Math. Anal.*, 7, 1-9, (2016).

[5] N. Y. Özgür, N. Mlaiki, N. Tas and N. Souayah. A new generalization of metric spaces: rectangular M-metric spaces, *Mathematical Sciences*, 12, 223-233, (2018).

[6] S. Czerwik. Contraction mappings in $b$-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1(1):5–11, 1993.

[7] R. George, S. Radenovic, K.P. Reshma, and S. Shukla. Rectangular b-metric space and contraction principles, *J. Nonlinear Sci. Appl.*, 8, 10051013, (2015).

[8] M. Imdad, M. Asim and R. Gubran. Common fixed point theorems for g-Generalized contractive mappings in b-metric spaces. *Indian Journal of Mathematics*, 60(1), 85-105 (2018).
[9] T. Kamran, M. Samreen and O.U. Ain. A Generalization of b-Metric Space and Some Fixed Point Theorems. *Mathematics*, 5, 19, (2017).
[10] S.G. Matthews. Partial metric topology. *Proc. 8th Summer Conference on General Topology and Application. Ann. New York Acad. Sci.* 728, 183197, (1994).
[11] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg. Some common fixed point results in ordered partial b-metric spaces. *Journal of Inequalities and Applications*, 2013(1):562, 2013.
[12] H. Piri and P. Kumam. Some fixed point theorems concerning F-contraction in complete metric spaces. *Fixed Point Theory and Applications*, 2014:210:1–11, October 2014.
[13] T. Suzuki. Generalized distance and existence theorems in complete metric spaces. *Journal of Mathematical Analysis and Applications*, 253(2):440–458, 2001.
[14] C. S. Wong. Generalized contractions and fixed point theorems. *Proceedings of the American Mathematical Society*, 42(2):409–417, 1974.

Mohammad Asim
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
*E-mail address*: mailtoasim27@gmail.com

A. Rauf Khan
Department of Applied sciences, University Polytechnic, Aligarh Muslim University, Aligarh-202002, India.
*E-mail address*: abdurrauf@rediffmail.com

Mohammad Imdad
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.
*E-mail address*: mhimdad@gmail.com