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ON A PARAMETRIC KIND OF GENOCCHI POLYNOMIALS

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Abstract. By defining two specific generating functions, we introduce a new parametric kind of Genocchi polynomials and study its basic properties in detail. As an interesting application, we also use the introduced polynomials in computing some new series of Taylor type involving Genocchi numbers.

1. Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [3, 10]. One of the special cases of Appell polynomials is Genocchi polynomials $G_n(x)$ where $G_n = G_n(0)$ is usually known as classical Genocchi numbers. These numbers have been extensively studied in different branches of mathematics such as, elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions) and quantum physics (quantum Groups). See [2, 1] in this regard.

The classical Genocchi numbers are defined by

$$\frac{2t}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad (|t| < \pi), \tag{1.2}$$

and are indeed a sequence of integers, because according to (1.2) we have

$$G_0 = 0 \text{ and } (G+1)^n + G_n = \begin{cases} 2, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

in which G^n is replaced by G_n. Therefore (for more details see [5, 4])

$$G_1 = 1, G_2 = -1, G_3 = 0, G_4 = 1, G_5 = 0, G_6 = -3, \cdots \quad \text{and} \quad G_{2n+1} = 0.$$

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Relationships between Genocchi, Euler and Bernoulli numbers are as follows:

$$G_{n+1} = (n+1)E_n, \quad G_{2n} = 2(1-2^n)B_{2n},$$

where E_n and B_n denote Euler and Bernoulli numbers respectively [8, 4].

As a special case of Apple polynomials, if $f(t) = \frac{2t}{e^t+1}$ is replaced in (1.1), then the Genocchi polynomials are generated by

$$\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad (|t| < \pi). \tag{1.3}$$

Hence, differentiating from both sides of (1.3) with respect to x yields

$$\frac{d}{dx}G_n(x) = nG_{n-1}(x) \quad \text{and} \quad \deg G_{n+1}(x) = n,$$

which directly leads to

$$\int_a^b G_n(x)\ dx = \frac{G_{n+1}(b) - G_{n+1}(a)}{n+1}.$$

In [9], the author introduced and investigated some generalizations and unifications of these polynomials by means of suitable generating functions. He proved several properties of these polynomials including some explicit representations in terms of the Hurwitz (or generalized) zeta function and the Gauss hypergeometric functions.

In [7], generating functions of the generalized Bernoulli polynomials, generalized Euler polynomials and generalized Genocchi polynomials associated with two positive real parameters and one complex parameter are unified and extended.

The present paper is organized as follows: In the next section, we introduce two parametric kinds of Genocchi polynomials and define their generating functions. Then we study some of their basic properties in detail. Finally in section 3, an interesting application of these polynomials are given to compute some new series of Taylor type involving Genocchi numbers.

2. A Parametric Kind of Genocchi Polynomials

If $p, q \in \mathbb{R}$, it is known that the Taylor expansion of the two functions $e^{pt}\cos qt$ and $e^{pt}\sin qt$ are respectively as follows [6]

$$e^{pt}\cos qt = \sum_{k=0}^{\infty} C_k(p,q)\frac{t^k}{k!}, \tag{2.1}$$

and

$$e^{pt}\sin qt = \sum_{k=0}^{\infty} S_k(p,q)\frac{t^k}{k!}, \tag{2.2}$$

where

$$C_k(p,q) = \sum_{j=0}^{[\frac{k}{2}]} (-1)^j \binom{k}{2j} p^{k-2j}q^{2j}, \tag{2.3}$$

and

$$S_k(p,q)=\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}q^{2j+1}. \tag{2.4}$$

By noting relations (2.1)-(2.4), we can introduce two parametric kinds of Genocchi polynomials as follows

$$\frac{2te^{pt}}{e^t+1}\cos qt=\sum_{n=0}^{\infty}G_n^{(c)}(p,q)\frac{t^n}{n!}\quad (|t|<\pi), \tag{2.5}$$

and

$$\frac{2te^{pt}}{e^t+1}\sin qt=\sum_{n=0}^{\infty}G_n^{(s)}(p,q)\frac{t^n}{n!}\quad (|t|<\pi). \tag{2.6}$$

It is clear that $G_n^{(c)}(p,0)=G_n(p)$. For instance, we have

$$G_2^{(c)}(p,q)=2p-1,$$
$$G_3^{(c)}(p,q)=3p^2-3p-3q^2,$$
$$G_4^{(c)}(p,q)=4p^3-6p^2-12q^2p+6q^2+1,$$
$$G_5^{(c)}(p,q)=5p^4-10p^3-30q^2p^2+(30q^2+5)p+5q^4,$$
$$G_6^{(c)}(p,q)=6p^5-15p^4-60q^2p^3+(90q^2+15)p^2+30q^4p-15q^4-15q^2-3,$$

and

$$G_2^{(s)}(p,q)=2q,$$
$$G_3^{(s)}(p,q)=6qp-3q,$$
$$G_4^{(s)}(p,q)=12qp^2-12qp-4q^3,$$
$$G_5^{(s)}(p,q)=20qp^3-30qp^2-20q^3p+10q^3+5q,$$
$$G_6^{(s)}(p,q)=30qp^4-60qp^3-60q^3p^2+(60q^3+30q)p+6q^5.$$

Proposition 2.1. *$G_n^{(c)}(p,q)$ and $G_n^{(s)}(p,q)$ defined in* (2.5) *and* (2.6) *can be directly represented in terms of Genocchi numbers as follows*

$$G_n^{(c)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}G_kC_{n-k}(p,q), \tag{2.7}$$

and

$$G_n^{(s)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}G_kS_{n-k}(p,q). \tag{2.8}$$

Proof. By noting the general identity

$$\left(\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{k!}\right)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}a_jb_{k-j}\right)\frac{t^k}{k!},$$

we have

$$\sum_{k=0}^{\infty} G_k^{(c)}(p,q)\frac{t^k}{k!} = \frac{2t}{e^t+1}\left(e^{pt}\cos qt\right) = \left(\sum_{k=0}^{\infty} G_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty} C_k(p,q)\frac{t^k}{k!}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}G_jC_{k-j}(p,q)\right)\frac{t^k}{k!},$$

which proves (2.7). The proof of (2.8) is similar. □

Proposition 2.2. *For every* $n \in \mathbb{Z}^+$ *we have*

$$G_n^{(c)}(1-p,q) = (-1)^{n+1}G_n^{(c)}(p,q), \tag{2.9}$$

and

$$G_n^{(s)}(1-p,q) = (-1)^{n}G_n^{(s)}(p,q). \tag{2.10}$$

Proof. Applying the generating function (2.5) gives

$$\sum_{n=0}^{\infty} G_n^{(c)}(1-p,q)\frac{t^n}{n!} = \frac{2te^{(1-p)t}}{e^t+1}\cos qt,$$

as well as

$$\sum_{n=0}^{\infty}(-1)^{n+1}G_n^{(c)}(p,q)\frac{t^n}{n!} = -\frac{-2te^{-pt}}{e^{-t}+1}\cos(-qt) = \frac{2te^{(1-p)t}}{e^t+1}\cos qt.$$

Similarly, property (2.10) can be proved. □

Corollary 2.1. *Relations* (2.9) *and* (2.10) *imply that*

$$G_{2n}^{(c)}(\frac{1}{2},q) = 0,$$
$$G_{2n+1}^{(s)}(\frac{1}{2},q) = 0,$$
$$\int_0^1 G_{2n}^{(c)}(p,q)\ \mathrm{d}p = 0,$$

and

$$\int_0^1 G_{2n+1}^{(s)}(p,q)\ \mathrm{d}p = 0.$$

Proposition 2.3. *For every* $n \in \mathbb{N}$, *the following identities hold*

$$G_n^{(c)}(1+p,q) + G_n^{(c)}(p,q) = 2nC_{n-1}(p,q), \tag{2.11}$$

and

$$G_n^{(s)}(1+p,q) + G_n^{(s)}(p,q) = 2nS_{n-1}(p,q). \tag{2.12}$$

Proof. We have

$$\sum_{n=0}^{\infty} G_n^{(c)}(1+p,q)\frac{t^n}{n!} = \frac{2te^{pt}(e^t+1-1)}{e^t+1}\cos qt = 2te^{pt}\cos qt - \frac{2te^{pt}}{e^t+1}\cos qt$$
$$= \sum_{n=0}^{\infty} 2C_n(p,q)\frac{t^{n+1}}{n!} - \sum_{n=0}^{\infty} G_n^{(c)}(p,q)\frac{t^n}{n!}$$
$$= \sum_{n=1}^{\infty} 2nC_{n-1}(p,q)\frac{t^n}{n!} - \sum_{n=0}^{\infty} G_n^{(c)}(p,q)\frac{t^n}{n!},$$

which proves (2.11). The proof of (2.12) is similar. □

Corollary 2.2. Relations (2.11) and (2.12) first imply that

$$G_{2n+1}^{(c)}(1,q) + G_{2n+1}^{(c)}(0,q) = 2(2n+1)(-1)^n q^{2n},$$

and

$$G_{2n}^{(s)}(1,q) + G_{2n}^{(s)}(0,q) = 4n(-1)^{n+1} q^{2n-1}.$$

Hence, combining proposition 2.2 with the above results respectively yields

$$G_{2n+1}^{(c)}(1,q) = G_{2n+1}^{(c)}(0,q) = (2n+1)(-1)^n q^{2n},$$

and

$$G_{2n}^{(s)}(1,q) = G_{2n}^{(s)}(0,q) = 2n(-1)^{n+1} q^{2n-1}.$$

Proposition 2.4. For every $n \in \mathbb{Z}^+$, the following identities hold

$$G_n^{(c)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(c)}(p,q) r^{n-k}, \tag{2.13}$$

and

$$G_n^{(s)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(s)}(p,q) r^{n-k}. \tag{2.14}$$

Proof. Apply (2.5) to obtain

$$\sum_{n=0}^{\infty} G_n^{(c)}(p+r,q)\frac{t^n}{n!} = \left(\frac{2te^{pt}}{e^t+1}\cos qt\right)e^{rt} = \left(\sum_{n=0}^{\infty} G_n^{(c)}(p,q)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} G_k^{(c)}(p,q) r^{n-k}\right)\frac{t^n}{n!},$$

which proves (2.13). The result (2.14) can be similarly proved. □

Proposition 2.5. We have

$$G_n^{(c)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} G_k^{(c)}(p,q) = 2nC_{n-1}(p,q), \tag{2.15}$$

and

$$G_n^{(s)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} G_k^{(s)}(p,q) = 2nS_{n-1}(p,q). \tag{2.16}$$

Proof. The result (2.15) is clear by (2.11) and (2.13). To obtain (2.16), combine (2.12) and (2.14). □

Corollary 2.3. Relations (2.15) *and* (2.16) *imply that*

$$G_n^{(c)}(0,q)+\sum_{k=0}^{n}\binom{n}{k}G_k^{(c)}(0,q)=2nq^{n-1}\cos(n-1)\frac{\pi}{2}=\begin{cases}0 & n=2m,\\ 2(-1)^m(2m+1)q^{2m} & n=2m+1,\end{cases}$$

and

$$G_n^{(s)}(0,q)+\sum_{k=0}^{n}\binom{n}{k}G_k^{(s)}(0,q)=2nq^{n-1}\sin(n-1)\frac{\pi}{2}=\begin{cases}4m(-1)^{m+1}q^{2m-1} & n=2m,\\ 0 & n=2m+1.\end{cases}$$

Proposition 2.6. For every $n\in\mathbb{N}$, the following partial differential equations hold

$$\frac{\partial}{\partial p}G_n^{(c)}(p,q)=nG_{n-1}^{(c)}(p,q), \tag{2.17}$$

$$\frac{\partial}{\partial q}G_n^{(c)}(p,q)=-nG_{n-1}^{(s)}(p,q), \tag{2.18}$$

$$\frac{\partial}{\partial p}G_n^{(s)}(p,q)=nG_{n-1}^{(s)}(p,q), \tag{2.19}$$

and

$$\frac{\partial}{\partial q}G_n^{(s)}(p,q)=nG_{n-1}^{(c)}(p,q). \tag{2.20}$$

Proof. Relation (2.5) yields

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial G_n^{(c)}(p,q)}{\partial p}\frac{t^n}{n!}&=\frac{2t^2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt=\sum_{n=0}^{\infty}G_n^{(c)}(p,q)\frac{t^{n+1}}{n!}\\&=\sum_{n=1}^{\infty}G_{n-1}^{(c)}(p,q)\frac{t^n}{(n-1)!}=\sum_{n=1}^{\infty}nG_{n-1}^{(c)}(p,q)\frac{t^n}{n!},\end{aligned}$$

proving (2.17). Other equations (2.18), (2.19) and (2.20) can be similarly derived. □

Proposition 2.7. If $G_n^{(c)}(p,q)$ and $G_n^{(s)}(p,q)$ are sorted in terms of the variable p, then they are polynomials of degree $n-1$ and $n-2$ respectively, such that we have

$$G_n^{(c)}(p,q)=np^{n-1}-\binom{n}{2}p^{n-2}+\cdots, \tag{2.21}$$

and

$$G_n^{(s)}(p,q)=2\binom{n}{2}qp^{n-2}-3\binom{n}{3}qp^{n-3}+\cdots. \tag{2.22}$$

Also, if they are sorted in terms of the variable q, then

$$G_n^{(c)}(p,q)=\begin{cases}n(-1)^{\frac{n-1}{2}}q^{n-1}+3(-1)^{\frac{n+1}{2}}\binom{n}{3}(p^2-p)q^{n-3}+\cdots & (n\ \mathit{odd}),\\ (-1)^{\frac{n+2}{2}}\binom{n}{2}(2p-1)q^{n-2}+(-1)^{\frac{n}{2}}\binom{n}{4}(4p^3-6p^2+1)q^{n-4}+\cdots & (n\ \mathit{even}),\end{cases} \tag{2.23}$$

and

$$G_n^{(s)}(p,q)=\begin{cases}(-1)^{\frac{n+2}{2}}nq^{n-1}+(-1)^{\frac{n}{2}}3\binom{n}{3}(p^2-p)q^{n-3}+\cdots\ (n\ even),\\ (-1)^{\frac{n+1}{2}}\binom{n}{2}(2p-1)q^{n-2}+(-1)^{\frac{n-1}{2}}\binom{n}{4}(4p^3-6p^2+1)q^{n-4}+\cdots\ (n\ odd).\end{cases}\tag{2.24}$$

Proof. We first prove (2.21) by induction. It is known from (2.15) that

$$G_1^{(c)}(p,q)=1,\ G_2^{(c)}(p,q)=2p-1\quad\text{and}\quad G_3^{(c)}(p,q)=3p^2-3p-3q^2.$$

Therefore (2.21) holds for $n=1,2,3$. Now assume that it is valid for $n-1$. By noting (2.17), we have

$$\frac{\partial}{\partial p}G_n^{(c)}(p,q)=n(n-1)p^{n-2}-n\binom{n-1}{2}p^{n-3}+\cdots.$$

To complete the proof, it is enough to integrate from both sides of the above equation with respect to the variable p to get the result (2.21). The result (2.22) can be similarly derived.
To prove (2.23), suppose that it first holds for $1,2,\cdots,n-1$. If $n=2m$, then from (2.15) we have

$$G_{2m}^{(c)}(p,q)=-\frac{1}{2}\sum_{k=0}^{2m-1}\binom{2m}{k}G_k^{(c)}(p,q)+2m\sum_{k=0}^{m-1}(-1)^k\binom{2m-1}{2k}p^{2m-1-2k}q^{2k}.\tag{2.25}$$

Hence, the coefficient of q^{2m-2} in the right hand side of (2.25) is equal to

$$-\frac{1}{2}\binom{2m}{2m-1}(2m-1)(-1)^{\frac{2m-2}{2}}+2m(-1)^{m-1}\binom{2m-1}{2m-2}p=(-1)^{m+1}\binom{2m}{2}(2p-1),$$

and the coefficient of q^{2m-4} is equal to

$$-\frac{1}{2}\left(\binom{2m}{2m-1}3(-1)^m\binom{2m-1}{3}(p^2-p)+\binom{2m}{2m-2}(-1)^m\binom{2m-2}{2}(2p-1)\right.$$
$$\left.+(-1)^{m-2}\binom{2m}{2m-3}(2m-3)\right)+2m(-1)^{m-2}\binom{2m-1}{2m-4}p^3=(-1)^m\binom{2m}{4}(4p^3-6p^2+1).$$

So, (2.23) is true for $n=2m$. In the second case, taking $n=2m+1$ in (2.15) gives

$$G_{2m+1}^{(c)}(p,q)=-\frac{1}{2}\sum_{k=0}^{2m}\binom{2m+1}{k}G_k^{(c)}(p,q)+(2m+1)\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}p^{2m-2k}q^{2k}.\tag{2.26}$$

Hence, the coefficient of q^{2m} in the right hand side of (2.26) is equal to

$$(-1)^m\binom{2m}{2m}p^{2m-2m}=(-1)^m(2m+1),$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2}\left(\binom{2m+1}{2m}(-1)^{m+1}\binom{2m}{2}(2p-1)+\binom{2m+1}{2m-1}(-1)^{m-1}(2m-1)\right)$$
$$+(2m+1)(-1)^{m-1}\binom{2m}{2m-2}p^2=3(-1)^{m+1}\binom{2m+1}{3}(p^2-p),$$

which completes the proof of (2.23). By combining (2.20) and (2.23), we can also obtain the result (2.24). □

Proposition 2.8. *The following identities hold*

$$G_n^{(c)}(p,q)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k}G_{n-2k}^{(c)}(p,0)q^{2k}, \tag{2.27}$$

and

$$G_n^{(s)}(p,q)=\sum_{k=0}^{[\frac{n-2}{2}]}(-1)^k\binom{n}{2k+1}G_{n-2k-1}^{(c)}(p,0)q^{2k+1}, \tag{2.28}$$

in which $G_{n-2k}^{(c)}(p,0)=G_{n-2k}(p)$ *and* $G_{n-2k-1}^{(c)}(p,0)=G_{n-2k-1}(p)$ *are the same as usual Genocchi polynomials.*

Proof. According to (2.18) and (2.20), first we have

$$\frac{\partial^{2k}}{\partial q^{2k}}G_n^{(c)}(p,q)=(-1)^k\frac{n!}{(n-2k)!}G_{n-2k}^{(c)}(p,q) \quad \text{for} \quad k=0,1,\cdots,[\frac{n-2}{2}],$$

and

$$\frac{\partial^{2k+1}}{\partial q^{2k+1}}G_n^{(c)}(p,q)=(-1)^{k+1}\frac{n!}{(n-2k-1)!}G_{n-2k-1}^{(s)}(p,q) \quad \text{for} \quad k=0,1,\cdots,[\frac{n-2}{2}],$$

because $G_n^{(c)}(p,q)$ is a polynomial of degree $n-1$ for even n and of degree $n-2$ for odd n in terms of the variable q according to the proposition 2.7. The Taylor expansion of $G_n^{(c)}(p,q)$ gives

$$G_n^{(c)}(p,q+h)=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial^k}{\partial q^k}G_n^{(c)}(p,q)h^k,$$

in which $h\in\mathbb{R}$. Since $G_n^{(s)}(p,0)=0$ for every $n\in\mathbb{Z}^+$, by replacing $q=0$ and $h=q$, we obtain the relation (2.27). In a similar way, equality (2.28) can be derived. □

Proposition 2.9. *If m is an odd number and $n\in\mathbb{Z}^+$, then we have*

$$G_n^{(c)}(mp,q)=m^{n-1}\sum_{k=0}^{m-1}(-1)^kG_n^{(c)}(p+\frac{k}{m},\frac{q}{m}), \tag{2.29}$$

and

$$G_n^{(s)}(mp,q)=m^{n-1}\sum_{k=0}^{m-1}(-1)^kG_n^{(s)}(p+\frac{k}{m},\frac{q}{m}). \tag{2.30}$$

Proof. First, consider the relation

$$\sum_{n=0}^{\infty} G_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!}=\frac{2t\mathrm{e}^{(p+\frac{k}{m})t}}{\mathrm{e}^t+1}\cos(\frac{q}{m}t).$$

If we multiply $(-1)^k$ in both sides of the above relation and take a sum over k from 0 to $m-1$, then we get

$$\sum_{k=0}^{m-1}(-1)^k\left(\sum_{n=0}^{\infty} G_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!}\right)=\frac{2t\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos(\frac{q}{m}t)\sum_{k=0}^{m-1}\left(-\mathrm{e}^{\frac{t}{m}}\right)^k$$
$$=m\frac{2\frac{t}{m}\mathrm{e}^{mp\frac{t}{m}}}{\mathrm{e}^{\frac{t}{m}}+1}\cos(q\frac{t}{m})\frac{1-(-1)^m\mathrm{e}^t}{\mathrm{e}^t+1}.$$

As m is odd, the relation (2.29) is true. In a similar way, equality (2.30) can be proved. □

For $m=3$, relations (2.29) and (2.30) respectively yield

$$G_n^{(c)}(1,3q)=3^{n-1}\left(G_n^{(c)}(\frac{1}{3},q)-G_n^{(c)}(\frac{2}{3},q)+G_n^{(c)}(1,q)\right),$$

and

$$G_n^{(s)}(1,3q)=3^{n-1}\left(G_n^{(s)}(\frac{1}{3},q)-G_n^{(s)}(\frac{2}{3},q)+G_n^{(s)}(1,q)\right).$$

Proposition 2.10. *For every $n\in\mathbb{N}$ and $q\in\mathbb{R}$, the two following propositions are valid:*

$\mathcal{P}_n$: *The function $p\mapsto(-1)^nG_{2n}^{(c)}(p,q)$ is positive on $(0,\frac{1}{2})$ and negative on $(\frac{1}{2},1)$. Moreover, $p=\frac{1}{2}$ is a unique simple root on $(0,1)$, i.e. the aforesaid function has no zero in the intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$.*

$\mathcal{Q}_n$: *The function $p\mapsto(-1)^nG_{2n+1}^{(c)}(p,q)$ is strictly increasing on $[0,\frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2},1]$ and always takes a positive value at $p=\frac{1}{2}$.*

Proof. The proposition $\mathcal{P}_1$ is clear, because $-G_2^{(c)}(p,q)=-(2p-1)=-2p+1$. Now define $f(p)=(-1)^nG_{2n+1}^{(c)}(p,q)$ to get $f'(p)=(2n+1)(-1)^nG_{2n}^{(c)}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly increasing on $[0,\frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2},1]$. Moreover, since $f(0)=(2n+1)q^{2n}\geq 0$ (by corollary 2.2) and f in $p=\frac{1}{2}$ has a maximum, one can conclude that $f(\frac{1}{2})>0$.
Finally define $g(p)=(-1)^{n+1}G_{2n+2}^{(c)}(p,q)$ to get $g'(p)=-(2n+2)(-1)^nG_{2n+1}^{(c)}(p,q)$. Since

$$g'(0)=g'(1)=-(2n+2)(2n+1)q^{2n}\leq 0,$$

and $G_{2n+1}^{(c)}(1-p,q)=G_{2n+1}^{(c)}(p,q)$, so by noting $\mathcal{Q}_n$ we have $\forall p\in(0,1):\ g'(p)<0$. Therefore, g takes the following table of variations

p	0		$\frac{1}{2}$		1
$g'(p)$		$-$	$-$	$-$	
$g(p)$		$\searrow$	0	$\searrow$	

As $g(\frac{1}{2})=0$ (by corollary 2.1) and $g'(\frac{1}{2})<0$, then $p=\frac{1}{2}$ is a simple root of g. So, the proof of $\mathcal{P}_{n+1}$ is complete. □

Proposition 2.11. *For every $n \in \mathbb{N}$ and $q \in \mathbb{R}$ we have*

$$\sup_{p\in[0,1]} |G^{(c)}_{2n-1}(p,q)| = \max\{|G^{(c)}_{2n-1}(0,q)|,\ |G^{(c)}_{2n-1}(\frac{1}{2},q)|\}, \tag{2.31}$$

and

$$\sup_{p\in[0,1]} |G^{(c)}_{2n}(p,q)| \leq n\max\{|G^{(c)}_{2n-1}(0,q)|,\ |G^{(c)}_{2n-1}(\frac{1}{2},q)|\}. \tag{2.32}$$

Proof. The result (2.31) is clear by noting the propositions 2.2 and 2.10. To prove (2.32), if $p \in [0, \frac{1}{2}]$ then we have

$$G^{(c)}_{2n}(p,q) = G^{(c)}_{2n}(p,q) - G^{(c)}_{2n}(\frac{1}{2},q) = 2n\int_{\frac{1}{2}}^{p} G^{(c)}_{2n-1}(t,q)\ \mathrm{d}t.$$

Therefore

$$\begin{aligned}|G^{(c)}_{2n}(p,q)| &\leq 2n\int_{p}^{\frac{1}{2}} |G^{(c)}_{2n-1}(t,q)|\ \mathrm{d}t \leq 2n(\frac{1}{2}-p)\sup_{t\in[p,\frac{1}{2}]}|G^{(c)}_{2n-1}(t,q)| \\ &\leq 2n(\frac{1}{2}-p)\max\{|G^{(c)}_{2n-1}(0,q)|,\ |G^{(c)}_{2n-1}(\frac{1}{2},q)|\},\end{aligned}$$

which is equivalent to

$$\sup_{p\in[0,\frac{1}{2}]} |G^{(c)}_{2n}(p,q)| \leq n\max\{|G^{(c)}_{2n-1}(0,q)|,\ |G^{(c)}_{2n-1}(\frac{1}{2},q)|\}.$$

On the other hand, $G^{(c)}_{2n}(1-p,q) = -G^{(c)}_{2n}(p,q)$ completes the proof of (2.32). □

Proposition 2.12. *For every $n \geq 2$ and $q > 0$, the two following propositions are valid:*

$\mathcal{P}_n$*: The function $p \mapsto (-1)^n G^{(s)}_{2n-1}(p,q)$ is negative on $[0,\frac{1}{2})$ and positive on $(\frac{1}{2},1]$. Moreover, $p=\frac{1}{2}$ is a unique simple root on $[0,1]$, i.e. the aforesaid function has no zero in the intervals $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$.*

$\mathcal{Q}_n$*: The function $p \mapsto (-1)^n G^{(s)}_{2n}(p,q)$ is strictly decreasing on $[0,\frac{1}{2}]$ and strictly increasing on $[\frac{1}{2},1]$ and always takes a negative value at $p=\frac{1}{2}$.*

Proof. The proposition $\mathcal{P}_2$ is clear, because

$$G^{(s)}_3(p,q) = (6p-3)q.$$

Now define $f(p) = (-1)^n G^{(s)}_{2n}(p,q)$ to get $f'(p) = 2n(-1)^n G^{(s)}_{2n-1}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly decreasing on $[0,\frac{1}{2}]$ and strictly increasing on $[\frac{1}{2},1]$. Moreover, since $f(0) = -2nq^{2n-1} < 0$ (by corollary 2.2) and f in $p=\frac{1}{2}$ has a minimum, one can conclude that $f(\frac{1}{2}) < 0$.

Finally define $g(p) = (-1)^{n+1}G^{(s)}_{2n+1}(p,q)$ to get $g'(p) = -(2n+1)(-1)^n G^{(s)}_{2n}(p,q)$. Since

$$g'(0) = g'(1) = 2n(2n+1)q^{2n-1} > 0,$$

and $G^{(s)}_{2n}(1-p,q) = G^{(s)}_{2n}(p,q)$, so by noting $\mathcal{Q}_n$ we have $\forall p \in [0,1]:\ g'(p) > 0$. Therefore, g takes the following table of variations

As $g(\frac{1}{2}) = 0$ (by corollary 2.1) and $g'(\frac{1}{2}) < 0$, so $p=\frac{1}{2}$ is a simple root of function g. So, the proof of $\mathcal{P}_{n+1}$ is complete. □

p	0		$\frac{1}{2}$		1
$g'(p)$		$+$	$+$	$+$	
$g(p)$		$\nearrow$	0	$\nearrow$	

Corollary 2.4. *For every $n \in \mathbb{N}$ and $q \in \mathbb{R}$ we have*

$$\sup_{p\in[0,1]} |G_{2n}^{(s)}(p,q)| = \max\{|G_{2n}^{(s)}(0,q)|,\ |G_{2n}^{(s)}(\frac{1}{2},q)|\},$$

and

$$\sup_{p\in[0,1]} |G_{2n+1}^{(s)}(p,q)| \leq \frac{2n+1}{2} \max\{|G_{2n}^{(s)}(0,q)|,\ |G_{2n}^{(s)}(\frac{1}{2},q)|\}.$$

Proposition 2.13. *Let m and n be two positive integers and*

$$I^{(c)} = \int_0^1 G_m^{(c)}(p,q)G_n^{(c)}(p,q)\ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(c)} = 0$ and if it is even then

$$I^{(c)} = \sum_{k=0}^{m+n-2} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} G_{n-j}^{(c)}(0,q) G_{m-k+j}^{(c)}(0,q) \right),$$

where $A = \max\{0, k-m\}$ and $B = \min\{n,k\}$.

Proof. First, suppose that $m+n$ is odd. By using (2.9) we have

$$I^{(c)} = \int_0^1 G_m^{(c)}(1-p,q)G_n^{(c)}(1-p,q)\ \mathrm{d}p = (-1)^{m+n}\int_0^1 G_m^{(c)}(p,q)G_n^{(c)}(p,q)\ \mathrm{d}p = -I^{(c)}.$$

Now, assume that $m+n$ is even. Since $\deg_p\big(G_m^{(c)}G_n^{(c)}\big) = m+n-2$ (from proposition 2.7), by noting (2.17) we obtain

$$\begin{aligned} G_m^{(c)}(p,q)G_n^{(c)}(p,q) &= \sum_{k=0}^{m+n-2} \left(\frac{\partial^k}{\partial p^k}\big(G_m^{(c)}(p,q)G_n^{(c)}(p,q)\big) \right)\Bigg|_{p=0} \frac{p^k}{k!} \\ &= \sum_{k=0}^{m+n-2} \left(\sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^j}{\partial p^j} G_n^{(c)}(p,q) \frac{\partial^{k-j}}{\partial p^{k-j}} G_m^{(c)}(p,q) \right)\Bigg|_{p=0} \right) \frac{p^k}{k!} \\ &= \sum_{k=0}^{m+n-2} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} G_{n-j}^{(c)}(0,q) G_{m-k+j}^{(c)}(0,q) \right) \frac{p^k}{k!}, \end{aligned}$$

which leads to the second result. □

Corollary 2.5. *Let $m, n \geq 2$ and*

$$I^{(s)} = \int_0^1 G_m^{(s)}(p,q)G_n^{(s)}(p,q)\ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(s)} = 0$ and if $m+n$ is even then

$$I^{(s)} = \sum_{k=0}^{m+n-4} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} G_{n-j}^{(s)}(0,q) G_{m-k+j}^{(s)}(0,q) \right),$$

where $A = \max\{0, k-m\}$ and $B = \min\{n,k\}$.

3. Some series of Taylor type involving Genocchi numbers

One of the applications of relations (2.5) and (2.6) is that they can be considered as the Taylor expansion of two special functions at $t=0$ involving Genocchi numbers. In other words, substituting the relations (2.7) and (2.8) in respectively (2.5) and (2.6) yield

$$f_c(t;p,q)=\frac{2t\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_kC_{n-k}(p,q)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}C_k(p,q)\right)\frac{t^n}{n!}, \tag{3.1}$$

and

$$f_s(t;p,q)=\frac{2t\mathrm{e}^{pt}}{\mathrm{e}^t+1}\sin qt=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_kS_{n-k}(p,q)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}S_k(p,q)\right)\frac{t^n}{n!}, \tag{3.2}$$

where $G_k=G_k(0)$ denotes Genocchi numbers and $C_k(p,q)$ and $S_k(p,q)$ are defined in (2.3) and (2.4). In order to evaluate the two functions f_c and f_s at some specific parameters, first let us prove the following identities

$$C_k(p,p)=2^{\frac{k}{2}}p^k\cos\frac{k\pi}{4}, \tag{3.3}$$
$$S_k(p,p)=2^{\frac{k}{2}}p^k\sin\frac{k\pi}{4}, \tag{3.4}$$
$$C_k(0,q)=q^k\cos\frac{k\pi}{2}, \tag{3.5}$$
$$S_k(0,q)=q^k\sin\frac{k\pi}{2}, \tag{3.6}$$

and

$$C_k(p,0)=p^k, \qquad S_k(p,0)=0. \tag{3.7}$$

It is easy to find out that

$$\cos k\theta+\mathrm{i}\sin k\theta=(\cos\theta+\mathrm{i}\sin\theta)^k$$
$$=\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}\sin^{2j}\theta\cos^{k-2j}\theta+\mathrm{i}\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}\sin^{2j+1}\theta\cos^{k-2j-1}\theta.$$

By replacing $\theta=\frac{\pi}{4}$ in the above relation, we obtain

$$\cos\frac{k\pi}{4}+\mathrm{i}\sin\frac{k\pi}{4}=2^{-\frac{k}{2}}\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}+\mathrm{i}\,2^{-\frac{k}{2}}\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1},$$

which leads to relations (3.3) and (3.4) respectively. Relations (3.5), (3.6) and (3.7) are also clear by noting relations (2.3) and (2.4).
Now, we can consider some particular examples.

Example 1. In this example, we present Taylor series of the hyperbolic secant function

$$\operatorname{sech} t = \frac{2}{e^t + e^{-t}} = \frac{2e^t}{e^{2t}+1}, \quad |t| < \frac{\pi}{2},$$

in terms of Genocchi numbers. For this purpose, replacing $t \to 2t$, $p = \frac{1}{2}$ and $q = 0$ in (3.1) gives

$$\begin{aligned} f_c(2t; \frac{1}{2}, 0) = \frac{4te^t}{e^{2t}+1} = 2t \operatorname{sech} t &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_k C_{n-k}(\frac{1}{2}, 0) \right) \frac{2^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_k 2^k \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore we have

$$t \operatorname{sech} t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_k 2^{k-1} \right) \frac{t^n}{n!}.$$

Since $t \operatorname{sech} t$ is an odd function, so

$$\sum_{k=0}^{2n} \binom{2n}{k} G_k 2^{k-1} = 0,$$

and we obtain

$$t \operatorname{sech} t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n+1} \binom{2n+1}{k} G_k 2^{k-1} \right) \frac{t^{2n+1}}{(2n+1)!},$$

which is equivalent to

$$\operatorname{sech} t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{2n+1} \binom{2n+1}{k} G_k 2^{k-1} \right) \frac{t^{2n}}{(2n+1)!}.$$

Example 2. Let $f(t) = \frac{t \cos t}{e^t+1}$ and $g(t) = \frac{t \sin t}{e^t+1}$. If in (3.1) we take $p = 0$ and $q = 1$, then by noting (3.5) we obtain

$$\begin{aligned} f_c(t; 0, 1) = \frac{2t}{e^t+1} \cos t &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_{n-k} \cos \frac{k\pi}{2} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} G_{n-2k} (-1)^k \right) \frac{t^n}{n!}, \end{aligned}$$

and in a similar way

$$\frac{2t}{e^t+1} \sin t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} G_{n-2k-1} \right) \frac{t^n}{n!}.$$

Example 3. Let $f(t) = \frac{t\mathrm{e}^t}{\mathrm{e}^t+1}\cos t$ and $g(t) = \frac{t\mathrm{e}^t}{\mathrm{e}^t+1}\sin t$. replacing $p = q = 1$ in (3.1) respectively gives

$$\frac{2t\mathrm{e}^t}{\mathrm{e}^t+1}\cos t = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}C_k(1,1)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}2^{\frac{k}{2}}\cos\frac{k\pi}{4}\right)\frac{t^n}{n!},$$

and

$$\frac{2t\mathrm{e}^t}{\mathrm{e}^t+1}\sin t = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}S_k(1,1)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}2^{\frac{k}{2}}\sin\frac{k\pi}{4}\right)\frac{t^n}{n!}.$$

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