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FIXED POINT RESULTS FOR COUPLINGS ON ABSTRACT METRIC SPACES AND AN APPLICATION

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Abstract. Recently, Choudhury et al. [13] introduced the concept of couplings between two nonempty subsets in metric spaces. The aim of this paper is to prove the existence and uniqueness of strong coupled fixed points for a class of couplings in the context of cone metric spaces. Our results unify and generalize many know results in literature. Some examples are provided to illustrate the obtained results.

1. Introduction and Preliminaries

Banach's contraction principle is one of the most important theorems in the field of fixed point theory. This famous result has been extended and generalized in many various directions. The concept of a cone metric space was reintroduced in 2007 by Huang and Zhang [17] as one of the generalizations of a metric space. Many (common) fixed point results are established in such spaces. For more details, we refer the reader to [2]-[10] and [20]-[29]. Now, we give some basic definitions which are used throughout the paper.

Definition 1.1. [14] *Let E be a real Banach space. A subset P of E is said to be a cone if and only if*

(i) *P is nonempty, closed and $P \neq \{0_E\}$;*
(ii) *$a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies that $ax + by \in P$;*
(iii) *$P \cap (-P) = \{0_E\}$;*

where 0_E is the zero vector of E.

For a cone P, we define a partial ordering with respect to P by

$$x \preceq y \Leftrightarrow y - x \in P.$$

We shall write $x \ll y$ if $y - x \in IntP$, where $IntP$ is the interior of P. We also write $x \prec y$ if $x \preceq y$ and $x \neq y$. The cone P in the normed space $(E, ||.||)$ is said to be normal if there is a number $\lambda \geq 0$ such that for all $x, y \in E$, we have

$$0_E \preceq x \preceq y \quad \text{implies} \quad ||x|| \leq \lambda ||y||.$$

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The least positive number satisfying this norm inequality is said to be a normal constant of P.

Example 1.2. *Let $E = \mathbb{R}$ and $P = [0, \infty)$. Then P is a normal cone such that the normal constant is equal to* 1.

Example 1.3. *Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Then P is a normal cone such that the normal constant is equal to* 1.

Definition 1.4. [17] *Let X be a nonempty set. Suppose that $d : X \times X \to E$ satisfies*

(d1) $0_E \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0_E$ *if and only if* $x = y$*;*
(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$*.*

Then d is said to be a cone metric on X and (X, d) is said to be a cone metric space.

Remark 1.5. *Each metric space is a cone metric space. It suffices to take $E = \mathbb{R}$ and $P = [0, \infty)$. The concept of a cone metric space is more general than a metric space.*

Example 1.6. [17] *Let $X = \mathbb{R}$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Consider the function $d : X \times X \to E$ defined by $d(x, y) = (a|x - y|, b|x - y|)$ for all $x, y \in X$, where $a, b > 0$. Then (X, d) is a cone metric space.*

Definition 1.7. [17] *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $0_E \ll c$, we say that $\{x_n\}$ is*

(C1) *Cauchy if there is some $N \in \mathbb{N}$ such that, for all $n, m \geq N$, $d(x_n, x_m) \ll c$;*
(C2) *convergent if there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) \ll c$. Then x is said to be the limit of the sequence $\{x_n\}$.*

Notice that every convergent sequence in a cone metric space X is a Cauchy sequence. A cone metric space X is called complete if every Cauchy sequence in X is convergent in X.

The following concept is introduced recently by Choudhury et al. [13].

Definition 1.8. [13] *Let A and B be two nonempty subsets of set X. A coupling with respect to A and B is a function $F : X \times X \to X$ such that $F(x, y) \in B$ and $F(y, x) \in A$ whenever $(x, y) \in A \times B$.*

The following examples illustrate Definition 1.8.

Example 1.9. *Let $X = \mathbb{R}$. Take $A = (-\infty, 0]$, $B = [1, \infty)$ and $F : X \times X \to X$ defined by*

$$F(x, y) = \begin{cases} 2 & \text{if } x < y \\ 0 & \text{if } x = y \\ -1 & \text{if } x > y. \end{cases}$$

For all $(x, y) \in A \times B$, we have $x < y$ and so $F(x, y) = 2 \in B$ and $F(y, x) = -1 \in A$. Then F is a coupling with respect to A and B.

Example 1.10. *Let $X = \mathbb{R}$. Take $A = (-\infty, -1]$, $B = [1, \infty)$ and $F(x, y) = y - x$ for all $x, y \in X$. For all $(x, y) \in A \times B$, we have $F(x, y) \geq 2$ and $F(y, x) \leq -2$. Then $F(x, y) \in B$ and $F(y, x) \in A$ and so F is a coupling with respect to A and B.*

Example 1.11. *Let $X = \mathbb{R}$. Take $A = [-1, 0]$ and $B = [2, 3]$. Define $F : X \times X \to X$ by*

$$F(x, y) = \begin{cases} \frac{y^2 - x^2}{9} & \text{if } x \geq y \\ 2 & \text{if } x < y. \end{cases}$$

For all $(x, y) \in A \times B$, we have $F(x, y) = 2 \in B$ and $F(y, x) = \frac{x^2 - y^2}{9} \in A$. Then F is a coupling with respect to A and B.

Example 1.12. *Let X be a nonempty set. Take $A = B = X$. Then every function $F : X \times X \to X$ is is a coupling with respect to X and X.*

Definition 1.13. [15] *Let X be a nonempty set. An element $(x, y) \in X \times X$ is a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.*

Definition 1.14. [12] *Let X be a nonempty set. An element $(x, y) \in X \times X$ is a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is a coupled fixed point and $x = y$, that is, if $x = F(x, x)$.*

Many results on coupled fixed point results have been given. For details, see [1, 3, 6, 7, 8, 16, 18, 19, 23, 24, 25]. While, in this paper, we prove the existence and uniqueness of strong coupled fixed points for a class of couplings in the context of cone metric spaces. Some examples are provided to illustrate the concepts and obtained results.

2. Main Results

Let us start with the following problem: Find an element $x \in X$ satisfying the following

$$\begin{cases} F(x, x) = x, \\ x \in A \cap B, \end{cases} \tag{2.1}$$

where (X, d) is a complete cone metric space with a cone P having nonempty interior, A and B be two nonempty closed subsets of X and $F : X \times X \to X$ is a coupling with respect to A and B satisfying

$$\begin{aligned} d(F(x, y), F(u, v)) \preceq a_1 d(x, u) + a_2 [d(F(x, y), x) + d(F(u, v), u)] \\ + a_3 d(y, v) + a_4 [d(F(x, y), u) + d(F(u, v), x)], \end{aligned} \tag{2.2}$$

for all $x, v \in A$ and $y, u \in B$, where a_i, $i = 1, \cdots, 4$ are nonnegative real numbers such that $a_1 + 2a_2 + a_3 + 2a_4 < 1$.

We state and prove our first result.

Theorem 2.1. *Let A and B be two nonempty closed subsets of a complete cone metric space (X, d) with a cone P having nonempty interior and $F : X \times X \to X$ be a coupling with respect to A and B satisfying the contraction condition (2.2). For any $(x_0, y_0) \in A \times B \cup B \times A$, let $x_{n+1} = F(y_n, x_n)$ and $y_{n+1} = F(x_n, y_n)$. Then the sequence $\{x_n\}$ converges to x, the unique solution of (2.1).*

Proof. Without loss of generality, let $(x_0, y_0) \in A \times B$. Since F is a coupling with respect to A and B, we have $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$. We shall show that $\{x_n\}$

and $\{y_n\}$ are Cauchy sequences. From (2.2), we have

$$\begin{aligned} d(x_{n+1}, y_{n+2}) &= d(F(x_{n+1}, y_{n+1}), F(y_n, x_n)) \\ &\preceq a_1 d(x_{n+1}, y_n) + a_2[d(y_{n+2}, x_{n+1}) + d(x_{n+1}, y_n)] \\ &+ a_3 d(x_n, y_{n+1}) + a_4[d(y_{n+2}, y_n) + d(x_{n+1}, x_{n+1})] \\ &\preceq (a_1 + a_2) d(x_{n+1}, y_n) + a_2 d(y_{n+2}, x_{n+1}) \\ &+ a_3 d(x_n, y_{n+1}) + a_4[d(y_{n+2}, x_{n+1}) + d(x_{n+1}, y_n)]. \end{aligned}$$

This leads to

$$(1 - a_2 - a_4) d(x_{n+1}, y_{n+2}) \preceq (a_1 + a_2 + a_4) d(y_n, x_{n+1}) + a_3 d(x_n, y_{n+1}).$$

Hence

$$d(x_{n+1}, y_{n+2}) \preceq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} d(y_n, x_{n+1}) + \frac{a_3}{1 - a_2 - a_4} d(x_n, y_{n+1}). \tag{2.3}$$

Again, from (2.2),

$$\begin{aligned} d(y_{n+1}, x_{n+2}) &= d(F(x_n, y_n), F(y_{n+1}, x_{n+1})) \\ &\preceq a_1 d(x_n, y_{n+1}) + a_2[d(y_{n+1}, x_n) + d(x_{n+2}, y_{n+1})] \\ &+ a_3 d(y_n, x_{n+1}) + a_4[d(y_{n+1}, y_{n+1}) + d(x_n, x_{n+2})]. \end{aligned}$$

Thus

$$d(y_{n+1}, x_{n+2}) \preceq \frac{a_1 + a_2 + a_4}{1 - a_2 - a_4} d(x_n, y_{n+1}) + \frac{a_3}{1 - a_2 - a_4} d(y_n, x_{n+1}). \tag{2.4}$$

Let $\theta_n := d(x_n, y_{n+1}) + d(y_n, x_{n+1})$ for $n \geq 0$. Combining (2.3) and (2.4), we obtain

$$\theta_{n+1} \preceq k\theta_n \quad \text{where} \quad k = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2 - a_4} < 1. \tag{2.5}$$

It follows that

$$0_E \preceq \theta_n \preceq k\theta_{n-1} \preceq \ldots \preceq k^n \theta_0, \quad \text{for all}\, n \geq 0. \tag{2.6}$$

On the other hand, from (2.2),

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(F(x_n, y_n), F(y_n, x_n)) \\ &\preceq (a_1 + a_3) d(x_n, y_n) + a_2[d(y_{n+1}, x_n) + d(x_{n+1}, y_n)] \\ &+ a_4[d(y_{n+1}, y_n) + d(x_{n+1}, x_n)] \\ &\preceq (a_1 + a_3 + 2a_4) d(x_n, y_n) + (a_2 + a_4)[d(x_n, y_{n+1}) + d(y_n, x_{n+1})] \\ &= (a_1 + a_3 + 2a_4) d(x_n, y_n) + (a_2 + a_4)\theta_n. \end{aligned}$$

Consider $\lambda_n = d(x_n, y_n)$. Choose $\alpha = a_1 + a_3 + 2a_4$ and $\beta = a_2 + a_4$. Then $\alpha < 1$ and $\beta < 1$. We have

$$\lambda_{n+1} \preceq \alpha\lambda_n + \beta\theta_n, \quad \text{for all}\, n \geq 0.$$

Using (2.6),

$$0_E \preceq \lambda_n \preceq \alpha^n \lambda_0 + \beta(k^{n-1} + \alpha k^{n-2} + \alpha^2 k^{n-3} + \cdots + \alpha^{n-1})\theta_0, \quad \text{for all}\, n \geq 0. \tag{2.7}$$

Now, from (2.5) and (2.6), we have

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\preceq d(x_n, y_n) + d(y_n, x_n) + d(x_n, y_{n+1}) + d(y_n, x_{n+1}) \\ &= 2\lambda_n + \theta_n \\ &\preceq 2\alpha^n \lambda_0 + 2\beta(k^{n-1} + \alpha k^{n-2} + \alpha^2 k^{n-3} + \cdots + \alpha^{n-1})\theta_0 \\ &+ k^n \theta_0. \end{aligned} \tag{2.8}$$

If $\lambda_0 = d(x_0, y_0) = 0_E$ and $\theta_0 = d(x_0, F(x_0, y_0)) + d(y_0, F(y_0, x_0)) = 0_E$, then $x_0 = y_0$ and $x_0 = F(x_0, y_0)$, that is, (x_0, y_0) is a strong coupled fixed point of F. From now on, suppose that $0_E \prec \lambda_0$ or $0_E \prec \theta_0$. For $m > n$, we have

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\preceq \sum_{i=n}^{m-1} [d(x_i, x_{i+1}) + d(y_i, y_{i+1})] \\ &\preceq 2\sum_{i=n}^{m-1} \alpha^i \lambda_0 + \sum_{i=n}^{m-1} k^i \theta_0 \\ &+ 2\beta \sum_{i=n}^{m-1} [k^{i-1} + \alpha k^{i-2} + \alpha^2 k^{i-3} + \cdots + \alpha^{i-1}]\theta_0 \\ &\preceq 2\sum_{i=n}^{\infty} \alpha^i \lambda_0 + \sum_{i=n}^{\infty} k^i \theta_0 \\ &+ 2\beta \sum_{i=n}^{\infty} [k^{i-1} + \alpha k^{i-2} + \alpha^2 k^{i-3} + \cdots + \alpha^{i-1}]\theta_0. \end{aligned}$$

We distinguish the following cases.
Case 1. If $\alpha \leq k$, we have $\frac{\alpha}{k} \leq 1$. So

$$k^{i-1} + \alpha k^{i-2} + \alpha^2 k^{i-3} + \cdots + \alpha^{i-1} = k^{i-1}[1 + \frac{\alpha}{k} + \cdots + (\frac{\alpha}{k})^{i-1}] \leq i k^{i-1}.$$

Having $\alpha, k \in [0, 1)$, we obtain

$$d(x_n, x_m) + d(y_n, y_m) \preceq 2\sum_{i=n}^{\infty} \alpha^i \lambda_0 + \sum_{i=n}^{\infty} k^i \theta_0 + 2\beta \sum_{i=n}^{\infty} i k^{i-1} \theta_0 \to 0_E, \quad \text{as } n \to \infty.$$

Case 2. If $\alpha > k$, we have $\frac{k}{\alpha} \leq 1$. So

$$k^{i-1} + \alpha k^{i-2} + \alpha^2 k^{i-3} + \cdots + \alpha^{i-1} = \alpha^{i-1}[1 + \frac{k}{\alpha} + \cdots + (\frac{k}{\alpha})^{i-1}] \leq i \alpha^{i-1}.$$

Similarly,

$$d(x_n, x_m) + d(y_n, y_m) \preceq 2\sum_{i=n}^{\infty} \alpha^i \lambda_0 + \sum_{i=n}^{\infty} k^i \theta_0 + 2\beta \sum_{i=n}^{\infty} i \alpha^{i-1} \theta_0 \to 0_E, \quad \text{as } n \to \infty.$$

Hence, for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$2\sum_{i=n}^{\infty} \alpha^i \lambda_0 + \sum_{i=n}^{\infty} k^i \theta_0 + 2\beta \sum_{i=n}^{\infty} [k^{i-1} + \alpha k^{i-2} + \alpha^2 k^{i-3} + \cdots + \alpha^{i-1}]\theta_0 \ll c.$$

Consequently, for every $c \in E$ with $0_E \ll c$, we get for all $m > n > N$,

$$d(x_n, x_m) + d(y_n, y_m) \ll c,$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and so are convergent. Since A and B are closed subsets of X, there exist $x \in A$ and $y \in B$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y. \tag{2.9}$$

From (2.7), for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$d(x_n, y_n) = \lambda_n \ll c,$$

which implies that

$$\lim_{n\to\infty} d(x_n, y_n) = 0_E. \tag{2.10}$$

We have

$$d(x, y) \preceq d(x, x_n) + d(x_n, y_n) + d(y_n, y).$$

From (2.9) and (2.10), we obtain $\lim_{n\to\infty} d(x, x_n) + d(x_n, y_n) + d(y_n, y) = 0_E$. Then for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$d(x, y) \preceq c.$$

Hence $d(x, y) = 0_E$, that is, $x = y$ and so $A \cap B \neq \emptyset$.

Now, we shall show that $x = F(x, y)$. From (2.2), by taking $u = y_n$, $v = x_n$ for $n \geq 0$, we have

$$\begin{aligned} d(x, F(x, y)) &\preceq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) \\ &= d(x, x_{n+1}) + d(F(x, y), F(y_n, x_n)) \\ &\preceq d(x, x_{n+1}) + a_1 d(x, y_n) + a_2[d(F(x, y), x) + d(x_{n+1}, y_n)] \\ &+ a_3 d(y, x_n) + a_4[d(F(x, y), y_n) + d(y, x_n)] \\ &\preceq d(x, x_{n+1}) + a_1 d(y, y_n) + a_2[d(F(x, y), x) + \theta_n] \\ &+ a_3 d(x, x_n) + a_4[d(F(x, y), x) + d(y, y_n) + d(x, x_n)], \end{aligned}$$

which implies that

$$(1 - a_2 - a_4) d(x, F(x, y)) \preceq d(x, x_{n+1}) + (a_1 + a_4) d(y, y_n) + a_2 \theta_n + (a_3 + a_4) d(x_n, x).$$

Since $\lim_{n\to\infty} d(x, x_{n+1}) + (a_1 + a_4) d(y, y_n) + a_2 \theta_n + (a_3 + a_4) d(x_n, x) = 0_E$. Then for every $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have

$$d(x, x_{n+1}) + (a_1 + a_4) d(y, y_n) + a_2 \theta_n + (a_3 + a_4) d(x_n, x) \ll (1 - a_2 - a_4) c.$$

Hence, for every $c \in E$ with $0_E \ll c$

$$(1 - a_2 - a_4) d(x, F(x, y)) \ll (1 - a_2 - a_4) c,$$

which implies that

$$d(x, F(x, y)) \ll c.$$

It follows that $d(x, F(x, y)) = 0_E$, that is, $x = F(x, y)$. We conclude that x is a strong coupled fixed point of F.

Suppose F has two strong coupled fixed points x, y in $A \cap B$, that is, $x = F(x, x)$ and $y = F(y, y)$ with $x, y \in A \cap B$. From (2.2), we have

$$\begin{aligned} d(x, y) = d(F(x, x), F(y, y)) &\preceq a_1 d(x, x) + a_2[d(F(x, x), x) + d(F(y, y), y)] \\ &+ a_3 d(x, y) + a_4[d(F(x, x), y) + d(F(y, y), x)] \\ &= (a_3 + 2a_4) d(x, y). \end{aligned}$$

Then

$$0_E \preceq (1 - a_3 - 2a_4) d(x, y) \preceq 0_E.$$

Since $1 - a_3 - 2a_4 > 0$, we get $d(x, y) = 0_E$, that is, $x = y$. This proves the uniqueness of the strong coupled fixed point. □

Using the same techniques, we have the following result.

Theorem 2.2. *Let A and B be two nonempty closed subsets of a complete cone metric space (X, d) with a cone P having nonempty interior. Let $F : A \times B \cup B \times A \to A \cup B$ be a coupling with respect to A and B satisfying the contraction condition (2.2). Then F has a unique strong coupled fixed point $x \in A \cap B$. For any $(x_0, y_0) \in A \times B \cup B \times A$, let $x_{n+1} = F(y_n, x_n)$ and $y_{n+1} = F(x_n, y_n)$. Then the sequence $\{x_n\}$ converges to x.*

The following example illustrates Theorem 2.2.

Example 2.3. *Let $X = \mathbb{R}$. Take $E = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Consider $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, 2|x - y|)$. Obviously, (X, d) is a complete cone metric space. Let $A = (-\infty, 1]$ and $B = [0, \infty)$. It is easy to show that A and B are closed subsets of X. Let $F : A \times B \cup B \times A \to A \cup B$ be defined by $F(x, y) = 1$ for all $(x, y) \in (A \times B) \cup (B \times A)$. Clearly, F is a coupling with respect to A and B. Letting $x, v \in A$ and $y, u \in B$, we have $F(x, y) = F(u, v) = 1$. Then for all a_i, $i = 1, 2, 3, 4$ with $a_1 + 2a_2 + a_3 + 2a_4 < 1$, we have*

$$\begin{aligned} d(F(x,y), F(u,v)) = d(1,1) = (0,0) &\preceq a_1 d(x,u) + a_2[d(F(x,y),x) + d(F(u,v),u)] \\ &\quad + a_3 d(y,v) + a_4[d(F(x,y),u) + d(F(u,v),x)]. \end{aligned}$$

All the required conditions of Theorem 2.2 are satisfied. In this case, F has a unique strong coupled fixed point in $A \cap B = [0, 1]$, which is $x = 1$.

The following example shows that under the conditions of Theorem 2.1, the coupling $F : X \times X \to X$ admits a least one strong coupled fixed point, i.e., the strong coupled fixed point of F in X is not perforce unique. However, following Theorem 2.2, the coupling F has a unique strong coupled fixed point in $A \cap B$.

Example 2.4. *Let $X = \mathbb{R}$. Take $E = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Consider $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, |x - y|)$. Obviously, (X, d) is a complete cone metric space. Let $A = (-\infty, 0]$ and $B = [0, \infty)$. It is easy to show that A and B are closed subsets of X. Define $F : X \times X \to X$ by*

$$F(x,y) = \begin{cases} \frac{y-x}{3} & \text{if } (x,y) \in (A \times B) \cup (B \times A) \\ 1 & \text{if not.} \end{cases}$$

Clearly, F is a coupling with respect to A and B.

Now, let $x, v \in A$ and $y, u \in B$. In this case, we have $F(x, y) = \frac{y-x}{3}$ and $F(u, v) = \frac{v-u}{3}$. Then

$$d(F(x,y), F(u,v)) = d(\frac{y-x}{3}, \frac{v-u}{3}) = \frac{1}{3}(|y - x - v + u|, |y - x - v + u|).$$

Moreover,

$$d(x,u) = (|x-u|, |x-u|), \quad d(y,v) = (|y-v|, |y-v|).$$

Hence for $a_1 = a_3 = \frac{1}{3}$ and $a_2 = a_4 = 0$, we have

$$\begin{aligned} 0_E &\preceq a_1 d(x,u) + a_3 d(y,v) - d(F(x,y), F(u,v)) \\ &= \frac{1}{3}(|x-u| + |y-v| - |y-x-v+u|, |x-u| + |y-v| - |y-x-v+u|), \end{aligned}$$

which implies that

$$\begin{aligned} d(F(x,y),F(u,v)) \preceq a_1 d(x,u) + a_2[d(F(x,y),x) + d(F(u,v),u)] \\ + a_3 d(y,v) + a_4[d(F(x,y),u) + d(F(u,v),x)]. \end{aligned}$$

Then all the required conditions of Theorem 2.1 are satisfied. In this case, we observe that $x = 0$ is the unique strong coupled fixed point of F in $A \cap B$. However, the coupling F has two strong coupled fixed points in X, which are $x = 0$ and $y = 1$.

We have the following immediate consequences.

Corollary 2.5. *(Banach type coupling) Let A and B be two nonempty closed subsets of a complete cone metric space (X,d) with a cone P having nonempty interior. Let $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$d(F(x,y),F(u,v)) \preceq k[d(x,u) + d(y,v)],$$

for all $x, v \in A$ and $y, u \in B$, where $k \in [0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. It suffices to take $a_1 = a_3 = k$ and $a_2 = a_4 = 0$ in Theorem 2.1. □

Corollary 2.6. *(Kannan type coupling) Let A and B be two nonempty closed subsets of a complete cone metric space (X,d) with a cone P having nonempty interior. Let $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$d(F(x,y),F(u,v)) \preceq k[d(x,F(x,y)) + d(u,F(u,v))],$$

for all $x, v \in A$ and $y, u \in B$, where $k \in [0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. It suffices to take $a_1 = a_3 = a_4 = 0$ and $a_2 = k$ in Theorem 2.1. □

Corollary 2.7. *(Chatterjea type coupling) Let A and B be two nonempty closed subsets of a complete cone metric space (X,d) with a cone P having nonempty interior. Let $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$d(F(x,y),F(u,v)) \preceq k[d(x,F(u,v)) + d(u,F(x,y))],$$

for all $x, v \in A$ and $y, u \in B$, where $k \in [0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. It suffices to take $a_1 = a_2 = a_3 = 0$ and $a_4 = k$ in Theorem 2.1. □

We also have the following results in the case of metric spaces.

Corollary 2.8. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) and $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq a_1 d(x,u) + a_2[d(F(x,y),x) + d(F(u,v),u)] \\ + a_3 d(y,v) + a_4[d(F(x,y),u) + d(F(u,v),x)], \end{aligned}$$

for all $x, v \in A$ and $y, u \in B$, where a_i, $i = 1, \cdots, 4$ are nonnegative real numbers such that $a_1 + 2a_2 + a_3 + 2a_4 < 1$. For any $(x_0, y_0) \in A \times B \cup B \times A$, let $x_{n+1} = F(y_n, x_n)$ and $y_{n+1} = F(x_n, y_n)$. Then the sequence $\{x_n\}$ converges to x, the unique strong coupled fixed point of F in $A \cap B$.

Corollary 2.9. *(Theorem 2.1,* [13]*) Let A and B be two nonempty closed subsets of a complete metric space (X, d) and $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)],$$

for all $x, v \in A$ and $y, u \in B$, where $k \in [0, 1)$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. It suffices to take $a_1 = a_3 = \frac{k}{2}$ and $a_2 = a_4 = 0$ in Corollary 2.8. □

Corollary 2.10. *(Theorem 3.1,* [13]*) Let A and B be two nonempty closed subsets of a complete metric space (X, d) and $F : X \times X \to X$ be a coupling with respect to A and B satisfying*

$$d(F(x,y), F(u,v)) \leq k[d(x, F(u,v)) + d(u, F(x,y))],$$

for all $x, v \in A$ and $y, u \in B$, where $k \in [0, \frac{1}{2})$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. It suffices to take $a_1 = a_2 = a_3 = 0$ and $a_4 = k$ in Corollary 2.8. □

3. Application

Let (X, d) be a metric space (it is a cone metric space). Let $F : X \times X \to X$ and $g : X \to \mathbb{R}$ be two given mappings. This section is devoted to study the existence of solutions of the following problem: Find an element $X \in X$ satisfying

$$\begin{cases} F(x,x) = x, \\ g(x) = 0. \end{cases} \tag{3.1}$$

We state and prove the following result.

Theorem 3.1. *Suppose the following conditions hold.*

(*i*) *There exists $x_1 \in X$ such that $g(x_1) \geq 0$;*
(*ii*) *g is continuous;*
(*iii*) *For all $x, y \in X$, we have $g(x)g(F(x,y)) \leq 0$;*
(*iv*) *There exist nonnegative real numbers a_i, $i = 1, \cdots, 4$ such that $a_1 + 2a_2 + a_3 + 2a_4 < 1$ for which, we have*

$$\begin{aligned} d(F(x,y), F(u,v)) \leq{}& a_1 d(x,u) + a_2[d(F(x,y), x) + d(F(u,v), u)] \\ &+ a_3 d(y,v) + a_4[d(F(x,y), u) + d(F(u,v), x)] \end{aligned}$$

for all $x, y \in X$ satisfying $g(x), g(v) \geq 0$ and $g(y), g(u) \leq 0$.

Then the problem (3.1) has a unique solution.

Proof. Take $A = \{x \in X : g(x) \geq 0\}$ and $B = \{x \in X : g(x) \leq 0\}$. From the assumptions (i), (ii), (iii), A and B are nonempty closed subsets of X. Let $x \in A$ and $y \in B$. In this case, we have $g(x) \geq 0$ and $g(y) \leq 0$. Using the condition (iii), we have $g(F(x,y)) \leq 0$ and $g(F(y,x)) \geq 0$, so $F(x,y) \in B$ and $F(y,x) \in A$. Then F is a coupling with respect to A and B.

Now, from assumption (iv), for $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} d(F(x,y), F(u,v)) \leq{}& a_1 d(x,u) + a_2[d(F(x,y), x) + d(F(u,v), u)] \\ &+ a_3 d(y,v) + a_4[d(F(x,y), u) + d(F(u,v), x)]. \end{aligned}$$

By Theorem 2.1, F has a unique strong coupled fixed point in $A \cap B = \{x \in X : g(x) = 0\}$. Consequently, the problem (3.1) has a unique solution.

□

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