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OPTIMAL BOUNDS FOR TOADER MEAN IN TERMS OF QUADRATIC, CONTRA-HARMONIC AND SECOND NEUMAN MEANS

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Abstract. In the article, we prove that the double inequalities

$$\alpha N_{AG}(a,b)+(1-\alpha)\,Q(a,b)<TD\,(a,b)<\beta N_{AG}(a,b)+(1-\beta)\,Q(a,b),$$

$$\lambda N_{AG}(a,b)+(1-\lambda)\,C(a,b)<TD(a,b)<\mu N_{AG}(a,b)+(1-\mu)\,C(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 3/10$, $\beta\leq 2\left(\sqrt{2}\pi-4\right)/\left[\left(2\sqrt{2}-1\right)\pi\right]=0.1542\cdots$, $\lambda\geq 9/16$ and $\mu\leq 4\,(\pi-2)\,/\,(3\pi)=0.4845\cdots$, where $Q(a,b)$, $C(a,b)$, $N_{AG}\,(a,b)$ and $TD(a,b)$ are the quadratic, contra-harmonic, second Neuman and Toader means of a and b, respectively.

1. Introduction

Let $a,b>0$ with $a\neq b$. Then the Toader mean $TD(a,b)$ [13, 14, 20, 49] is defined by

$$TD(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2t+b^2\sin^2t}dt=\begin{cases}\frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-(b/a)^2}\right), & a>b,\\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-(a/b)^2}\right), & a<b,\end{cases}\tag{1.1}$$

where $\mathcal{E}\,(r)=\int_0^{\pi/2}\left(1-r^2\sin^2t\right)^{1/2}dt$ is the complete elliptic integral of the second kind. The Toader mean $TD(a,b)$ satisfies

$$TD(a,b)=R_E\left(a^2,b^2\right)$$

and

$$TD(1,r)=\frac{2}{\pi}\mathcal{E}\left(\sqrt{1-r^2}\right),$$

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where

$$R_E(a,b)=\frac{1}{\pi}\int_0^{+\infty}\left(\frac{a}{t+a}+\frac{b}{t+b}\right)\frac{t}{\sqrt{(t+a)(t+b)}}dt$$

stands for the symmetric complete elliptic integral of the second kind [35, 42], therefore it cannot be expressed in terms of the elementary transcendental functions.

Recently, the bounds for the Toader mean have attracted the attention of many researchers [31, 32, 33, 34, 39].

Let

$$\begin{aligned} G(a,b)&=\sqrt{ab}, \quad A(a,b)=(a+b)/2,\\ Q(a,b)&=\sqrt{(a^2+b^2)/2}, \quad C(a,b)=\left(a^2+b^2\right)/(a+b) \end{aligned} \tag{1.2}$$

and

$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases}$$

be the geometric mean [9, 10, 12, 21, 22, 24, 28, 40, 41, 47, 61], arithmetic mean [18, 27, 45, 64, 65], quadratic mean [72], contra-harmonic mean [7, 16, 29, 30], and the pth power mean [8, 11, 15, 17, 23, 25, 26, 36, 37, 38, 48, 56, 57, 62, 63, 71] of a and b. Then $M_p(a,b)$ is symmetric and homogeneous of degree one, and continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$.

It is well known that

$$G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b)<Q(a,b)=M_2(a,b)<C(a,b) \tag{1.3}$$

for all $a,b>0$ and $a\neq b$.

Vuorinen [50] conjectured that the inequality

$$M_{3/2}(a,b)<TD(a,b) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$. This conjecture was proved by Barnard, Pearce and Richards in [5].

Alzer and Qiu [3] presented the best possible upper power mean bound for the Toader mean as follows:

$$TD(a,b)<M_{\log 2/(\log\pi-\log 2)}(a,b) \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

Let $a,b>0$ with $a\neq b$ and $v=(a-b)/(a+b)$. Then the second Neuman mean $N_{AG}(a,b)$ [44, 46] is given by

$$N_{AG}(a,b)=\frac{1}{2}A(a,b)\left[1+\left(1-v^2\right)\frac{\tanh^{-1}(v)}{v}\right]. \tag{1.6}$$

Neuman [43] proved that the double inequality

$$G(a,b)<N_{AG}(a,b)<A(a,b) \tag{1.7}$$

holds for $a,b>0$ with $a\neq b$.

From (1.3)-(1.5) and (1.7) we clearly see that

$$N_{AG}(a,b)<A(a,b)<TD(a,b)<Q(a,b)<C(a,b) \tag{1.8}$$

for $a,b>0$ with $a\neq b$.

Motivated by inequality (1.8), it is natural to ask what are the least values α and λ and the greatest values β and μ such that the double inequalities

$$\alpha N_{AG}(a,b)+(1-\alpha)Q(a,b)<TD(a,b)<\beta N_{AG}(a,b)+(1-\beta)Q(a,b),$$

$$\lambda N_{AG}(a,b)+(1-\lambda)\,C(a,b)<TD(a,b)<\mu N_{AG}(a,b)+(1-\mu)\,C(a,b)$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this article is to answer this questions.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Let $r\in(0,1)$. Then the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [6] of the first and second kinds are defined by

$$\mathcal{K}(r)=\int_0^{\pi/2}\left[1-r^2\sin^2(t)\right]^{-1/2}dt,\quad \mathcal{K}\left(0^+\right)=\frac{\pi}{2},\quad \mathcal{K}\left(1^-\right)=\infty,$$

$$\mathcal{E}(r)=\int_0^{\pi/2}\left[1-r^2\sin^2(t)\right]^{1/2}dt,\quad \mathcal{E}\left(0^+\right)=\frac{\pi}{2},\quad \mathcal{E}\left(1^-\right)=1.$$

Many remarkable inequalities and properties for the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{K}(r)$, and other related special functions can be found in the literature [1, 2, 19, 51, 52, 53, 54, 55, 58, 59, 60, 66, 67, 68, 69, 70]

Lemma 2.1. *(See [4]) $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the formulas*

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-\left(1-r^2\right)\mathcal{K}(r)}{r\left(1-r^2\right)},\quad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-\left(1-r^2\right)\mathcal{K}(r)}{1+r}$$

Lemma 2.2. *(See [4]). The function $\left[\mathcal{E}(r)-r'^2\mathcal{K}(r)\right]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$.*

Lemma 2.3. *The function*

$$f_1(r)=\frac{(1+r^2)\tanh^{-1}(r)-r}{2r^3} \tag{2.1}$$

is strictly increasing on $(0,1)$.

Proof. It follows from (2.1) that

$$f_1'(r)=\frac{\left(3+r^2\right)}{2r^4}\left[\frac{r\left(3-r^2\right)}{\left(1-r^2\right)\left(3+r^2\right)}-\tanh^{-1}(r)\right]. \tag{2.2}$$

Let

$$f_2(r)=\frac{r\left(3-r^2\right)}{\left(1-r^2\right)\left(3+r^2\right)}-\tanh^{-1}(r). \tag{2.3}$$

Then simple computations lead to

$$f_2(0^+)=0, \tag{2.4}$$

$$f_2'(r)=\frac{16r^4}{\left(1-r^2\right)^2\left(3+r^2\right)^2}>0 \tag{2.5}$$

for all$r\in(0,1)$.

From equation (2.4) and (2.5) we know that $f_2(r)>0$ for all $r\in(0,1)$. Therefore, Lemma 2.3 follows from (2.2) and (2.3) together with $f_2(r)>0$. □

Lemma 2.4. *Let $p \in (0,1)$, $r \in (0,1)$ and*

$$f(r) = \frac{2}{\pi}\left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right] + p\frac{(1+r^2)\tanh^{-1}(r) - r}{2r^3} - \frac{(1-p)}{\sqrt{1+r^2}}. \quad (2.6)$$

Then the following statements are true:

(1) *If $p = 3/10$, then $f(r) > 0$ for all $r \in (0,1)$;*

(2) *If $p = 2(\sqrt{2}\pi - 4)/[(2\sqrt{2}-1)\pi] = 0.1542\cdots$, then there exists $r_0 \in (0,1)$ such that $f(r) < 0$ for $r \in (0, r_0)$ and $f(r) > 0$ for $r \in (r_0, 1)$.*

Proof. From (2.6), Lemma 2.2, Lemma 2.3 and the function $r \mapsto 1/\sqrt{1+r^2}$ is strictly decreasing on $(0,1)$ we know that $f(r)$ is strictly increasing on $(0,1)$.

For part (1), if $p = 3/10$, then (2.6) becomes

$$f(r) = \frac{2}{\pi}\left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right] + \frac{3}{10}\frac{(1+r^2)\tanh^{-1}(r) - r}{2r^3} - \frac{7}{10\sqrt{1+r^2}}. \quad (2.7)$$

It follows from (2.7) and the monotonicity of $f(r)$ that

$$f(r) > f(0^+) = 0 \quad (2.8)$$

for all $r \in (0,1)$.

For part (2), if $p = 2(\sqrt{2}\pi - 4)/[(2\sqrt{2}-1)\pi]$, then (2.6) leads to

$$f(r) = \frac{2}{\pi}\left[\frac{\mathcal{E}(r) - \left(1-r^2\right)\mathcal{K}(r)}{r^2}\right] + \frac{2\left(\sqrt{2}\pi - 4\right)}{\pi\left(2\sqrt{2}-1\right)}\frac{(1+r^2)\tanh^{-1}(r) - r}{2r^3} \quad (2.9)$$

$$- \frac{8-\pi}{\pi\left(2\sqrt{2}-1\right)}\frac{1}{\sqrt{1+r^2}},$$

$$f\left(0^+\right) = -\frac{80 - 14\sqrt{2}\pi - 3\pi}{6\pi\left(2\sqrt{2}-1\right)} = -0.2429\cdots < 0, \quad (2.10)$$

$$f\left(1^-\right) = \infty. \quad (2.11)$$

Therefore, part (2) follows from (2.10) and (2.11) together with the monotonicity of $f(r)$. □

Lemma 2.5. *Let $p \in (0,1)$, $r \in (0,1)$ and*

$$g(r) = \frac{2}{\pi}\left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right] + p\frac{(1+r^2)\tanh^{-1}(r) - r}{2r^3} - 2(1-p). \quad (2.12)$$

Then the following statements are true:

(1) *If $p = 9/16$, then $g(r) > 0$ for all $r \in (0,1)$;*

(2) *If $p = (4\pi - 8)/(3\pi) = 0.4845\cdots$, then there exists $r_1 \in (0,1)$ such that $g(r) < 0$ for $r \in (0, r_1)$ and $g(r) > 0$ for $r \in (r_1, 1)$.*

Proof. From Lemma 2.2, Lemma 2.3 and (2.12) we know that $g(r)$ is strictly increasing on $(0,1)$.

For part (1), if $p = 9/16$, then (2.12) becomes

$$g(r) = \frac{2}{\pi}\left[\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2}\right] + \frac{9}{16}\frac{(1+r^2)\tanh^{-1}(r) - r}{2r^3} - \frac{7}{8}. \quad (2.13)$$

It follows from (2.13) and the monotonicity of $g(r)$ that

$$g(r) > g(0^+) = 0 \quad (2.14)$$

for all $r \in (0,1)$.

For part (2), if $p=(4\pi-8)/(3\pi)$, then (2.12) leads to

$$g(r)=\frac{2}{\pi}\left[\frac{\mathcal{E}(r)-\left(1-r^2\right)\mathcal{K}(r)}{r^2}\right]+\frac{4\pi-8}{3\pi}\frac{(1+r^2)\tanh^{-1}(r)-r}{2r^3}-\frac{2(8-\pi)}{3\pi},$$

$$g\left(0^+\right)=\frac{37\pi-128}{18\pi}=-0.2079<0, \tag{2.15}$$

$$g\left(1^-\right)=\infty. \tag{2.16}$$

Therefore, part (2) follows from (2.15) and (2.16) together with the monotonicity of $g(r)$. □

3. Main results

Theorem 3.1. *The double inequality*

$$\alpha N_{AG}(a,b)+(1-\alpha)Q(a,b)<TD(a,b)<\beta N_{AG}(a,b)+(1-\beta)Q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\geq 3/10$ and $\beta\leq\left[2(\sqrt{2}\pi-4)\right]/\left[(2\sqrt{2}-1)\pi\right]=0.1542\cdots$.

Proof. Since $Q(a,b)$, $TD(a,b)$ and $N_{AG}(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$ and $p\in(0,1)$. Then (1.1), (1.2) and (1.6) together with Lemma 2.1 lead to

$$\frac{Q(a,b)-TD(a,b)}{Q(a,b)-N_{AG}(a,b)}=\frac{\sqrt{1+r^2}-\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{\sqrt{1+r^2}-\frac{1}{2}\left[1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\right]}, \tag{3.1}$$

$$\lim_{r\to0^+}\frac{\sqrt{1+r^2}-\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{\sqrt{1+r^2}-\frac{1}{2}\left[1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\right]}=\frac{3}{10}, \tag{3.2}$$

$$\lim_{r\to1^-}\frac{\sqrt{1+r^2}-\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{\sqrt{1+r^2}-\frac{1}{2}\left[1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\right]}=\frac{2\left(\sqrt{2}\pi-4\right)}{\left(2\sqrt{2}-1\right)\pi}=0.1542\cdots, \tag{3.3}$$

$$TD(a,b)-[pN_{AG}(a,b)+(1-p)Q(a,b)] \tag{3.4}$$

$$=A(a,b)\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)-\frac{1}{2}p\left(1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\right)-(1-p)\sqrt{1+r^2}\right].$$

Let

$$F(r)=\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right) \tag{3.5}$$

$$-\frac{1}{2}p\left(1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\right)-(1-p)\sqrt{1+r^2}.$$

Then simple computations lead to

$$F\left(0^+\right)=0, \tag{3.6}$$

$$F\left(1^-\right)=\frac{4}{\pi}-\sqrt{2}+\left(\sqrt{2}-\frac{1}{2}\right)p, \tag{3.7}$$

$$F'(r)=rf(r) \tag{3.8}$$

where $f(r)$ is defined by (2.6).

We divide the proof into four cases.

Case 1 $p = 3/10$. Then Lemma 2.4(1) and (3.8) lead to the conclusion that $F(r)$ is strictly increasing on $(0,1)$. Therefore,

$$TD(a,b) > \frac{3}{10}N_{AG}(a,b) + \frac{7}{10}Q(a,b) \tag{3.9}$$

follows from (3.4)-(3.6) together with the monotonicity of $F(r)$.

Case 2 $p = \left[2\left(\sqrt{2}\pi - 4\right)\right] / \left[\left(2\sqrt{2}-1\right)\pi\right]$. Then (3.7) becomes

$$F\left(1^-\right) = 0. \tag{3.10}$$

It follows from Lemma 2.4(2) and (3.8) that there exists $r_0 \in (0,1)$ such that $F(r)$ is strictly decreasing on $(0, r_0]$ and strictly increasing on $[r_0, 1)$. Therefore,

$$TD(a,b) < \left[\frac{2\left(\sqrt{2}\pi - 4\right)}{\left(2\sqrt{2}-1\right)\pi}\right] N_{AG}(a,b) + \left[\frac{8-\pi}{\left(2\sqrt{2}-1\right)\pi}\right] Q(a,b) \tag{3.11}$$

follows from (3.4)-(3.6) and (3.10) together with the piecewise monotonicity of $F(r)$.

Case 3 $p = p_0 < 3/10$. Then it follows from (1.8), (3.1) and (3.2) that there exists $\delta_0 \in (0,1)$ such that

$$TD(a,b) < p_0 N_{AG}(a,b) + (1-p_0)Q(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_0)$.

Case 4 $p = p^* > \left[2\left(\sqrt{2}\pi - 4\right)\right] / \left[\left(2\sqrt{2}-1\right)\pi\right]$. Then it follows from (1.8), (3.1) and (3.3) that there exists $\delta^* \in (0,1)$ such that

$$TD(a,b) > p^* N_{AG}(a,b) + (1-p^*)\, Q(a,b)$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta^*, 1)$.

Therefore, Theorem 3.1 follows easily from Cases 1-4 and (1.8). □

Theorem 3.2. *The double inequality*

$$\lambda N_{AG}(a,b) + (1-\lambda)\, C(a,b) < TD(a,b) < \mu N_{AG}(a,b) + (1-\mu)\, C(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\lambda \geq 9/16$ *and* $\mu \leq (4\pi - 8)/(3\pi) = 0.4845\cdots$.

Proof. Since $C(a,b)$, $T(a,b)$ and $N_{AG}(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0,1)$ and $p \in (0,1)$. Then from (1.1), (1.2) and (1.6) together with Lemma 2.1 we have

$$\frac{C(a,b) - TD(a,b)}{C(a,b) - N_{AG}(a,b)} = \frac{\left(1+r^2\right) - \frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{(1+r^2) - \frac{1}{2}\left[1 + (1-r^2)\frac{\tanh^{-1}(r)}{r}\right]}, \tag{3.12}$$

$$\lim_{r\to 0^+} \frac{\left(1+r^2\right) - \frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{(1+r^2) - \frac{1}{2}\left[1 + (1-r^2)\frac{\tanh^{-1}(r)}{r}\right]} = \frac{9}{16}, \tag{3.13}$$

$$\lim_{r\to 1^-} \frac{\left(1+r^2\right) - \frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{(1+r^2) - \frac{1}{2}\left[1 + (1-r^2)\frac{\tanh^{-1}(r)}{r}\right]} = \frac{4\pi - 8}{3\pi}, \tag{3.14}$$

$$T(a,b) - [pN_{AG}(a,b) + (1-p)\, C(a,b)] \tag{3.15}$$

$$= A(a,b)\left[\frac{2}{\pi}\left(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right) - \frac{1}{2}p\left(1 + (1-r^2)\frac{\tanh^{-1}(r)}{r}\right) - (1-p)(1+r^2)\right].$$

Let

$$G(r)=\frac{2}{\pi}\big(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\big)-\frac{1}{2}p\Big(1+(1-r^2)\frac{\tanh^{-1}(r)}{r}\Big)-(1-p)(1+r^2). \quad (3.16)$$

Then simple computations lead to

$$G\left(0^+\right)=0, \quad (3.17)$$

$$G\left(1^-\right)=\frac{4}{\pi}-2+\frac{3}{2}p, \quad (3.18)$$

$$G'(r)=rg(r), \quad (3.19)$$

where $g\left(r\right)$ is defined by (2.12).

We divide the proof into four cases.

Case 1 $p=9/16$. Then Lemma 2.5(1) and (3.19) lead to the conclusion that $G\left(r\right)$ is strictly increasing on $(0,1)$. Therefore,

$$TD(a,b)>\frac{9}{16}N_{AG}(a,b)+\frac{7}{16}C(a,b) \quad (3.20)$$

follows from (3.15)-(3.17) together with the monotonicity of $G\left(r\right)$.

Case 2 $p=4\left(\pi-2\right)/\left(3\pi\right)$. Then (3.18) becomes

$$G\left(1^-\right)=0. \quad (3.21)$$

It follows from Lemma 2.5(2) and (3.19) that there exists $r_1\in(0,1)$ such that $G\left(r\right)$ is strictly decreasing on $(0,r_1]$ and strictly increasing on $[r_1,1)$. Therefore,

$$TD(a,b)<\frac{4\left(\pi-2\right)}{3\pi}N_{AG}(a,b)+\left(\frac{8-\pi}{3\pi}\right)C(a,b) \quad (3.22)$$

follows from (3.15)-(3.17) and (3.21) together with the piecewise monotonicity of $G\left(r\right)$.

Case 3 $p=p_0<9/16$. Then from (1.8), (3.12) and (3.13) we clearly see that there exists $\delta_0\in(0,1)$ such that

$$TD(a,b)<p_0N_{AG}(a,b)+(1-p_0)C(a,b)$$

for all $a>b>0$ with $(a-b)/(a+b)\in(0,\delta_0)$.

Case 4 $p=p^*>4\left(\pi-2\right)/\left(3\pi\right)$. Then inequality (1.8) and equations (3.12) and (3.14) imply that there exists $\delta^*\in(0,1)$ such that

$$TD(a,b)>p^*N_{AG}(a,b)+(1-p^*)C(a,b)$$

for all $a>b>0$ with $(a-b)/(a+b)\in(1-\delta^*,1)$.

Therefore, Theorem 3.2 follows from Cases 1-4 and (1.8). □

Let $r\in(0,1)$, $x=r^2/\big(1+\sqrt{1-r^2}\big)^2$, $a=1$, $b=\sqrt{1-r^2}$, $\alpha=3/10$, $\beta=\left[2\left(\sqrt{2}\pi-4\right)\right]/\left[\left(2\sqrt{2}-1\right)\pi\right]$, $\lambda=9/16$ and $\mu=4\left(\pi-2\right)/\left(3\pi\right)$. Then Theorems 3.1 and 3.2 lead to Corollary 3.3 immediately.

Corollary 3.3. *The double inequality*

$$\frac{\pi}{4(1+x)}\max\left\{\frac{3}{10}\left[1+(1-x^2)\frac{\tanh^{-1}(x)}{x}\right]+\frac{7}{5}\sqrt{1+x^2},\right.$$
$$\left.\frac{9}{16}\left[1+(1-x^2)\frac{\tanh^{-1}(x)}{x}\right]+\frac{7}{8}(1+x^2)\right\}<\mathcal{E}(r)$$

$$< \frac{1}{1+x}\min\left\{\frac{\sqrt{2}\pi-4}{2\sqrt{2}-1}\left[1+(1-x^2)\frac{\tanh^{-1}(x)}{x}\right]+\frac{8-\pi}{2\sqrt{2}-1}\sqrt{1+x^2},\right.$$
$$\left.\frac{2(\pi-2)}{3}\left[1+(1-x^2)\frac{\tanh^{-1}(x)}{x}\right]+\frac{8-\pi}{3}(1+x^2)\right\}$$

holds for all $r \in (0,1)$.

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