Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 9 Issue 4(2018), Pages 39-52.

# NEW FORMULAE BETWEEN SQUARES OF SOME JACOBI POLYNOMIALS AND SQUARES OF CERTAIN FRACTIONAL JACOBI FUNCTIONS

WALEED ABD-ELHAMEED

Abstract. This paper is dedicated to deriving a new formula expressing explicitly the squares of Jacobi polynomials of certain parameters in terms of the squares of certain fractional Jacobi functions. The derived formula is given in terms of values of terminating hypergeometric functions of the type ${}_5F_4$ at the point $x = 1$. Several reduced formulae which are given in forms free of any hypergeometric functions can be deduced with the aid of some standard reduction formulae such as Pfaff-Saalschütz's and Watson's identities. Moreover, some other reduced formulae can be deduced by means of some algorithmic methods, and in particular, the algorithms of Zeilberger, Petkovsek and van Hoeij.

## 1. Introduction

The Jacobi polynomials are of fundamental importance in theoretical mathematical analysis as well as applied analysis. There are six important families in the general class of Jacobi polynomials. The four families, namely, Legendre, ultraspherical and Chebyshev polynomials of first and second kinds are symmetric Jacobi polynomials (i.e., the two indices in such cases coincide), however the two families of third and fourth kinds are nonsymmetric Jacobi polynomials. All the six special families of the general class of Jacobi polynomials have their roles and importance from theoretical and practical points of view. In this respect, there is a huge number of articles employing the four kinds of symmetric Jacobi polynomials in various applications. For example, the Legendre polynomials are utilized for solving linear

2000 *Mathematics Subject Classification.* 33F10, 33C20, 33Cxx, 68W30.
*Key words and phrases.* Jacobi polynomials, fractional Jacobi functions, generalized hypergeometric functions, symbolic computation.

Submitted February 23, 2016. Published August 19, 2018.
Communicated by Virginia Kiryakova.

and nonlinear sixth-order two point boundary value problems via an elegant harmonic numbers operational matrix of derivatives in Abd-Elhameed [1]. Moreover, there are several articles employ ultraspherical and Chebyshev polynomials of the first and second kinds in several applications, (see, for example, [26, 11, 14]). Recently, some authors are interested in using Chebyshev polynomials of the third and fourth kinds in obtaining spectral solutions for some boundary value problems (see, for example [12, 13]).

Fractional calculus is a pivotal branch of mathematical analysis. There are old and recent intensive studies concerning fractional calculus theoretically and practically. The fractional calculus is interested in investigating the derivatives and integrals to an arbitrary order (real or complex order). The fractional calculus is crucial in many disciplines such as economics, physics, and statistics. For example, the half-order derivatives and integrals are used in the formulation of certain electrochemical problems. For some applications, see [7, 23, 41, 21].

The hypergeometric functions play a prominent role in the area of special functions and their applications (see, for example Askey [4]). Many relations and transformation formulae between special functions are given in terms of certain kinds of these functions. Moreover, almost all of the elementary functions of mathematics are either hypergeometric or ratios of hypergeometric functions.

There are several articles interested in developing new formulae between various orthogonal polynomials. Most of these formulae are given in terms of hypergeometric functions. In this respect, the two problems of linearizing products of various orthogonal polynomials and the connection coefficients between them are of interest. There is a number of old and recent articles in this direction. For example, the two problems of linearization and connection coefficients of ultraspherical and Jacobi polynomials have been studied by many authors, see, for instance, Askey and Gasper [5], Gasper [18, 19], Hylleraas [22], Rahman [34], Chaggara and Koepf [6], Abd-Elhameed et al. [2], Doha and Abd-Elhameed [10]. For some other studies about the linearization and connection coefficients for various orthogonal polynomials, see for example, Doha [8, 9], , Doha and Ahmed [15], Markett [27], Maroni and da Rocha [28], Sánchez-Ruiz [36], Sánchez-Ruiz and Dehesa [37], and Tcheutia [38].

The main aim of this research paper is to establish some new formulae relating Jacobi polynomials of certain parameters with the squares of some fractional Jacobi functions. Moreover, linearization formulae of Jacobi polynomials of certain parameters can be also obtained.

The paper is organized as follows. Section 2 is dedicated to presenting some preliminaries including some useful properties of Jacobi polynomials, fractional Jacobi functions and also transformation formulae between certain hypergeometric functions. In Section 3, we are interested in establishing a new formula relating Jacobi polynomials of certain parameters with the squares of certain fractional Jacobi functions. With the aid of some standard formulae, or via utilizing algorithmic methods, such as Zeilberger's and Petkovsek's algorithms, some formulae written explicitly in reduced forms which are free of any hypergeometric functions are deduced.

## 2. Preliminaries and useful transformations

This section is dedicated to presenting some useful properties of Jacobi polynomials and fractional Jacobi functions. Moreover, important standard formulae of hypergeometric functions and also transformation formulae between certain types of hypergeometric functions are presented.

2.1. **Some properties of the classical Jacobi polynomials.** The Jacobi polynomials $P_n^{(\mu,\nu)}(x)$ (see, Andrews et al. [3], and Olver et al. [32]), provide a class of polynomials defined on $[-1,1]$ by the following Rodrigues' type formula:

$$(1-x)^{\mu}\,(1+x)^{\nu}\,P_n^{(\mu,\nu)}(x)=\frac{(-1)^n}{2^n\,n!}\,\frac{d^n}{dx^n}\left[(1-x)^{n+\mu}(1+x)^{n+\nu}\right],$$

or in terms of the Gauss hypergeometric function

$$P_n^{(\mu,\nu)}(x)=\frac{(\mu+1)_n}{n!}\;{}_2F_1\left(\begin{array}{c}-n,n+\mu+\nu+1\\ \mu+1\end{array}\middle|\,\frac{1-x}{2}\right),$$

where

$$(z)_k=\frac{\Gamma(z+k)}{\Gamma(z)}.$$

We recall here that the generalized hypergeometric function is defined as

$${}_pF_q\left(\begin{array}{c}[a_p]\\ [b_q]\end{array}\middle|\,x\right)=\sum_{m=0}^{\infty}\frac{(a_1,a_2,\ldots,a_p)_m\;x^m}{(b_1,b_2,\ldots,b_p)_m\;m!},$$

where the symbols $[a_p]$ and $[b_q]$ denote, respectively, to the two sets $\{a_1,a_2,\ldots,a_p\}$ and $\{b_1,b_2,\ldots,b_q\}$ of real or complex parameters, where $b_j\neq 0$, for all $1\leq j\leq q$, $(a_1,a_2,\ldots,a_p)_m=\prod_{i=1}^{p}(a_i)_m$, and $(b_1,b_2,\ldots,b_p)_m=\prod_{j=1}^{q}(b_j)_m$.

It is very useful to use the normalized Jacobi polynomials defined as:

$$R_n^{(\mu,\nu)}(x)=\frac{P_n^{(\mu,\nu)}(x)}{P_n^{(\mu,\nu)}(1)}=\frac{n!}{(\mu+1)_n}P_n^{(\mu,\nu)}(x)={}_2F_1\left(\begin{array}{c}-n,n+\mu+\nu+1\\ \mu+1\end{array}\middle|\,\frac{1-x}{2}\right). \tag{2.1}$$

These polynomials satisfy the following orthogonality relation on $[-1,1]$:

$$\int_{-1}^{1}(1-x)^{\mu}(1+x)^{\nu}\ R_i^{(\mu,\nu)}\ (x)R_j^{(\mu,\nu)}(x)\,dx=\begin{cases}0, & j\neq i,\\ h_i^{(\mu,\nu)}, & j=i,\end{cases} \tag{2.2}$$

where

$$h_i^{(\mu,\nu)}=\frac{2^{\mu+\nu+1}\ i!\ \Gamma(i+\nu+1)\ [\Gamma(\mu+1)]^2}{(2i+\mu+\nu+1)\,\Gamma(i+\mu+\nu+1)\,\Gamma(i+\mu+1)}. \tag{2.3}$$

The following six orthogonal polynomial families can be obtained as direct special cases of the normalized Jacobi polynomials as follows:

$$\begin{aligned} C_n^{(\alpha)}(x)&=R_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x), & T_n(x)&=R_n^{(-\frac{1}{2},-\frac{1}{2})}(x),\\ U_n(x)&=(n+1)\,R_n^{(\frac{1}{2},\frac{1}{2})}(x), & V_n(x)&=R_n^{(-\frac{1}{2},\frac{1}{2})}(x),\\ W_n(x)&=(2n+1)\,R_n^{(\frac{1}{2},-\frac{1}{2})}(x), & P_n(x)&=R_n^{(0,0)}(x), \end{aligned}$$

where $C_n^{(\alpha)}(x)$, $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$ and $P_n(x)$ are the ultraspherical, Chebyshev of the first, second, third and fourth kinds, and Legendre polynomials, respectively.

The following identity is also important:

$$R_n^{(\mu,\nu)}(-x)=\frac{(-1)^n\,\Gamma(\mu+1)\,\Gamma(n+\nu+1)}{\Gamma(\nu+1)\,\Gamma(n+\mu+1)}\ R_n^{(\nu,\mu)}(x). \tag{2.4}$$

For more properties of the classical Jacobi polynomials in general and their special polynomials in particular, one can be referred to the useful books of Andrews et al. [3] and Mason and Handscomb [29].

2.2. **Fractional Jacobi functions.** This subsection concentrates on presenting some properties of the fractional Jacobi functions. For more properties of the fractional derivatives and integrals, one can be referred, for example to [31, 33].

The fractional Jacobi functions are defined as (see, Mirevski et al. [30], and Gogovcheva and Boyadjiev [20])

**Definition 1.** *For all $\alpha>0,\mu>-1,\nu>-1$, the fractional Jacobi functions $P_\alpha^{(\mu,\nu)}(x)$ are defined as*

$$P_\alpha^{(\mu,\nu)}(x)=\frac{(-2)^{-\alpha}}{\Gamma(\alpha+1)}\,(1-x)^{-\mu}(1+x)^{-\nu}\,D^{\alpha}\left[(1-x)^{\alpha+\mu}(1+x)^{\alpha+\nu}\right]. \tag{2.5}$$

Also, Gogovcheva and Boyadjiev in [20] have stated and proved the following two theorems.

**Theorem 1.** *The fractional Jacobi functions defined in* (2.5) *have the following power series representation:*

$$P_\alpha^{(\mu,\nu)}(x) = 2^{-\alpha} \sum_{m=0}^{\infty} \binom{\alpha+\mu}{\alpha-m} \binom{\alpha+\nu}{m} (x-1)^m \, (x+1)^{\mu-m}, \tag{2.6}$$

*where*

$$\binom{\mu}{\nu} = \frac{\Gamma(\mu+1)}{\Gamma(\nu+1)\,\Gamma(\mu-\nu+1)},$$

*is the binomial coefficient with real arguments.*

**Theorem 2.** *The Gauss hypergeometric representation for the Jacobi functions defined in* (2.5) *is given by*

$$P_\alpha^{(\mu,\nu)}(x) = \binom{\alpha+\mu}{\alpha} \, {}_2F_1 \left( \begin{array}{c} -\alpha, \alpha+\mu+\nu+1 \\ \mu+1 \end{array} \middle| \, \frac{1-x}{2} \right). \tag{2.7}$$

**Remark 1.** *It is worthy to mention here that the properties of Jacobi functions are similar to those of the classical Jacobi polynomials.*

The following properties of the fractional Jacobi functions are also of interest:

(1) The fractional Jacobi functions satisfy the following linear differential equation of the second order:

$$(1-x^2)y''(x) + [\nu - \mu - (\mu+\nu+2)\,x]\; y'(x) + \alpha(\alpha+\mu+\nu+1)\, y(x) = 0. \tag{2.8}$$

(2) For every integer $m \geq 1$ and $m-1 \leq \alpha < m$, the following properties hold:
(i) $\lim\limits_{\alpha \to m} P_\alpha^{(\mu,\nu)}(x) = P_m^{(\mu,\nu)}(x)$.
(ii) $P_\alpha^{(\mu,\nu)}(1) = \binom{\alpha+\mu}{\alpha}$.
(iii) $P_\alpha^{(\mu,\nu)}(-1) = \binom{\alpha+\nu}{\alpha}$.

For more properties of the fractional Jacobi functions, see, [30] and [20].
Now, we introduce the so-called normalized fractional Jacobi functions defined as:

$$R_\alpha^{(\mu,\nu)}(x) = \frac{1}{\binom{\alpha+\mu}{\alpha}} \, P_\alpha^{(\mu,\nu)}(x).$$

These functions of course generalize the normalized Jacobi polynomials $R_m^{(\mu,\nu)}(x)$ defined in (2.1). In addition, the following six fractional indices functions are special cases of the fractional Jacobi functions:

$$C_\alpha^{(\mu)}(x) = R_\alpha^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(x), \qquad T_\alpha(x) = R_\alpha^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$U_\alpha(x) = (\alpha+1)\, R_\alpha^{(\frac{1}{2},\frac{1}{2})}(x), \qquad V_\alpha(x) = R_\alpha^{(-\frac{1}{2},\frac{1}{2})}(x),$$

$$W_\alpha(x) = (2\,\alpha+1)\, R_\alpha^{(\frac{1}{2},-\frac{1}{2})}(x), \qquad P_\alpha(x) = R_\alpha^{(0,0)}(x),$$

where $C_{\alpha}^{(\alpha)}(x),\ T_{\alpha}(x),\ U_{\alpha}(x),\ V_{\alpha}(x),\ W_{\alpha}(x)$ and $P_{\alpha}(x)$ are the fractional ultraspherical, Chebyshev of the first, second, third and fourth kinds, and the fractional Legendre functions.

**Remark 2.** *All relations and properties of the normalized Jacobi polynomials can be easily transformed to give their counterparts of the normalized fractional Jacobi functions. In this respect, the Gauss hypergeometric representation for the normalized fractional Jacobi functions is*

$$R_{\alpha}^{(\mu,\nu)}(x) = {}_2F_1\left(\begin{array}{c} -\alpha, \alpha+\mu+\nu+1 \\ \mu+1 \end{array}\middle|\ \frac{1-x}{2}\right). \tag{2.9}$$

2.3. **Some transformation formulae.** The following four theorems are of great importance in the sequel.

**Theorem 3.** *Pfaff-Saalschütz identity (see,* [16, 32]*)*
*For $c+d=a+b+1-r,\ r=0,1,2,\ldots,$ one has*

$${}_3F_2\left(\begin{array}{c} -r, a, b \\ c, d \end{array}\middle|\ 1\right) = \frac{(c-a)_r\ (c-b)_r}{(c)_r\ (c-a-b)_r}. \tag{2.10}$$

**Theorem 4.** *Watson's identity (see,* [40]*)*

$$\begin{aligned} &{}_3F_2\left(\begin{array}{c} -r, r+2\,a+2\,b-1, a \\ 2\,a, a+b \end{array}\middle|\ 1\right) \\ &= \begin{cases} \dfrac{r!\,\Gamma(a+\frac{r}{2})\,\Gamma(b+\frac{r}{2})\,\Gamma(2\,a)\,\Gamma(a+b)}{(\frac{r}{2})!\,\Gamma(a+b+\frac{r}{2})\,\Gamma(2\,a+r)\,\Gamma(a)\,\Gamma(b)}, & r \text{ even}, \\ 0, & r \text{ odd}. \end{cases} \end{aligned} \tag{2.11}$$

**Theorem 5.** [35] *The following identity (Clausen's identity) holds:*

$${}_3F_2\left(\begin{array}{c} 2\,a, 2\,b, a+b \\ 2\,a+2\,b, a+b+\frac{1}{2} \end{array}\middle|\ w\right) = \left[{}_2F_1\left(\begin{array}{c} a, b \\ a+b+\frac{1}{2} \end{array}\middle|\ w\right)\right]^2. \tag{2.12}$$

**Theorem 6.** *For all nonnegative integers $r,p,q,r,s,t,u,$ one has (see,* [17] *and* [25]*)*

$$\begin{aligned} &{}_{p+r+1}F_{q+s}\left(\begin{array}{c} -r, [a_p], [c_r] \\ [b_q], [d_s] \end{array}\middle|\ y\,z\right) = \sum_{m=0}^{r}\binom{r}{m}\frac{(a_p)_m\ (\alpha_t)_m\ y^m}{(b_q)_m\ (\beta_u)_m\ (m+\xi)_m}\times \\ &{}_{p+t+1}F_{q+u+1}\left(\begin{array}{c} m-r, [m+a_p], [m+\alpha_t] \\ 2\,m+\xi+1, [m+b_q], [m+\beta_u] \end{array}\middle|\ y\right)\times \\ &{}_{r+u+2}F_{s+t}\left(\begin{array}{c} -m, m+\xi, [c_r], [\beta_u] \\ [d_s], [\alpha_t] \end{array}\middle|\ z\right). \end{aligned} \tag{2.13}$$

## 3. Some formulae between squares of certain Jacobi polynomials and the squares of certain fractional Jacobi functions

This section is dedicated to establishing a new formula which connects the squares of some Jacobi polynomials with the squares of certain fractional Jacobi functions. This formula is given in terms of a terminating hypergeometric function of the type ${}_5F_4(1)$. Moreover, in this section, and thanks to some standard reduction formulae, some new formulae free of any hypergeometric functions are derived in detail. In addition, some other reduced formulae are obtained via employing some computer algebra algorithms.

**Theorem 7.** *For all nonnegative integers $n$ and $k$, the following formula is valid:*

$$\begin{aligned}\left(R_n^{(\mu,-\frac{1}{2})}(x)\right)^2 &= \sum_{k=0}^{2n} \frac{\binom{2n}{k}\left(\mu+\frac{1}{2}\right)_k(\gamma+1)_k(2\gamma+1)_k(2n+2\mu+1)_k}{(\mu+1)_k(2\mu+1)_k\left(\gamma+\frac{1}{2}\right)_k(k+2\gamma+1)_k}\\ &\times\ {}_5F_4\left(\left.\begin{matrix} k-2n,\mu+k+\frac{1}{2},2\mu+k+2n+1,\gamma+k+1,2\gamma+k+1\\ \mu+k+1,2\mu+k+1,\gamma+k+\frac{1}{2},2\gamma+2k+2\end{matrix}\right|1\right)\\ &\times\left(R_{\frac{k}{2}}^{(\gamma,-\frac{1}{2})}(x)\right)^2.\end{aligned} \tag{3.1}$$

*Proof.* With the aid of Gauss hypergeometric representations of the Jacobi polynomials $R_n^{(\mu,-\frac{1}{2})}(x)$, and the fractional Jacobi functions $R_{\frac{k}{2}}^{(\gamma,-\frac{1}{2})}(x)$, we can write

$$R_n^{(\mu,-\frac{1}{2})}(x) = {}_2F_1\left(\left.\begin{matrix}-n,n+\mu+\frac{1}{2}\\ \mu+1,\end{matrix}\right|\frac{1-x}{2}\right), \tag{3.2}$$

$$R_{\frac{k}{2}}^{(\gamma,-\frac{1}{2})}(x) = {}_2F_1\left(\left.\begin{matrix}-\frac{k}{2},\frac{k}{2}+\gamma+\frac{1}{2}\\ \gamma+1,\end{matrix}\right|\frac{1-x}{2}\right), \tag{3.3}$$

and in virtue of Clausen's identity (2.12), one has

$$\left(R_n^{(\mu,-\frac{1}{2})}(x)\right)^2 = {}_3F_2\left(\left.\begin{matrix}-2n,2n+2\,\mu+1,\mu+\frac{1}{2}\\ \mu+1,2\,\mu+1\end{matrix}\right|\frac{1-x}{2}\right), \tag{3.4}$$

$$\left(R_{\frac{k}{2}}^{(\gamma,-\frac{1}{2})}(x)\right)^2 = {}_3F_2\left(\left.\begin{matrix}-k,k+2\,\gamma+1,\gamma+\frac{1}{2}\\ \gamma+1,2\,\gamma+1\end{matrix}\right|\frac{1-x}{2}\right). \tag{3.5}$$

Now, if Theorem 6 is applied accompanied with the following choices of the parameters involved in transformation (2.13):

$$p=q=2,\ r=s=0,\ t=2,\ u=1,\ [c_r]=[d_s]=\emptyset,\ a_1=2\,n+2\,\mu+1, a_2=\mu+\frac{1}{2},$$
$$b_1=2\,\mu+1,\ b_2=\mu+1,\ \alpha_1=2\,\gamma+1,\ \alpha_2=\gamma+1,\ \beta_1=\gamma+\frac{1}{2}, \xi=2\,\gamma+1,$$
$$y=1,\ z=\tfrac{1-x}{2},$$

then the following transformation formula is obtained:

$$
\begin{aligned}
&{}_3F_2\left(\begin{array}{c}-2n, 2n+2\,\mu+1, \mu+\frac{1}{2}\\ \mu+1, 2\,\mu+1\end{array}\middle|\,\frac{1-x}{2}\right)=\\
&\sum_{k=0}^{2n}\frac{\binom{2n}{k}\left(\mu+\frac{1}{2}\right)_k(\gamma+1)_k(2\gamma+1)_k(2n+2\mu+1)_k}{(\mu+1)_k(2\mu+1)_k\left(\gamma+\frac{1}{2}\right)_k(k+2\gamma+1)_k}\\
&\times\ {}_5F_4\left(\begin{array}{c}k-2n, \mu+k+\frac{1}{2}, 2\mu+k+2n+1, \gamma+k+1, 2\gamma+k+1\\ \mu+k+1, 2\mu+k+1, \gamma+k+\frac{1}{2}, 2\gamma+2k+2\end{array}\middle|\,1\right)\\
&\times\ {}_3F_2\left(\begin{array}{c}-k, k+2\,\gamma+1, \gamma+\frac{1}{2}\\ \gamma+1, 2\,\gamma+1\end{array}\middle|\,\frac{1-x}{2}\right).
\end{aligned}
\tag{3.6}
$$

Relation (3.6) together with Eqs. (3.4) and (3.5) immediately yield

$$
\begin{aligned}
&\left(R_n^{(\mu,-\frac{1}{2})}(x)\right)^2=\sum_{k=0}^{2n}\frac{\binom{2n}{k}\left(\mu+\frac{1}{2}\right)_k(\gamma+1)_k(2\gamma+1)_k(2n+2\mu+1)_k}{(\mu+1)_k(2\mu+1)_k\left(\gamma+\frac{1}{2}\right)_k(k+2\gamma+1)_k}\\
&\times\ {}_5F_4\left(\begin{array}{c}k-2n, \mu+k+\frac{1}{2}, 2\mu+k+2n+1, \gamma+k+1, 2\gamma+k+1\\ \mu+k+1, 2\mu+k+1, \gamma+k+\frac{1}{2}, 2\gamma+2k+2\end{array}\middle|\,1\right)\\
&\times\left(R_{\frac{k}{2}}^{(\gamma,-\frac{1}{2})}(x)\right)^2.
\end{aligned}
$$

Theorem 7 is now proved. □

**Corollary 1.** *In case of* $\mu=-\frac{1}{2}$ *and* $\gamma=-\frac{1}{4}$, *formula* (3.1) *reduces to*

$$
T_n^2(x)=\frac{1}{2}+\frac{1}{2}\sum_{k=0}^{2n}\frac{\binom{2n}{k}\ \left(\frac{3}{4}\right)_k\left(k+\frac{3}{4}\right)_{2n-k}\left(k-2n+\frac{3}{2}\right)_{2n-k}(2n)_k}{\left(\frac{1}{4}\right)_k\left(k+\frac{1}{2}\right)_k\ \left(2k+\frac{3}{2}\right)_{2n-k}\left(\frac{3}{4}-2n\right)_{2n-k}}\ \left(R_{\frac{k}{2}}^{(-\frac{1}{4},-\frac{1}{2})}(x)\right)^2.
\tag{3.7}
$$

*Proof.* Taking into account the Legendre's duplication formula:

$$
\frac{\Gamma(2z)}{\Gamma(z)}=\frac{2^{2\,z-1}\Gamma(z+\frac{1}{2})}{\sqrt{\pi}},
$$

It is easy to see that

$$
\lim_{\mu\to\frac{-1}{2}}\frac{(\mu+\frac{1}{2})_k}{(2\,\mu+1)_k}=\begin{cases}1 & k=0,\\ \frac{1}{2} & k\geq 1,\end{cases}
$$

and hence formula (3.1) for the case corresponds to the choice: $\mu=-\frac{1}{2}$ and $\gamma=-\frac{1}{4}$, is turned into

$$
T_n^2(x)=\frac{1}{2}+\frac{1}{2}\sum_{k=0}^{2n}\frac{\binom{2n}{k}\ \left(\frac{3}{4}\right)_k(2n)_k}{\left(\frac{1}{4}\right)_k\left(k+\frac{1}{2}\right)_k}\ {}_3F_2\left(\begin{array}{c}k-2n, k+\frac{3}{4}, k+2n\\ k+\frac{1}{4}, 2k+\frac{3}{2}\end{array}\middle|\,1\right)\ \left(R_{\frac{k}{2}}^{(-\frac{1}{4},-\frac{1}{2})}(x)\right)^2.
\tag{3.8}
$$

The ${}_3F_2(1)$ in (3.8) is one-balanced, then with the aid of Pfaff-Saalschütz identity (2.10), it can be summed to give

$$ {}_3F_2\left(\begin{array}{c} k-2n, k+\frac{3}{4}, k+2n \\ k+\frac{1}{4}, 2k+\frac{3}{2} \end{array}\middle| 1\right) = \frac{\left(k+\frac{3}{4}\right)_{2n-k}\left(k-2n+\frac{3}{2}\right)_{2n-k}}{\left(2k+\frac{3}{2}\right)_{2n-k}\left(\frac{3}{4}-2n\right)_{2n-k}}, $$

and this leads to the following formula:

$$ T_n^2(x) = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{2n} \frac{\binom{2n}{k}\ \left(\frac{3}{4}\right)_k \left(k+\frac{3}{4}\right)_{2n-k}\left(k-2n+\frac{3}{2}\right)_{2n-k}(2n)_k}{\left(\frac{1}{4}\right)_k\left(k+\frac{1}{2}\right)_k\ \left(2k+\frac{3}{2}\right)_{2n-k}\left(\frac{3}{4}-2n\right)_{2n-k}}\ \left(R_{\frac{k}{2}}^{(-\frac{1}{4},-\frac{1}{2})}(x)\right)^2. $$

Corollary 1 is proved. □

**Corollary 2.** *In case of* $\mu = \frac{-1}{2}$ *and* $\gamma = \frac{1}{2}$, *formula* (3.1) *reduces to*

$$ T_n^2(x) = \frac{1}{4}\left(2 + W_{n-1}^2(x) - 2\,W_{n-\frac{1}{2}}^2(x) + W_n^2(x)\right). \tag{3.9} $$

*Proof.* Substituting $\mu = \frac{-1}{2}$ and $\gamma = -\frac{1}{2}$ into relation (3.1) leads to the following formula:

$$ T_n^2(x) = \frac{1}{2} + \frac{n}{2}\sum_{k=0}^{2n} \frac{(k+1)^2\,(k+2n-1)!}{(2k)!\,(2n-k)!}\ {}_4F_3\left(\begin{array}{c} k-2n, k+\frac{3}{2}, k+2, k+2n \\ k+\frac{1}{2}, k+1, 2k+3 \end{array}\middle| 1\right) \times \left(R_{\frac{k}{2}}^{(\frac{1}{2},-\frac{1}{2})}(x)\right)^2. \tag{3.10} $$

It is easy to see that the ${}_4F_3(1)$ in (3.10) can be reduced to the following form:

$$ {}_4F_3\left(\begin{array}{c} k-2n, k+\frac{3}{2}, k+2, k+2n \\ k+\frac{1}{2}, k+1, 2k+3 \end{array}\middle| 1\right) = \begin{cases} 1, & \text{if } k = 2n, \\ \dfrac{-1}{2\,n}, & \text{if } k = 2n-1, \\ \dfrac{1}{2\,n(4\,n-3)}, & \text{if } k = 2n-2, \\ 0, & \text{otherwise,} \end{cases} $$

and hence, the following formula is obtained

$$ T_n^2(x) = \frac{1}{2} + \frac{(2n-1)^2}{4}\left(R_{n-1}^{(\frac{1}{2},-\frac{1}{2})}(x)\right)^2 - 2\,n^2\left(R_{n-\frac{1}{2}}^{(\frac{1}{2},-\frac{1}{2})}(x)\right)^2 + \frac{(2n+1)^2}{4}\left(R_n^{(\frac{1}{2},-\frac{1}{2})}(x)\right)^2, $$

which can be written alternatively in the form

$$ T_n^2(x) = \frac{1}{4}\left(2 + W_{n-1}^2(x) - 2\,W_{n-\frac{1}{2}}^2(x) + W_n^2(x)\right). $$

□

**Corollary 3.** *In case of* $\mu=\frac{1}{2}$ *and* $\gamma=-\frac{1}{2}$, *formula* (3.1) *reduces to:*

$$W_n^2(x)=(1-2\,n)(1+2\,n)+4\sum_{k=1}^{2n}(2n+1-k)\;T_{\frac{k}{2}}^2(x). \tag{3.11}$$

*Proof.* If we note the limit:

$$\lim_{\gamma\to\frac{-1}{2}}\frac{(2\,\gamma+1)_k}{(\gamma+\frac{1}{2})_k}=\begin{cases}1 & k=0,\\ 2 & k\geq 1,\end{cases}$$

and also the identity:

$$W_n(x)=(2n+1)\,R_n^{(\frac{1}{2},-\frac{1}{2})}(x),$$

then formula (3.1) for the case corresponds to the choice: $\gamma=-\frac{1}{2}$ and $\mu=\frac{1}{2}$, is turned into

$$\begin{aligned}W_n^2(x)=&-(2n+1)^2+2\,(2n+1)^2\sum_{k=0}^{2n}\frac{\binom{2n}{k}\;\left(\frac{1}{2}\right)_k\;(2n+2)_k}{(k+1)\;\left(\frac{3}{2}\right)_k\;(k)_k}\\ &\times\;{}_4F_3\left(\left.\begin{matrix}k-2n,k+\frac{1}{2},k+1,k+2n+2\\ k+\frac{3}{2},k+2,2k+1\end{matrix}\right|1\right)T_{\frac{k}{2}}^2(x).\end{aligned} \tag{3.12}$$

To the best of our knowledge, the ${}_4F_3(1)$ in (3.12) has no standard reduction formula in literature, but we can employ computer algebra to obtain a reduction formula for this ${}_4F_3(1)$. For this purpose, we set

$$G_{i,n}=\;{}_4F_3\left(\left.\begin{matrix}-i,-i+2n+\frac{1}{2},-i+2n+1,-i+4n+2\\ -i+2n+\frac{3}{2},-i+2n+2,-2i+4n+1\end{matrix}\right|1\right).$$

The application of Zeilberger's algorithm enables one to obtain the following recurrence relation of order two which is satisfied by $G_{i,n}$:

$$\begin{aligned}&(i-1)\,i\,(i-4n-3)\,(i-4n-2)\,G_{i-2,n}\\ &+4\,i(i-4n-2)\,(2i-4n-5)\,(i-2n-3)\,G_{i-1,n}\\ &+4(2i-4n-5)(2i-4n-3)(i-2n-3)(i-2n-2)\,G_{i,n}=0,\end{aligned} \tag{3.13}$$

with the following initial values:

$$G_{0,n}=1,\;G_{1,n}=\frac{1}{2\,n+1}.$$

The exact solution of the second-order recurrence relation (3.13) can be obtained via utilizing any suitable symbolic algorithm. For example, the algorithm of Petkovsek (see, Koepf [24]), or the improved version of van Hoeij (see, [39]) may be used for this specific purpose. Explicitly, the solution of (3.13) is given by

$${}_4F_3\left(\left.\begin{matrix}-i,-i+2n+\frac{1}{2},-i+2n+1,-i+4n+2\\ -i+2n+\frac{3}{2},-i+2n+2,-2i+4n+1\end{matrix}\right|1\right)=\frac{(i+1)!\;(4\,n-2\,i+2)!}{2\,(2n+1)\,(4\,n-i+1)!},$$

and hence the ${}_4F_3(1)$ in (3.12) has the following reduction formula:

$$ {}_4F_3\left(\left.\begin{array}{c} k-2n, k+\frac{1}{2}, k+1, k+2n+2 \\ k+\frac{3}{2}, k+2, 2k+1 \end{array}\right| 1\right) = \frac{(2\,k+2)!\;(2n-k+1)!}{2\,(2n+1)\,(2\,n+k+1)!}, \tag{3.14} $$

and hence, the following formula is obtained:

$$ W_n^2(x) = (1-2\,n)(1+2\,n) + 4\sum_{k=1}^{2n}(2n+1-k)\;T_{\frac{k}{2}}^2(x). $$

Corollary 3 is proved. □

Now, and based on the simple identity:

$$ \left(T_{\frac{k}{2}}(x)\right)^2 = \frac{1+T_k(x)}{2}, \tag{3.15} $$

the following linearization formula of Chebyshev polynomials of the fourth kind can be obtained. This result is given in the following corollary.

**Corollary 4.** *For every nonnegative integer $n$, the following linearization formula holds:*

$$ W_n^2(x) = (2n+1) + 2\sum_{k=1}^{2n}(2n+1-k)\;T_k(x). \tag{3.16} $$

**Corollary 5.** *In case of $\gamma = \mu+1$, formula (3.1) is turned into*

$$ \begin{aligned} &\left(R_n^{(\mu,-\frac{1}{2})}(x)\right)^2 = \frac{(\mu+n+1)^2(2\mu+2n+1)^2}{(\mu+1)^2(2\mu+4n+1)^2}\;\left(R_n^{(\mu+1,-\frac{1}{2})}(x)\right)^2 \\ &- \frac{n(2n-1)(2\mu+2n+1)^2}{(\mu+1)^2(2\mu+4n-1)(2\mu+4n+1)}\;\left(R_{n-\frac{1}{2}}^{(\mu+1,-\frac{1}{2})}(x)\right)^2 \\ &+ \frac{n(2n-1)\left(4\mu(\mu+2)+8n^3+2(2\mu-5)n^2+(3-2\mu)n+3\right)}{(\mu+1)^2(2\mu+4n-3)(2\mu+4n+1)^2}\;\left(R_{n-1}^{(\mu+1,-\frac{1}{2})}(x)\right)^2 \\ &+ \frac{(2\mu+1)(2\mu+3)n(2n-1)!}{(\mu+1)^2(2\mu+4n+1)\Gamma(2n+2\mu+1)}\sum_{k=0}^{2n-3}\frac{\Gamma(k+2\mu+3)}{k!\,(2\mu+2k+1)(2\mu+2k+5)}\times \\ &\left(R_{\frac{k}{2}}^{(\mu+1,-\frac{1}{2})}(x)\right)^2. \end{aligned} \tag{3.17} $$

*Proof.* Substituting $\gamma = \mu+1$, into relation (3.1) yields

$$ \begin{aligned} &\left(R_n^{(\mu,-\frac{1}{2})}(x)\right)^2 = \sum_{k=0}^{2n}\frac{\binom{2n}{k}\;\left(\mu+\frac{1}{2}\right)_k\;(\mu+2)_k\;(2\mu+3)_k\;(2n+2\mu+1)_k}{(\mu+1)_k\;\left(\mu+\frac{3}{2}\right)_k\,(2\mu+1)_k(k+2\mu+3)_k} \\ &\times\;{}_5F_4\left(\left.\begin{array}{c} k-2n, \mu+k+\frac{1}{2}, \mu+k+2, 2\mu+k+3, 2\mu+k+2n+1 \\ \mu+k+1, \mu+k+\frac{3}{2}, 2\mu+k+1, 2\mu+2k+4 \end{array}\right| 1\right) \\ &\times\left(R_{\frac{k}{2}}^{(\mu+1,-\frac{1}{2})}(x)\right)^2. \end{aligned} \tag{3.18} $$

Here, and for the sake of reducing the ${}_5F_4(1)$ in (3.18), we make use of Zeilberger's algorithm. If we set

$$M_{i,n,\mu} = {}_5F_4\left(\left.\begin{array}{c} -i, \mu-i+2n+\frac{1}{2}, 2\mu-i+4n+1, \gamma-i+2n+1, 2\gamma-i+2n+1 \\ \mu-i+2n+1, 2\mu-i+2n+1, \gamma-i+2n+\frac{1}{2}, 2\gamma-2i+4n+2 \end{array}\right| 1\right),$$

then the application of Zeilberger's algorithm enables one to obtain the following recurrence relation of order one which is satisfied by $M_{i,n,\mu}$:

$$\begin{aligned} &2\,(-2\mu+2i-4n-3)(-2\mu+2i-4n-1)(-2\mu+i-2n)(-\mu+i-2n)\,M_{i+1,n,\mu} \\ &+(i+1)(-2\mu+i-4n)(-2\mu+2i-4n-5)(-2\mu+i-2n-2)\,M_{i,n,\mu}=0, \end{aligned}$$

with the initial value $M_{0,n,\mu}=1$.

The solution of this recurrence relation is

$$M_{i,n,\mu} = {}_5F_4\left(\left.\begin{array}{c} -i, \mu-i+2n+\frac{1}{2}, 2\mu-i+4n+1, \gamma-i+2n+1, 2\gamma-i+2n+1 \\ \mu-i+2n+1, 2\mu-i+2n+1, \gamma-i+2n+\frac{1}{2}, 2\gamma-2i+4n+2 \end{array}\right| 1\right)$$

$$= \begin{cases} 1, & \text{if } i=0, \\ \dfrac{1-2n}{(\mu+n)(2\mu+4n+1)}, & \text{if } i=1, \\ \dfrac{4\mu(\mu+2)+8n^3+2(2\mu-5)n^2+(3-2\mu)n+3}{(\mu+n)(\mu+2n-1)(2\mu+2n-1)(2\mu+4n+1)^2}, & \text{if } i=2, \\ \dfrac{2(2\mu+1)(2\mu+3)\ i!\ \Gamma(2\mu-2i+4n+2)}{(2\mu+4n+1)(2\mu-i+2n+1)(2\mu-i+2n+2)(2\mu-2i+4n+5)\Gamma(2\mu-i+4n+1)}, \\ \quad \text{if } 3\le i\le 2n. \end{cases}$$

The last reduction formula along with formula (3.18)- after performing some manipulations- lead to formula (3.17). □

**Acknowledgment:** The author is very grateful to the anonymous referee for carefully reading the paper and for his comments which improved the manuscript in its present form.

## References

[1] W.M. Abd-Elhameed, *On solving linear and nonlinear sixth-order two point boundary value problems via an elegant harmonic numbers operational matrix of derivatives*, CMES-Comp. Model. Eng. **101** (2014) 159–185.

[2] W.M. Abd-Elhameed, E.H. Doha, H. M. Ahmed, *Linearization formulae for certain Jacobi polynomials*, Ramanujan J. **39 1**(2016) 155–168.

[3] G.E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

[4] R. Askey, *Orthogonal polynomials and special functions*, SIAM, 1975.

[5] R. Askey, G. Gasper, *Linearization of the product of Jacobi polynomials. III*, Can. J. Math. **23** (1971) 332–338.

[6] H. Chaggara, W. Koepf, *On linearization coefficients of Jacobi polynomials*, Appl. Math. Lett. **23** (2010) 609–614.

[7] J. Crank, *The Mathematics of Diffusion*, Clarendon Press, Oxford, 1979.

[8] E.H. Doha, *On the connection coefficients and recurrence relations arising from expansions in series of Laguerre polynomials*, J. Phys. A: Math. Gen. **36** (2003) 5449–5462.

[9] E. Doha, *On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials*, J. Phys. A: Math. Gen. **37 3** (2004) 657–675.

[10] E.H. Doha, W.M. Abd-Elhameed, *New linearization formulae for the products of Chebyshev polynomials of third and fourth kind*, Rocky Mt. J. Math. To appear.

[11] E.H. Doha, W.M. Abd-Elhameed, *Efficient spectral ultraspherical-dual-Petrov–Galerkin algorithms for the direct solution of (2n+1) th-order linear differential equations*, Math. Comput. Simulat. **79** (2009) 3221–3242.

[12] E.H. Doha, W.M. Abd-Elhameed, *Integrals of Chebyshev polynomials of third and fourth kinds: An application to solution of boundary value problems with polynomial coefficients*, J. Contemp. Math. Anal. **49** (2014) 296–308.

[13] E.H. Doha, W.M. Abd-Elhameed, M.A. Bassuony, *On the coefficients of differentiated expansions and derivatives of Chebyshev polynomials of the third and fourth kinds*, Acta Math. Sci. **35** (2015) 326–338.

[14] E.H. Doha, W.M. Abd-Elhameed, Y.H. Youssri, *Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane–Emden type*, New Astron. **23** (2013) 113–117.

[15] E.H. Doha, H.M. Ahmed, *Recurrences and explicit formulae for the expansion and connection coefficients in series of Bessel polynomials*, J. Phys. A: Math. Gen. **37** (2004) 8045–8063.

[16] A. Erdélyi, W. Magnus, W., F. Oberhettinger, and F.G. Tricomi., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York-Toronto-London, 1953.

[17] J.L. Fields, J. Wimp, *Expansions of hypergeometric functions in hypergeometric functions*, Math. Comp. **15** (1961) 390–395.

[18] G. Gasper, *Linearization of the product of Jacobi polynomials I*, Can. J. Math. **22** (1970) 171–175.

[19] G. Gasper, *Linearization of the product of Jacobi polynomials II*, Can. J. Math. **22** (1970) 582–593.

[20] E. Gogovcheva, L. Boyadjiev, *Fractional extensions of Jacobi polynomials and Gauss hypergeometric function*, Fract. Calc. Appl. Anal. **8** (2005) 431–438.

[21] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, 2000.

[22] E.A. Hylleraas, *Linearization of products of Jacobi polynomials.*, Math. Scand. **10** (1962) 189–200.

[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006.

[24] W. Koepf, *Hypergeometric summation: An Algorithmic Approach to Summation and Special Function Identities*, Braunschweig, Germany: Vieweg, 1998.

[25] Y.L. Luke, *The special functions and their approximations*, Academic press, New York, 1969.

[26] K. Maleknejad, S. Sohrabi, Y. Rostami, *Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials*, Appl. Math Comput. **188** (2007) 123–128.

[27] C. Markett, *Linearization of the product of symmetric orthogonal polynomials*, Constr. Approx. **10** (1994) 317–338.

[28] P. Maroni, Z. da Rocha, *Connection coefficients between orthogonal polynomials and the canonical sequence: an approach based on symbolic computation*, Numer. Algor. **47** (2008) 291–314.

[29] J.C. Mason, D.C. Handscomb, *Chebyshev polynomials*, Chapman and Hall, New York, NY, CRC, Boca Raton, 2003.

[30] S.P. Mirevski, L. Boyadjiev, R. Scherer, *On the Riemann–Liouville fractional calculus, g-Jacobi functions and F-Gauss functions*, Appl. Math. Comput. **187** (2007) 315–325.

[31] K.B. Oldham, J. Spanier, *The fractional Calculus*, Academic Press, New York, USA, 1974.

[32] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge, DLMF, Digital Library of Mathematical Functions. http://dlmf.nist.gov, 2010.

[33] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, mathematics in science and engineering*, vol. 198, Academic press, 1999.

[34] M. Rahman, *A non-negative representation of the linearization coefficients of the product of Jacobi polynomials*, Can. J. Math. **33** (1981) 915–928.

[35] E.D. Rainville, *Special functions*, The Maximalan Company, New York, 1960.

[36] J. Sánchez-Ruiz, *Linearization and connection formulae involving squares of Gegenbauer polynomials*, Appl. Math. Lett. **14** (2001) 261–267.

[37] J. Sánchez-Ruiz, J.S. Dehesa, *Some connection and linearization problems for polynomials in and beyond the Askey scheme*, J. Comput. Appl. Math. **133** (2001) 579–591.

[38] D.D. Tcheutia, *On connection, linearization and duplication coefficients of classical orthogonal polynomials*, PhD thesis, Universität Kassel 2014. Available at https://kobra.bibliothek.uni-kassel.de/handle/urn:nbn:de:hebis:34-2014071645714., 2014.

[39] M. van Hoeij, *Finite singularities and hypergeometric solutions of linear recurrence equations*, J. Pure Appl. Algebra **139** (1999) 109–131.

[40] G.N. Watson, *A note on generalized hypergeometric series*, Proc. London Math. Soc. **2** (1925) 23.

[41] X.J. Yang, D. Baleanu, H.M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, New York, 2015.

Waleed Abd-Elhameed

. Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

*E-mail address*: walee_9@yahoo.com