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THE C^*-ALGEBRA VALUED CARISTI'S FIXED POINT THEOREM: A GRAPHIC VERSION

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Abstract. In this paper, we introduce the notion of G-continuity and G-lower semi continuity on C^*-valued metric space. Moreover, we present an extension of the Caristi's fixed point theorem in the setting of a C^*-valued metric space, endowed with a graph. It also extends some old extensions of Banach contraction principle in metric spaces with graph.

1. Introduction

The Banach fixed point theorem (BFT) [7] is a expedient tool to prove the existence of unique solution of many non-linear differential/integral equations. It states that a self mapping T on a complete metric space M has a unique fixed point provided that there exists a $\kappa \in (0,1)$ such that

$$d(Ta_1, Ta_2) \leq \kappa d(a_1, a_2) \quad \text{for all } a_1, a_2 \in M. \tag{1.1}$$

Since the contractive condition (1.1) holds for all $(a_1, a_2) \in M \times M$, one innovative direction to generalize the BFT is to impose a suitable condition on ordered pairs from $M \times M$ such that if (1.1) holds only on a subset of $M \times M$ then the mapping T still has a fixed point. This work is motivated by some recent work on extensions of Banach contraction principlr to metric space with partial order. As a first step Ran and Reurings [33] proved that the mapping T has a fixed point if in addition M is a partially ordered set and the contractive condition (1.1) holds for those ordered pairs from $M \times M$ whose elements are comparable. Jachymski [18], give a sweeping idea of extending this result and replaced the ordered pairs with the edges of a graph, set of whose vertices coincides with the metric space M. He showed that the mapping T has a fixed point if the contractive condition holds for those ordered pairs whose elements are the edges of the graph. Afterwards, many researchers extended his idea in different directions, see for example ([3], [4], [5], [17], [20], [35] and [36]). Many thought-provoking generalizations of a metric space had been introduced by different authors in the quest to generalize the BFT, see for example an excellent survey by An *et al.* [6]. Ma *et al.* [26] recently acquaint with an interesting generalized metric space which he named as C^*-algebra-valued

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metric space or C^*-valued metric space and established the BFP with this new setting. Batul and Kamran [8], and Shehwar and Kamran [37] further generalized some results by Ma *et al.* [26]. Later many researchers followed the same technique to prove certain fixed point results [27, 28]. One may also generalize the BFT by weakening the contractive condition (1.1). The initiative in this direction was taken by Edelstein [14, 15] and Raktoch [32]. Then a number of results ([1], [11], [22], [31]) were obtained by various authors by weakening the condition (1.1). Excellent review articles e.g. Collaco and Silva [12] , and Rhoades [34], give a comparison of different contractive conditions and corresponding results, obtained by several authors. In this regard, Caristi's fixed point theorem ([10], [23], [24]) is a precious extension of the BFT. This crucial result says that if (M, d) is a complete metric space and $T : M \to M$ a self mapping, then T has a fixed point if there exists a lower semi continuous map $\phi : M \to [0, \infty)$ such that

$$d(m, Tm) \leq \phi(m) - \phi(Tm) \quad \text{for all } m \in M. \tag{1.2}$$

This general fixed point result has found many applications in nonlinear analysis. It is shown, for example, that this theorem yeilds essentially all the inwardness results [16] of geometric fixed point theory in Banach spaces. The inwardness conditions are ones which asserts that points in domain are mapped in domain. Possibly, the most weakest of the inwardness conditions is the Leray-Schauder boundary condition, is the assumption that a map points x of any where except to outward part of the ray originating at some interior point of M through x.
The proof of Caristi's theorem use different techniques [9, 25]. It is noteworthy that Caristi's result has a closed connection with Ekeland's variational principle.
Recently, Shehwar *et al.* [38] gave a C^*-valued metric space version of the Caristi's fixed point theorem. The proof given by them uses the idea of minimal elements.

In the present paper, we give the graphic version of the Caristi's fixed point theorem in context of C^*-valued metric spaces. The main technique of the proof is different from that of [38]. Moreover, the results have been illustrated by constructing a nontrivial example for each of the novel results.

2. Preliminaries

In this section, we recollect some definitions and results which will be utilized in the rest of the article. Following Jachymski [18], we will use Δ as the diagonal of the Cartesian product $M \times M$ in a metric space (M, d). We will consider G as a directed graph whose vertices set $V(G)$ is same as M and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We adopt the perception that G has no parallel edges so we can recognize G with the pair $(V(G), E(G))$. Moreover, sometimes we may treat G as weighted graph by assigning a weight to each of the edge which can be the distance between its vertices.

Now, we conjure up some imperative notions from C^*-algebra that may be found in [13]. A complex algebra $\mathcal{A}$ is called a $*$-algebra with conjugate linear involution $*$ such that for any $a, b \in \mathcal{A}$, $a^{**} = a$ and $(ab)^* = b^*a^*$. In addition, if $\mathcal{A}$ is a Banach space and for $a \in \mathcal{A}$, $\|a^*a\| = \|a\|^2$, then $\mathcal{A}$ is called a C^*-algebra. The set $\mathrm{S}_p(a) = \{\lambda \in \mathbb{C} : \lambda I - a \text{ is noninvertible}\}$, where I is multiplicative identity of $\mathcal{A}$, is called spectrum of an element $a \in \mathcal{A}$, . An element $a \in \mathcal{A}$ is called positive element of $\mathcal{A}$ if a is self adjoint i.e., $a = a^*$ and $\mathrm{S}_p(a) \subset [0, \infty)$. The set $\mathcal{A}_+$ denotes the set of positive elements in $\mathcal{A}$. We will write $a \succeq b$ iff $a - b \in \mathcal{A}_+$. Each positive element a of a C^*-algebra has a unique positive square root. If a and b are self conjugate elements of a C^*-algebra and $\theta \preceq a \preceq b$ then $\|a\| \preceq \|b\|$, where θ is the zero element of the C^*-algebra $\mathcal{A}$.

Let M be a nonempty set. A mapping $d : M \times M \to \mathcal{A}$ is called a C^*-valued metric [26] on M if it satisfies the following conditions: i) $d(a_1, a_2) \succeq \theta$ for all $a_1, a_2 \in M$; ii) $d(a_1, a_2) = \theta \Leftrightarrow a_1 = a_2$; iii) $d(a_1, a_2) = d(a_2, a_1)$ for all $a_1, a_2 \in M$; iv) $d(a_1, a_2) \preceq d(a_1, a_3) + d(a_3, a_2)$ for all $a_1, a_2, a_3 \in M$. Then $(M, \mathcal{A}, d)$ is called a C^*-valued metric space. Let $a \in (M, \mathcal{A}, d)$, a sequence $\{a_n\}$ in $(M, \mathcal{A}, d)$ is said to be convergent with respect to $\mathcal{A}$, if for any $\epsilon > 0$ there exists a positive integer N such that $\|d(a_n, a)\| \leqslant \epsilon$ for all $n > N$. A sequence $\{a_n\}$ is called a Cauchy sequence with respect to $\mathcal{A}$ if for any $\epsilon > 0$ there exists a positive integer N such that $\|d(a_n, a_m)\| \leqslant \epsilon$ for all $n, m > N$. If every Cauchy sequence with respect to $\mathcal{A}$ is convergent then $(M, \mathcal{A}, d)$ is said to be a complete C^*-valued metric space. Let $(M, \mathcal{A}, d)$ be a C^*-valued metric space. A mapping $T : M \to M$ is said to be a C^*-algebra-valued contraction mapping [26] on M if there exists an $A \in \mathcal{A}$ with $\|A\| < 1$ such that $d(Ta_1, Ta_2) \preceq A^* d(a_1, a_2) A$, for all $a_1, a_2 \in M$. A mapping $g : M \to \mathcal{A}$ is said to be lower semi continuous [8] at a_0 with respect to with respect to $\mathcal{A}$ if $\|g(x_0)\| \leq \lim_{x \to x_0} \inf \|g(x)\|$.

3. Main Results

We begin this section by introducing couple of definitions in the context of a C^*-valued metric space.

Definition 3.1. Consider a C^*-valued metric space $(M, \mathcal{A}, d)$ accompanied with a graph $G = (V(G), E(G))$. Let the set of its vertices be $V(G) = M$. A mapping $T : M \to M$ is said to be G-continuous if for each sequence $\{a_n\}$ in M such that $(a_n, a_{n+1}) \in E(G)$ and $a_n \to a$ as $n \to \infty$, we have $Ta_n \to Ta$ as $n \to \infty$.

Definition 3.2. Consider a C^*-valued metric space $(M, \mathcal{A}, d)$ accompanied with a graph $G = (V(G), E(G))$ with $V(G) = M$. We say that a mapping $g : M \to \mathcal{A}_+$ is G-lower semi continuous with respect to $\mathcal{A}_+$, if for each sequence $\{a_n\}$ in M such that $(a_n, a_{n+1}) \in E(G)$ and for $a_n \to a$ as $n \to \infty$, we have $\|g(a)\| \leq \lim\limits_{n\to\infty} inf \|g(a_n)\|$.

We are now ready to state the first main result of this paper.

Theorem 3.3. *Let $(M, \mathcal{A}, d)$ be a complete C^*-valued metric space endowed with the graph $G = (V(G), E(G))$ with $V(G) = M$. Let $T : M \to M$ be an edge preserving mapping and $\phi : M \to \mathbb{A}_+$ be a G-lower semi continuous mapping satisfying the following conditions:*

- **:** (i) *for each $a \in M$ we have*

$$d(a, Ta) \preceq \phi(a) - \phi(Ta) \quad \text{whenever } (a, Ta) \in E(G), \tag{3.1}$$

- **:** (ii) *there exists $a_0 \in M$ such that $(a_0, Ta_0) \in E(G)$,*
- **:** (iii) *T is G-continuous.*

Then T has a fixed point.

Proof. By hypothesis (ii), we have $a_0 \in M$ such that $(a_0, Ta_0) \in E(G)$. Let $Ta_0 = a_1$, then from (i) we get $d(a_0, a_1) \preceq \phi(a_0) - \phi(a_1)$. Since T is edge preserving, $(a_1, Ta_1) \in E(G)$. Let $Ta_1 = a_2$. Continuing in this way, we get a sequence $\{a_n\}$ in M such that $(a_n, a_{n+1}) \in E(G)$ and

$$d(a_n, a_{n+1}) \preceq \phi(a_n) - \phi(a_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{3.2}$$

This implies that

$$\phi(a_{n+1}) \preceq \phi(a_n). \tag{3.3}$$

Therefore, $\{\phi(a_n)\}$ is a non-increasing sequence in $\mathcal{A}_+$. Thus there exists an $a \succeq \theta$ such that

$$\lim_{n\to\infty} \phi(a_n) = a. \tag{3.4}$$

Let $m, n \in \mathbb{N}$ with $m > n$. By using triangular inequality and (3.2), we have

$$\begin{aligned} d(a_n, a_m) &\preceq d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \cdots d(a_{m-1}, a_m) \\ &\preceq \phi(a_n) - \phi(a_{n+1}) + \phi(a_{n+1}) - \phi(a_{n+2}) + \cdots + \phi(a_{m-1}) - \phi(a_m) \\ &\preceq \phi(a_n) - \phi(a_m). \end{aligned}$$

Since $d(a_n, a_m)$ and $\phi(a_n) - \phi(a_m)$ are positive elements of the C^*-algebra $\mathcal{A}$, it further implies that

$$\|d(a_n, a_m)\| \le \|\phi(a_n) - \phi(a_m)\|.$$

Now taking (3.4) into account and letting $n \to \infty$, we have

$$\lim_{n,m\to\infty} \|d(a_n, a_m)\| = 0.$$

Therefore, $\{a_n\}$ is a Cauchy sequence in M and by completeness of M, we have $a_n \to a^* \in M$. G-continuity of T further implies that

$$a^* = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} T(a_n) = T(a^*).$$

□

Example 3.4. Let $M = \mathbb{C}$ and consider the algebra $\mathcal{A} = \mathbb{C}^2$ with point-wise operations of addition, scalar multiplication and multiplication. Note that positive elements for algebra of complex numbers are non negative reals ([13] ,[29]) and hence $\mathcal{A}_+ = \{(z_1, z_2) : z_1,\ z_2 \text{ are non-negative reals}\}$. Let $G = (V(G), E(G))$ be a graph with $V(G) = M$ and

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) : n = 1, 2, \cdots \right\} \cup \Big\{(\xi, \xi) : \xi \in \mathbb{R}\Big\}. \tag{3.5}$$

Note that $\|w\| = \max(|z_1|, |z_2|)$ where $w = (z_1, z_2)$ defines a norm on $\mathcal{A}$ and $* : \mathcal{A} \to \mathcal{A}$, given by $w^* = (\bar{z_1}, \bar{z_2})$, defines a convolution on $\mathbb{C}^2$. Further,

$$\begin{aligned} \|ww^*\| &= \| \left(|z_1|^2, |z_2|^2\right) \| &(3.6)\\ &= \max\left(|z_1|^2, |z_2|^2\right) &(3.7)\\ &= (\max(|z_1|, |z_2|)^2 &(3.8)\\ &= \|w\|^2. &(3.9) \end{aligned}$$

Thus $\mathcal{A}$ becomes a C^*-algebra. Define $d : M \times M \to \mathcal{A}_+$ by $d(z_1, z_2) = (|z_1 - z_2|, 0)$. Clearly $(M, \mathcal{A}, d)$ is a complete C^*-algebra valued metric space. Now define a G-lower semi continuous map $\phi : M \to \mathcal{A}_+$ by $\phi(z) = (|z|, 0)$ and $T : M \to M$ by $Tz = \dfrac{z^2}{2}$. We have

$$d\left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) = \left(\left|\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right|, 0\right) \tag{3.10}$$

$$= \left(\left|\frac{1}{2^n}\right| - \left|\frac{1}{2^{2n+1}}\right|, 0\right) \tag{3.11}$$

$$= \phi\left(\frac{1}{2^n}\right) - \phi\left(\frac{1}{2^{2n+1}}\right). \tag{3.12}$$

Hence all the conditions of Theorem 3.3 are satisfied and T has a fixed point. Taking $z_1 = 2 + i,\ z_2 = 2 - i,\ d(Tz_1, Tz_2) = \left(\left|\frac{(2+i)^2}{2} - \frac{(2-i)^2}{2}\right|, 0\right) = (4, 0) = \left(\sqrt{2}, 0\right) d(z_1, z_2) \left(\sqrt{2}, 0\right)$. Note that $d(Tz_1, Tz_2) \preceq a^* d(z_1, z_2) a$ only when $\|a\| \geq \| \left(\sqrt{2}, 0\right) \| > 1$. Thus the contractive condition introduced by Ma et al. [26] is not satisfied.

The second main result of this article, is as follows.

Theorem 3.5. *Let $(M, \mathcal{A}, d)$ be a complete C^*-valued metric space endowed with the graph $G = (V(G), E(G))$ with $E(G) = M$. Let $T : M \to M$ be an edge preserving map and let $\phi : M \to \mathcal{A}_+$ be a lower semi continuous mapping such that for each $a \in M$ we have,*

$$d(Ta, T^2a) \preceq \phi(a) - \phi(Ta) \quad \textit{whenever } (a, Ta) \in E(G). \tag{3.13}$$

Assume that

- : (a) *there exists $a_0 \in M$ such that $(a_0, Ta_0) \in E(G)$,*
- : (b) *there exists a function $g : M \to \mathcal{A}_+$ defined $g(a) = d(a, Ta)$ which is G-lower semi continuous.*

Then T has a fixed point.

Proof. By hypothesis (a) we have $a_0 \in M$ such that $(a_0, Ta_0) \in E(G)$. Let $a_1 = Ta_0$, then from (3.13), we have

$$d(Ta_0, T^2a_0) \preceq \phi(a_0) - \phi(Ta_0)$$
$$\Rightarrow d(a_1, Ta_1) \preceq \phi(a_0) - \phi(a_1).$$

Since T is edge preserving and $(a_0, a_1) \in E(G) \ \Rightarrow \ (Ta_0, Ta_1) = (a_1, a_2) \in E(G)$ with $Ta_1 = a_2$. Continuing in the same way, we get a sequence $\{a_n\}$ such that $a_{n+1} = Ta_n$ with $(a_n, a_{n+1}) \in E(G)\ \forall\ n \in \mathbb{N}$ and

$$d(a_n, a_{n+1}) \preceq \phi(a_{n-1}) - \phi(a_n). \tag{3.14}$$

Now working on the same lines as in the proof of Theorem 3.3, it follows that $\{a_n\}$ is a Cauchy sequence in M. Since M is complete, so we have $a_n \to a^* \in M$ with $(a_n, a_{n+1}) \in E(G)$. By hypothesis (b), $g(a) = d(a, Ta)$ is G-lower semi continuous.

Thus we have

$$\begin{aligned}\|d(a^*, Ta^*)\| = \|g(a^*)\| &\le \lim_{n\to\infty} \inf \|g(a_n)\| \\ &= \lim_{n\to\infty} \inf \|d(a_n, Ta_n)\| \\ &\le \lim_{n\to\infty} \inf \|\phi(a_{n-1}) - \phi(a_n)\| = 0.\end{aligned}$$

Hence $Ta^* = a^*$. □

Example 3.6.
Let $\mathcal{A} = M_{2\times 2}(\mathbb{R})$ be the algebra of all 2×2 matrices with real entries and usual matrix operations. Define $d : \mathbb{R} \times \mathbb{R} \to \mathcal{A}$ by

$$d(x, y) = \begin{pmatrix} |\xi - \eta| & 0 \\ 0 & |\xi - \eta| \end{pmatrix}. \tag{3.15}$$

It is easy to check that $(\mathbb{R}, \mathcal{A}, d)$ is a complete C^*-valued metric space. Define $T : \mathbb{R} \to \mathbb{R}$ by

$$T\xi = \frac{\xi^2}{2},$$

and consider the graph $G = (V(G), E(G))$, where $V(G) = \mathbb{R}$ and

$$E(G) = \left\{ \left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) : n = 1, 2, \cdots \right\} \cup \Big\{(\xi, \xi) : \xi \in \mathbb{R}\Big\}. \tag{3.16}$$

Note that, for each $n \in \mathbb{N}$,

$$\left(T\frac{1}{2^n}, T\frac{1}{2^{2n+1}}\right) = \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{4n+3}}\right) \in E(G).$$

Define $\phi : M \to \mathbb{A}_+$ by

$$\phi(\xi) = \begin{pmatrix} |\xi| & 0 \\ 0 & |\xi| \end{pmatrix}. \tag{3.17}$$

It can be easily seen that

$$\begin{aligned} d\left(\frac{1}{2^n}, \frac{1}{2^{2n+1}}\right) &= \begin{pmatrix} \left|\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right| & 0 \\ 0 & \left|\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right| \end{pmatrix} \\ &= \phi\left(\frac{1}{2^n}\right) - \phi\left(\frac{1}{2^{2n+1}}\right). \end{aligned}$$

Hence all the conditions of Theorem 3.5 are satisfied and as a result 0 is the fixed point of T. Note that the contractive condition introduced by Ma et al. [26] is not satisfied here, for example, at $\xi = 0,\ \eta = 5$.

Remark 3.7. One of the stimulating application of metric fixed point theorem is its use in finding the solutions for differential equations. The general approach is converting such equations to a problem of finding fixed point of certain mapping. In most of the cases, the metric space for such mappings is, in fact, a function space. A suitable distance function on such function spaces allows us to easily use Banach contraction principle or one of the many other fixed point theorems, to solve such problems. But, such function spaces inherit a natural 'order' from the order on $\mathbb{R}$. This important feature of these function spaces is mostly ignored while applying conventional fixed point theorems by focusing only on the underlying metric structure. In [30, 33], the authors have used order together with the metric conditions and establish some interesting examples of solving certain boundary value problems. Consider the following periodic boundary value problem

$$x'(\tau) = f(\tau, x(\tau)), \qquad \tau \in [0, P], \tag{3.18}$$

where $P > 0$ and $f : [0, P] \times \mathbb{R} \to \mathbb{R}$ is continuous function. Clearly any solution of this problem must be continuously differentiable on $[0, P]$. So the space to consider for this problem is $C^1([0, P], \mathbb{R})$. The above problem can be written as integral equation

$$x(\tau) = \int_0^R H(\tau, s) \left(f(s, x(s) + \lambda x(s)) \right) ds, \tag{3.19}$$

where $\lambda > 0$ and the green function is given by

$$H(\tau, s) = \begin{cases} \dfrac{e^{\lambda(P+s-\tau)}}{e^{\lambda P - 1}} & 0 \le s < \tau \le P, \\ \dfrac{e^{\lambda(s-\tau)}}{e^{\lambda P - 1}} & 0 \le \tau < s \le P. \end{cases}$$

Define a mapping $T : C([0, P], \mathbb{R}) \to C([0, P], \mathbb{R})$ by

$$T(x)(\tau) = \int_0^R H(\tau, s) \left(f(s, x(s) + \lambda x(s)) \right) ds. \tag{3.20}$$

Where $x(\tau) \in C([0, P], \mathbb{R})$ is fixed point of T, then $x(\tau) \in C^1([0, P], \mathbb{R})$ is the solution to the boundary value problem. Under appropriate assumptions, the mapping T satisfies the following properties:

(1) if $x(\tau) \le y(\tau)$, then we have $T(x) \le T(y)$;
(2) if $x(\tau) \le y(\tau)$, then $\|T(x) - T(y)\| \le k\|x - y\|$ for a constant $k < 1$.

Note that the above contractive condition does not hold for all functions in $C([0, P], \mathbb{R})$ but it is valid only for the functions that are comparable. Therefore, in [30] authors have to use a weaker version of Banach contraction principle to prove the existence of the solution of (3.18). This motivates us to establish our main result in the setting of C^*-algebra valued metric spaces.

Remark 3.8. Recently, Kadelburg and Radenovic [31] and Alsulami et al. [19] have observed that the results established by Ma et al. [26] are direct consequences of the corresponding results in standard metric spaces. Their observation is based on the same results from C^*-algebras that are used in [26]. Our results in this paper follow the techniques used by Ma et al. to establish a complete and detailed proof of Theorem 3.3. A short version of the proof can also be obtained by using the idea presented in [19, 31].

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Author's contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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