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COUPLED FIXED POINT THEOREMS IN A_b-METRIC SPACES SATISFYING RATIONAL INEQUALITY

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Abstract. In this paper, we prove some fixed point theorems for self mapping that satisfies a new type of rational contraction in the n-dimensions A_b-metric spaces, which was introduced by Manoj et. al. [4].

1. Introduction and preliminaries

First we would like to give the reader a brief on some of the history of different metric spaces. Different generalizations of metric spaces was given, for instance, In the year 1989, I.A. Bhaktin [1] introduced of b-metric spacse in which the triangle inequality of the ordinary metric spaces is replaced by a more general form. Also, Sedghi et. al. [2] introduced another generalized concept of metric spaces and he called it S-metric spaces. Next, the notion of S_b-metric and partial S_b-metric were introduced see [6], [7]. Recently, Abbas et. al. in [3] introduce A-metric spaces, which an n-dimensions metric spaces. For more results on other generalized metric spaces we refer the reader to the following references [6, 8, 9, 10, 11, 12, 13, 14]
Notation: Throughout this paper, n is a fixed positive integer that denote the dimension of the space.

Definition 1.1. [3] Let X be a nonempty set. A function $A : X^n \to [0, \infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:

(A1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$,
(A2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$,

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(A3)

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) &\leq [A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ &\quad +A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ &\quad +A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ &\quad \vdots \\ &\quad +A(x_{n-1}, x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ &\quad +A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a).] \end{aligned}$$

The pair (X, A) is called an A-metric space.

Definition 1.2. [1] Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $b \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$

The pair (X, d) is called a b-metric space.

Inspired by Abbas et. al. [3] and Bakhtin [1], Manoj et. al. [4] introduced a generalized form of n-tuple metric spaces, which they named it A_b-metric space and defined as follows.

Definition 1.3. [4] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is called an A_b-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:

(A_b1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$,
(A_b2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$,
(A_b3)

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) &\leq b[A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ &\quad +A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ &\quad +A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ &\quad \vdots \\ &\quad +A(x_{n-1}, x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ &\quad +A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A) is called an A_b-metric space.

Definition 1.4. An A_b-metric space is said to be symmetric if $A(x_1, x_1, \ldots, (x_1)_{n-1}, x_2) = A(x_2, x_2, \ldots, (x_2)_{n-1}, x_1)$.

Remark: Note that A-metric spaces is a special case of A_b-metric space and that is when $b = 1$. Also, b-metric spaces and S_b-metric spaces are special cases of A_b-metric spaces with $n = 2$ and 3 respectively. Ordinary metric spaces and S-metric space are also special cases of A_b-metric space with $b = 1$ and respective values of n as 2 and 3.

Examle 1.5. [4] Let $X = [1, +\infty)$. Define $A_b : X^n \to [1, +\infty)$ by

$$A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$$

for all $x_i \in X, i = 1, 2, \ldots n$.

Then (X, A_b) is an A_b-metric space with $s = 2 > 1$

Examle 1.6. [4] Let $X = \mathbb{R}$. Define $A_b : X^n \to [1, +\infty)$ by

$$\begin{aligned} A_b(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) &= |\sum_{i=n}^{2} x_i - (n-1)x_1|^2 + |\sum_{i=n}^{3} x_i - (n-2)x_2|^2 + \ldots \\ &\quad + |\sum_{i=n}^{n-3} x_i - 3x_{n-3}|^2 + |\sum_{i=n}^{n-2} x_i - 2x_{n-2}|^2 \\ &\quad + |x_n - x_{n-1}|^2 \end{aligned}$$

for all $x_i \in X, i = 1, 2, \ldots n$.

Then (X, A_b) is an A_b-metric space with $s = 2 > 1$

Examle 1.7. [5] Let $X = [1, \infty)$. Define a function $A_b : X^n \to [0, \infty)$ by $A_b(x_1, x_2, \cdots, x_n) = |x_1 - \max\{x_2, \cdots, x_n\}|^2$ For all $x_1, x_2, \cdots, x_n \in X$. Then (X, A_b) is an A_b-metric on X with $b = 2$, and it not difficult to see that (X, A_b) is not an $A-$metric space on X.

Examle 1.8. [5] Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_k\} \subset \mathbb{R} : \sum_{i=1}^{k} \lim |x_i|^p < \infty\}$. Let $A_b : X^n \to [0, \infty)$ defined by $A_b(x_1, x_2, \cdots, x_n) = (\sum_{k=1}^{\infty} \lim |x_1 - \max\{x_2, \cdots, x_n\}|^p)^{1/p}$, where $x_1 = \{x_{1_k}\} \in l^p$, $x_i = \{x_{i_k}\} \in l^p, i = 2, \cdots, n$. Then (X, A_b) is an A_b-metric on X, with constant $b = 2^{\frac{1}{p}} > 1$

Lemma 1.9. [4] *Let (X, A) be A_b-metric space. Then $A(x, x, x, ..., x, y) \leq bA(y, y, y, ..., y, x)$ for all $x, y \in X$.*

Lemma 1.10. [5] *Let (X, A) be A_b-metric space. Then for all $x, y, z \in X$ we have $A(x, x, x, ..., x, z) \leq (n-1)bA(x, x, x, ..., x, y) + b^2 A(y, y, y, ..., y, z)$.*

Definition 1.11. The A_b-metric space (X, A) is said to be bounded if there exists a constant $r > 0$ such that $A(x, x, x, ..., x, y) \leq r$ for all $x, y \in X$. Otherwise, X is unbounded.

Definition 1.12. [5] Given a point x_0 in A_b-metric space (X, A) and a positive real number r, the set $B(x_0, r) = \{y \in X : A(y, y, y, ..., y, x_0) < r\}$ is called an open ball centered at x_0 with radius r.

The set $\overline{B(x_0, r)} = \{y \in X : A(y, y, y, ..., y, x_0) \leq r\}$ is called a closed ball centered at x_0 with radius r.

Definition 1.13. [5] A subset G in A_b-metric space (X, A) is said to be an open set if for each $x \in G$ there exists an $r > 0$ such that $B(x, r) \subset G$. A subset $F \subset X$ is called closed if $X \backslash F$ is open.

Lemma 1.14. [5] *In any A_b-metric space (X, A), each open ball is an open set in X and each closed ball is also a closed set in X.*

Theorem 1.15. [5] *Let (X, A) be A_b-metric space, then:*

(i) *An arbitrary union and finite intersection of open balls $B(x, r) \in X$ is open.*
(ii) *An arbitrary intersection and finite union of closed balls $B(x, r) \in X$ is closed.*

Theorem 1.16. [5] *The collection $T = \{B(x, r) : x \in X, r > 0\}$ of all balls in A_b-metric space (X, A) is a basis for a topology τ on X.*

Definition 1.17. [5] Let (X, A) be A_b-metric space. A sequence $\{x_k\}$ in X is said to converge to a point $x \in X$. If $A(x_k, x_k, x_k, ..., x_k, x) \to 0$ as $k \to \infty$.

That is, for each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$ we have $A(x_k, x_k, x_k, ..., x_k, x) \leq \varepsilon$ and we write $\lim_{k\to\infty} x_k = x$.

Lemma 1.18. [5] *Let (X, A) be A_b-metric space. If the sequence $\{x_k\}$ in X converges to a point x, then x is unique.*

Definition 1.19. [5] Let (X, A) be A_b-metric space. A sequence $\{x_k\}$ in X is called a Cauchy sequence if $A(x_k, x_k, x_k, ..., x_k, x_m) \to 0$ as $k, m \to \infty$.

That is, for each $\epsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k, m \in \mathbb{N}$ we have $A(x_k, x_k, x_k, ..., x_k, x_m) \leq \epsilon$ and we write $\lim_{k\to\infty} x_k = x$.

Lemma 1.20. [5] *Every convergent sequence in A_b-metric space is a Cauchy sequence.*

Remark 1.21. [5] The converse of Lemma 1.17 does not hold in general. A Cauchy sequence in an A_b-metric space does not need to be convergent.

Definition 1.22. [5] The A_b-metric space (X, A) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.23. [5] Let (X, A_X) and (Z, A_Z) be A_b-metric spaces. A function $f : X \to Z$ is continuous at a point $x_0 \in X$. If $f^{-1}(G)$ is open in X, for each open set G in Z. The function f is continuous on X if it is continuous at each points of X.

Theorem 1.24. [5] *Let (X, A_X) and (Z, A_Z) be A_b-metric spaces. A function $f : X \to Z$ is continuous at a point $x_0 \in X$ iff it is sequentially continuous at x_0.*

N. Mlaiki and Y. Rohen [5] proved some coupled fixed point theorems in partially ordered A_b-metric spaces.

In 1987, Guo and Lakshmikantham [13] introduced the idea of coupled fixed point.

Following definition was given by Guo and Lakshmikantham.

Definition 1.25. [15] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $S : X \times X \to X$ if

$$S(x, y) = x, \text{ and } S(y, x) = y.$$

2. Main results

In this section, we prove some coupled fixed point theorems in A_b-metric space.

Theorem 2.1. *Let (X, A) be a complete symmetric A_b-metric space with parameter $b \geq 1$ and let the mappings $f, g : X^2 \to X$ satisfying*

$$\begin{aligned}
A(f(x,y),\dots,f(x,y),g(u,v)) \quad \leq \quad & a_1 \frac{A(x,\dots,x,u)+A(y,\dots,y,v)}{2} \\
& + a_2 \frac{A(f(x,y),\dots,f(x,y),g(u,v))A(x,\dots,x,u)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \\
& + a_3 \frac{A(f(x,y),\dots,f(x,y),g(u,v))A(y,\dots,y,v)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \\
& + a_4 \frac{A(x,\dots,x,f(x,y))A(x,\dots,x,u)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \\
& + a_5 \frac{A(x,\dots,x,f(x,y))A(y,\dots,y,v)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \\
& + a_6 \frac{A(u,\dots,u,g(u,v))A(x,\dots,x,u)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \\
& + a_7 \frac{A(u,\dots,u,g(u,v))A(y,\dots,y,v)}{1+A(x,\dots,x,u)+A(y,\dots,y,v)} \qquad (1)
\end{aligned}$$

for all $x, y, u, v \in X$ and $a_1, a_2, \dots, a_7 \geq 0$ with $a_1 + a_2 + \cdots + a_7 < 1$ and $b < \frac{1-a_2-a_3-a_6-a_7}{a_1+a_4+a_5}$. Then f and g have a common coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be arbitrary points. Define

$$\begin{aligned}
x_{2k+1} = f(x_{2k}, y_{2k}) \quad &, \quad y_{2k+1} = f(y_{2k}, x_{2k}) \\
x_{2k+2} = g(x_{2k+1}, y_{2k+1}) \quad &, \quad y_{2k+2} = g(y_{2k+1}, x_{2k+1})
\end{aligned}$$

for $k = 0, 1, 2, \ldots$. Then

$$\begin{aligned}
& A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) = A(f(x_{2k}, y_{2k}), \ldots, f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1})) \\
\leq\ & a_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2} \\
& + a_2 \frac{A(f(x_{2k}, y_{2k}), \ldots, f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1})) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_3 \frac{A(f(x_{2k}, y_{2k}), \ldots, f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1})) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_4 \frac{A(x_{2k}, \ldots, x_{2k}, f(x_{2k}, y_{2k})) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_5 \frac{A(x_{2k}, \ldots, x_{2k}, f(x_{2k}, y_{2k})) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_6 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, g(x_{2k+1}, y_{2k+1})) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_7 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, g(x_{2k+1}, y_{2k+1})) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
=\ & a_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2} \\
& + a_2 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_3 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_4 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_5 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_6 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) A(x_{2k}, \ldots, x_{2k}, x_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
& + a_7 \frac{A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{1 + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})} \\
\leq\ & a_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2} \\
& + (a_2 + a_3) A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + (a_4 + a_5) A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) \\
& + (a_6 + a_7) A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (1 - a_2 - a_3 - a_6 - a_7) A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \quad & \leq \quad (\frac{a_1}{2} + a_4 + a_5) A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) \\
& \quad + \frac{a_1}{2} A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) \\
\Rightarrow A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) \quad & \leq \quad \frac{\frac{a_1}{2} + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) \\
& \quad + \frac{\frac{a_1}{2}}{1 - a_2 - a_3 - a_6 - a_7} A(y_{2k}, \ldots, y_{2k}, y_{2k+1})
\end{aligned} \tag{2}$$

Proceeding similarly one can prove that

$$\begin{aligned} A(y_{2k+1},\ldots,y_{2k+1},y_{2k+2}) &\leq \frac{\frac{a_1}{2}+a_4+a_5}{1-a_2-a_3-a_6-a_7}A(y_{2k},\ldots,y_{2k},y_{2k+1}) \\ &\quad + \frac{\frac{a_1}{2}}{1-a_2-a_3-a_6-a_7}A(x_{2k},\ldots,x_{2k},x_{2k+1}) \quad (3) \end{aligned}$$

Adding (2) and (3) we get

$$\begin{aligned} A(x_{2k+1},\ldots,x_{2k+1},x_{2k+2}) &+ A(y_{2k+1},\ldots,y_{2k+1},y_{2k+2}) \\ &\leq \frac{a_1+a_4+a_5}{1-a_2-a_3-a_6-a_7}[A(x_{2k},\ldots,x_{2k},x_{2k+1})+A(y_{2k},\ldots,y_{2k},y_{2k+1})] \\ \Rightarrow A(x_{2k+1},\ldots,x_{2k+1},x_{2k+2}) &+ A(y_{2k+1},\ldots,y_{2k+1},y_{2k+2}) \\ &\leq h[A(x_{2k},\ldots,x_{2k},x_{2k+1})+A(y_{2k},\ldots,y_{2k},y_{2k+1})] \end{aligned}$$

where $h=\frac{a_1+a_4+a_5}{1-a_2-a_3-a_6-a_7}<1$.

Also, we can show that

$$\begin{aligned} A(x_{2k+2},\ldots,x_{2k+2},x_{2k+3}) &+ A(y_{2k+2},\ldots,y_{2k+2},y_{2k+3}) \\ &\leq h[A(x_{2k+1},\ldots,x_{2k+1},x_{2k+2})+A(y_{2k+1},\ldots,y_{2k+1},y_{2k+2})] \\ &\leq h^2[A(x_{2k},\ldots,x_{2k},x_{2k+1})+A(y_{2k},\ldots,y_{2k},y_{2k+1})] \end{aligned}$$

Continuing in this way, we have

$$\begin{aligned} A(x_i,\ldots,x_i,x_{i+1}) &+ A(y_i,\ldots,y_i,y_{i+1}) \\ &\leq h[A(x_{i-1},\ldots,x_{i-1},x_i)+A(y_{i-1},\ldots,y_{i-1},y_i)] \\ &\leq h^2[A(x_{i-2},\ldots,x_{i-2},x_{i-1})+A(y_{i-2},\ldots,y_{i-2},y_{i-1})] \\ &\leq \cdots \leq h^i[A(x_0,\ldots,x_0,x_1)+A(y_0,\ldots,y_0,y_1)] \end{aligned}$$

If $A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})=A_i$, then

$$A_i \leq hA_{i-1} \leq h^2A_{i-2} \leq \cdots \leq h^iA_0.$$

Now, we have to prove that $\{x_i\}$ and $\{y_i\}$ are two Cauchy sequences in A_b-metric space (X,A).

For all $i, m \in \mathbb{N}$ with $i \leq m$, we have

$$\begin{aligned}
& A(x_i, \dots, x_i, x_m) + A(y_i, \dots, y_i, y_m) \\
\leq\ & b[(n-1)A(x_i, \dots, x_i, x_{i+1}) + A(x_{i+1}, \dots, x_{i+1}, x_m) \\
& +(n-1)A(y_i, \dots, y_i, y_{i+1}) + A(y_{i+1}, \dots, y_{i+1}, y_m)] \\
=\ & (n-1)b[A(x_i, \dots, x_i, x_{i+1}) + A(y_i, \dots, y_i, y_{i+1})] \\
& +b[A(x_{i+1}, \dots, x_{i+1}, x_m) + A(y_{i+1}, \dots, y_{i+1}, y_m)] \\
\leq\ & (n-1)b[A(x_i, \dots, x_i, x_{i+1}) + A(y_i, \dots, y_i, y_{i+1})] \\
& +b^2[(n-1)A(x_{i+1}, \dots, x_{i+1}, x_{i+2}) + A(x_{i+2}, \dots, x_{i+2}, x_m) \\
& +(n-1)A(y_{i+1}, \dots, y_{i+1}, y_{i+2}) + A(y_{i+2}, \dots, y_{i+2}, y_m)] \\
=\ & (n-1)b[A(x_i, \dots, x_i, x_{i+1}) + A(y_i, \dots, y_i, y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1}, \dots, x_{i+1}, x_{i+2}) + A(y_{i+1}, \dots, y_{i+1}, y_{i+2})] \\
& +(n-1)b^2[A(x_{i+2}, \dots, x_{i+2}, x_m) + A(y_{i+2}, \dots, y_{i+2}, y_m)] \\
\leq\ & (n-1)b[A(x_i, \dots, x_i, x_{i+1}) + A(y_i, \dots, y_i, y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1}, \dots, x_{i+1}, x_{i+2}) + A(y_{i+1}, \dots, y_{i+1}, y_{i+2}) \\
& + \dots + (n-1)b^{m-i-1}[A(x_{m-2}, \dots, x_{m-2}, x_{m-1}) \\
& +A(y_{m-2}, \dots, y_{m-2}, y_{m-1})] \\
& +b^{m-i}[A(x_{m-1}, \dots, x_{m-1}, x_m) + A(y_{m-1}, \dots, y_{m-1}, y_m)] \\
\leq\ & (n-1)b[A(x_i, \dots, x_i, x_{i+1}) + A(y_i, \dots, y_i, y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1}, \dots, x_{i+1}, x_{i+2}) + A(y_{i+1}, \dots, y_{i+1}, y_{i+2})] \\
& +(n-1)b^3[A(x_{i+2}, \dots, x_{i+2}, x_{i+3}) + A(y_{i+2}, \dots, y_{i+2}, y_{i+3})] \\
& + \dots + (n-1)b^{m-i-1}[A(x_{m-2}, \dots, x_{m-2}, x_{m-1}) \\
& +A(y_{m-2}, \dots, y_{m-2}, y_{m-1})] \\
& +(n-1)b^{m-i}[A(x_{m-1}, \dots, x_{m-1}, x_m) + A(y_{m-1}, \dots, y_{m-1}, y_m)] \\
\leq\ & (n-1)\{bh^i + b^2h^{i+1} + b^3h^{i+2} + \dots + b^{m-i}h^{m-1}\}A_0 \\
<\ & (n-1)bh^i[1 + bh + (bh)^2 + \dots]A_0 \\
=\ & \frac{(n-1)bh^i}{1-bh}A_0 \to 0 \text{ as } i \to \infty
\end{aligned}$$

which shows that $\{x_i\}$ and $\{y_i\}$ are Cauchy sequences in X. As X is complete A_b-metric space, so there exists $x, y \in X$ such that $x_i \to x$ and $y_i \to y$ as $i \to \infty$.

Now we will prove that $x = f(x, y)$ and $y = f(y, x)$. On the contrary suppose that $x \neq f(x, y)$ and $y \neq f(y, x)$. Then $A(x, \dots, x, f(x, y)) = l_1 > 0$ and

$A(y,\dots,y,f(y,x))=l_2>0$. Using inequality (1)

$$\begin{aligned}
l_1 &= A(x,\dots,x,f(x,y))\\
&\leq b[(n-1)A(x,\dots,x,x_{i+1})+A(x_{i+1},\dots,x_{i+1},f(x,y))]\\
&= b[(n-1)A(x,\dots,x,x_{i+1})+A(f(x_i,y_i),\dots,f(x_i,y_i),f(x,y))]\\
&\leq (n-1)bA(x,\dots,x,x_{i+1})+b\Big[a_1\frac{A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}{2}\\
&\quad +a_2\frac{A(f(x_i,y_i),\dots,f(x_i,y_i),f(x,y))A(x_i,\dots,x_i,x)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\\
&\quad +a_3\frac{A(f(x_i,y_i),\dots,f(x_i,y_i),f(x,y))A(y_i,\dots,y_i,y)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\\
&\quad +a_4\frac{A(x_i,\dots,x_i,f(x_i,y_i))A(x_i,\dots,x_i,x)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\\
&\quad +a_5\frac{A(x_i,\dots,x_i,f(x_i,y_i))A(y_i,\dots,y_i,y)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\\
&\quad +a_6\frac{A(x,\dots,x,f(x,y))A(x_i,\dots,x_i,x)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\\
&\quad +a_7\frac{A(x,\dots,x,f(x,y))A(y_i,\dots,y_i,y)}{1+A(x_i,\dots,x_i,x)+A(y_i,\dots,y_i,y)}\Big]
\end{aligned}$$

Since $\{x_i\}$ and $\{y_i\}$ are convergent to x and y, therefore by taking limit as $i\to\infty$ we get $l_1\leq 0$, which is a contradiction, so $A(x,\dots,x,f(x,y))=0$ which gives $x=f(x,y)$.

Similarly, we can prove that $y=f(y,x)$. Also, we can prove that $x=g(x,y)$ and $y=g(y,x)$. Hence (x,y) is a common coupled fixed point of f and g.

In order to prove the uniqueness of the coupled fixed point, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (1), we have

$$\begin{aligned}
A(x,\ldots,x,p) &= A(f(x,y),\ldots,f(x,y),g(p,q)) \\
&\le a_1\frac{A(x,\ldots,x,p)+A(y,\ldots,y,q)}{2} \\
&\quad +a_2\frac{A(f(x,y),\ldots,f(x,y),g(p,q))A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_3\frac{A(f(x,y),\ldots,f(x,y),g(p,q))A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_4\frac{A(x,\ldots,x,f(x,y))A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_5\frac{A(x,\ldots,x,f(x,y))A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_6\frac{A(p,\ldots,p,g(p,q))A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_7\frac{A(p,\ldots,p,g(p,q))A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&= a_1\frac{A(x,\ldots,x,p)+A(y,\ldots,y,q)}{2} \\
&\quad +a_2\frac{A(x,\ldots,x,p)A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_3\frac{A(x,\ldots,x,p)A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_4\frac{A(x,\ldots,x,x)A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_5\frac{A(x,\ldots,x,x)A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_6\frac{A(p,\ldots,p,p)A(x,\ldots,x,p)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
&\quad +a_7\frac{A(p,\ldots,p,p)A(y,\ldots,y,q)}{1+A(x,\ldots,x,p)+A(y,\ldots,y,q)} \\
\Rightarrow A(x,\ldots,x,p) &\le a_1\frac{A(x,\ldots,x,p)+A(y,\ldots,y,q)}{2} \\
&\quad +a_2A(x,\ldots,x,p)+a_3A(x,\ldots,x,p) \\
\Rightarrow (1-a_2-a_3)A(x,\ldots,x,p) &\le \frac{a_1}{2}A(x,\ldots,x,p)+\frac{a_1}{2}A(y,\ldots,y,q) \\
\Rightarrow (1-\frac{a_1}{2}-a_2-a_3)A(x,\ldots,x,p) &\le \frac{a_1}{2}A(y,\ldots,y,q) \\
\Rightarrow A(x,\ldots,x,p) &\le \frac{a_1}{2-a_1-2a_2-2a_3}A(y,\ldots,y,q) \qquad (4)
\end{aligned}$$

Similarly,

$$A(y,\ldots,y,q) \le \frac{a_1}{2-a_1-2a_2-2a_3}A(x,\ldots,x,p) \qquad (5)$$

Adding (4) and (5) we have

$$A(x,\ldots,x,p)+A(y,\ldots,y,q)\le \frac{a_1}{2-a_1-2a_2-2a_3}[A(x,\ldots,x,p)+A(y,\ldots,y,q)]$$
$$\Rightarrow [1-\frac{a_1}{2-a_1-2a_2-2a_3}][A(x,\ldots,x,p)+A(y,\ldots,y,q)]\le 0$$
$$\Rightarrow \frac{2(1-a_1-a_2-a_3)}{2-a_1-2a_2-2a_3}[A(x,\ldots,x,p)+A(y,\ldots,y,q)]\le 0.$$

Since $a_1+a_2+a_3<1$, $\frac{2(1-a_1-a_2-a_3)}{2-a_1-2a_2-2a_3}>0$.
Hence $A(x,\ldots,x,p)+A(y,\ldots,y,q)=0$, which implies that $x=p$ and $y=q\Rightarrow (x,y)=(p,q)$.
Thus f and g have unique coupled common fixed point. This complete the proof. □

Corollary 2.2. *Let (X,A) be a complete symmetric A_b-metric space with parameter $b\ge 1$ and let the mappings $f:X^2\to X$ satisfying*

$$\begin{aligned}A(f(x,y),\ldots,f(x,y),f(u,v)) \;\le\;& a_1\frac{A(x,\ldots,x,u)+A(y,\ldots,y,v)}{2}\\
&+a_2\frac{A(f(x,y),\ldots,f(x,y),f(u,v))A(x,\ldots,x,u)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\\
&+a_3\frac{A(f(x,y),\ldots,f(x,y),f(u,v))A(y,\ldots,y,v)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\\
&+a_4\frac{A(x,\ldots,x,f(x,y))A(x,\ldots,x,u)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\\
&+a_5\frac{A(x,\ldots,x,f(x,y))A(y,\ldots,y,v)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\\
&+a_6\frac{A(u,\ldots,u,f(u,v))A(x,\ldots,x,u)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\\
&+a_7\frac{A(u,\ldots,u,f(u,v))A(y,\ldots,y,v)}{1+A(x,\ldots,x,u)+A(y,\ldots,y,v)}\end{aligned}$$

for all $x,y,u,v\in X$ and $a_1,a_2,\ldots,a_7\ge 0$ with $a_1+a_2+\cdots+a_7<1$ and $s<\frac{1-a_2-a_3-a_6-a_7}{a_1+a_4+a_5}$. Then f has a unique coupled fixed point in X.

Theorem 2.3. *Let (X,A) be a complete symmetric A_b-metric space with parameter $b\ge 1$ and let the mappings $f,g:X^2\to X$ satisfy*

$$\begin{aligned}&A(f(x,y),\ldots,f(x,y),g(u,v))\le \beta_1\frac{A(x,\ldots,x,u)+A(y,\ldots,y,v)}{2}\\
&+\beta_2\frac{A(x,\ldots,x,f(x,y))A(u,\ldots,u,g(u,v))}{1+b[A(x,\ldots,x,g(x,y))+A(u,\ldots,u,f(u,v))+A(x,\ldots,x,u)+A(y,\ldots,y,v)]}\end{aligned}\tag{6}$$

for all $x,y,u,v\in X$ and β_1,β_2 are non-negative real numbers with $\beta_1+\beta_2<1$ and $b<\frac{1-\beta_2}{\beta_1}$. Then f and g have a unique common coupled fixed point.

Proof. Let $x_0,y_0\in X$ be arbitrary points. Define

$$x_{2k+1}=f(x_{2k},y_{2k}) \quad,\quad y_{2k+1}=f(y_{2k},x_{2k})$$
$$x_{2k+2}=g(x_{2k+1},y_{2k+1}) \quad,\quad y_{2k+2}=g(y_{2k+1},x_{2k+1})$$

for $k = 0, 1, 2, \ldots$. Then

$$\begin{aligned}
&A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})\\
&= A(f(x_{2k}, y_{2k}), \ldots, f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))\\
&\leq \beta_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2}\\
&+\beta_2 \frac{A(x_{2k}, \ldots, x_{2k}, f(x_{2k}, y_{2k}))A(x_{2k+1}, \ldots, x_{2k+1}, g(x_{2k+1}, y_{2k+1}))}{1 + b[A(x_{2k}, .., g(x_{2k+1}, y_{2k+1})) + A(x_{2k+1}, ., x_{2k+1}, f(x_{2k}, y_{2k})) + A(x_{2k}, ., x_{2k+1}) + A(y_{2k}, ., y_{2k+1})]}\\
&= \beta_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2}\\
&+\beta_2 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1})A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})}{1 + b[A(x_{2k}, \ldots, x_{2k+2}) + A(x_{2k+1}, \ldots, x_{2k+1}) + A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})]}\\
&\leq \beta_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2} + \beta_2 A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (1-\beta_2)A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) &\leq \beta_1 \frac{A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})}{2}\\
\Rightarrow A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) &\leq \frac{\beta_1}{2(1-\beta_2)}[A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})]
\end{aligned} \tag{7}$$

Similarly we can show that

$$A(y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2}) \leq \frac{\beta_1}{2(1-\beta_2)}[A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})] \tag{8}$$

Adding (7) and (8) we have

$$\begin{aligned}
A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) &+ A(y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})\\
&\leq \frac{\beta_1}{(1-\beta_2)}[A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})]\\
&= h[A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})]
\end{aligned}$$

where $h = \frac{\beta_1}{(1-\beta_2)}$.
Similarly, we can show that

$$\begin{aligned}
A(x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3}) &+ A(y_{2k+2}, \ldots, y_{2k+2}, y_{2k+3})\\
&\leq h[A(x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2}) + A(y_{2k+1}, \ldots, y_{2k+1}, y_{2k+2})]\\
&\leq h^2[A(x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(y_{2k}, \ldots, y_{2k}, y_{2k+1})]
\end{aligned}$$

Now, if $A(x_i, \ldots, x_i, x_{i+1}) + A(y_i, \ldots, y_i, y_{i+1}) = A_i$ then

$$A_i \leq hA_{i-1} \leq h^2 A_{i-2} \leq \cdots \leq h^i A_0$$

So, for $m > i$ we have

$$
\begin{aligned}
& A(x_i,\ldots,x_i,x_m)+A(y_i,\ldots,y_i,y_m) \\
\leq\ & b[(n-1)A(x_i,\ldots,x_i,x_{i+1})+A(x_{i+1},\ldots,x_{i+1},x_m) \\
& +(n-1)A(y_i,\ldots,y_i,y_{i+1})+A(y_{i+1},\ldots,y_{i+1},y_m)] \\
=\ & (n-1)b[A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})] \\
& +b[A(x_{i+1},\ldots,x_{i+1},x_m)+A(y_{i+1},\ldots,y_{i+1},y_m)] \\
\leq\ & (n-1)b[A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})] \\
& +b^2[(n-1)A(x_{i+1},\ldots,x_{i+1},x_{i+2})+A(x_{i+2},\ldots,x_{i+2},x_m) \\
& +(n-1)A(y_{i+1},\ldots,y_{i+1},y_{i+2})+A(y_{i+2},\ldots,y_{i+2},y_m)] \\
=\ & (n-1)b[A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1},\ldots,x_{i+1},x_{i+2})+A(y_{i+1},\ldots,y_{i+1},y_{i+2})] \\
& +b[A(x_{i+2},\ldots,x_{i+2},x_m)+A(y_{i+2},\ldots,y_{i+2},y_m)] \\
\leq\ & (n-1)b[A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1},\ldots,x_{i+1},x_{i+2})+A(y_{i+1},\ldots,y_{i+1},y_{i+2}) \\
& +\cdots+(n-1)b^{m-i-1}[A(x_{m-2},\ldots,x_{m-2},x_{m-1}) \\
& +A(y_{m-2},\ldots,y_{m-2},y_{m-1})] \\
& +b^{m-i}[A(x_{m-1},\ldots,x_{m-1},x_m)+A(y_{m-1},\ldots,y_{m-1},y_m)] \\
\leq\ & (n-1)b[A(x_i,\ldots,x_i,x_{i+1})+A(y_i,\ldots,y_i,y_{i+1})] \\
& +(n-1)b^2[A(x_{i+1},\ldots,x_{i+1},x_{i+2})+A(y_{i+1},\ldots,y_{i+1},y_{i+2})] \\
& +\cdots+(n-1)b^{m-i-1}[A(x_{m-2},\ldots,x_{m-2},x_{m-1}) \\
& +A(y_{m-2},\ldots,y_{m-2},y_{m-1})] \\
& +(n-1)b^{m-i}[A(x_{m-1},\ldots,x_{m-1},x_m)+A(y_{m-1},\ldots,y_{m-1},y_m)] \\
\leq\ & (n-1)\{bh^i+b^2h^{i+1}+b^3h^{i+2}+\cdots+b^{m-i}h^{m-1}\}A_0 \\
<\ & (n-1)bh^i[1+bh+(bh)^2+\ldots]A_0 \\
=\ & \frac{(n-1)bh^i}{1-bh}A_0 \to 0 \text{ as } i\to\infty
\end{aligned}
$$

Therefore $\{x_i\}$ and $\{y_i\}$ are Cauchy sequences in X. Since X is complete A_b-metric space, so there exists $x, y \in X$ such that $x_i \to x$ and $y_i \to y$ as $i \to \infty$.

Now we will show that $x = f(x,y)$ and $y = f(y,x)$. Suppose on the contrary that $x \neq f(x,y)$ and $y \neq f(y,x)$, so that $A(x,\ldots,x,f(x,y)) = l_1 > 0$ and $A(y,\ldots,y,f(y,x)) = l_2 > 0$.

Consider the following and using inequality (6), we get

$$\begin{aligned}
l_1 &= A(x,\ldots,x,f(x,y)) \\
&\leq b[(n-1)A(x,\ldots,x,x_{i+1}) + A(x_{i+1},\ldots,x_{i+1},f(x,y))] \\
&= b[(n-1)A(x,\ldots,x,x_{i+1}) + A(f(x_i,y_i),\ldots,f(x_i,y_i),f(x,y))] \\
&\leq (n-1)bA(x,\ldots,x,x_{i+1}) + b\Big[\beta_1 \frac{A(x_i,\ldots,x_i,x)+A(y_i,\ldots,y_i,y)}{2} \\
&\quad + \beta_2 \frac{A(x_i,\ldots,x_i,f(x_i,y_i))A(x,\ldots,x,f(x,y))}{1+b[A(x_i,\ldots,x_i,f(x,y))+A(x,\ldots,x,f(x_i,y_i))+A(x_i,\ldots,x_i,x)+A(y_i,\ldots,y_i,y)]}\Big] \\
&= (n-1)bA(x,\ldots,x,x_{i+1}) + \frac{b\beta_1}{2}[A(x_i,\ldots,x_i,x)+A(y_i,\ldots,y_i,y)] \\
&\quad + \beta_2 \frac{A(x_i,\ldots,x_i,x_{i+1})A(x,\ldots,x,f(x,y))}{1+b[A(x_i,\ldots,x_i,f(x,y))+A(x,\ldots,x,x_{i+1})+A(x_i,\ldots,x_i,x)+A(y_i,\ldots,y_i,y)]}
\end{aligned}$$

Taking the limit as $i \to \infty$, we get

$$A(x,\ldots,x,f(x,y)) \leq 0$$

Therefore

$$A(x,\ldots,x,f(x,y)) = 0$$

which implies that $x = f(x,y)$. Similarly, we can prove that $y = f(y,x)$. Also, we can prove that $x = g(x,y)$ and $y = g(y,x)$. Hence, (x,y) is a common coupled fixed point of f and g.

In order to prove that the uniqueness of the common coupled fixed point of f and g, if possible let (p,q) be the second common coupled fixed point of f and g.

Then by using inequality (6), we have

$$\begin{aligned}
&A(x,\ldots,x,p) = A(f(x,y),\ldots,f(x,y),g(p,q)) \\
&\leq \frac{\beta_1}{2}[A(x,\ldots,x,p)+A(y,\ldots,y,q)] \\
&+\beta_2 \frac{A(x,\ldots,x,f(x,y))A(p,\ldots,p,g(p,q))}{1+b[A(x,\ldots,x,g(p,q))+A(p,\ldots,p,f(x,y))+A(x,\ldots,x,p)+A(y,\ldots,y,q)]} \\
&\Rightarrow A(x,\ldots,x,p) \leq \frac{\beta_1}{2}[A(x,\ldots,x,p)+A(y,\ldots,y,q)] \\
&\Rightarrow (1-\frac{\beta_1}{2})A(x,\ldots,x,p) \leq \frac{\beta_1}{2}A(y,\ldots,y,q) \\
&\Rightarrow A(x,\ldots,x,p) \leq \frac{\beta_1}{2-\beta_1}A(y,\ldots,y,q) \qquad (9)
\end{aligned}$$

Similarly

$$A(y,\ldots,y,q) \leq \frac{\beta_1}{2-\beta_1}A(x,\ldots,x,p) \qquad (10)$$

Adding (9) and (10) we have

$$\begin{aligned} A(x,\ldots,x,p)+A(y,\ldots,y,q) &\leq \frac{\beta_1}{2-\beta_1}[A(x,\ldots,x,p)+A(y,\ldots,y,q)] \\ \Rightarrow (1-\frac{\beta_1}{2-\beta_1})[A(x,\ldots,x,p)+A(y,\ldots,y,q)] &\leq 0 \\ \Rightarrow \frac{2(1-\beta_1)}{2-\beta_1}[A(x,\ldots,x,p)+A(y,\ldots,y,q)] &\leq 0 \end{aligned}$$

But $\frac{2(1-\beta_1)}{2-\beta_1} > 0$. Therefore $A(x,\ldots,x,p)+A(y,\ldots,y,q)=0$. Which implies that $x=p$ and $y=q \Rightarrow (x,y)=(p,q)$. Thus f and g have a unique common coupled fixed point as desired. □

Next, we present the following immediate corollary.

Corollary 2.4. *Let (X,A) be a complete symmetric A_b-metric space with parameter $b \geq 1$ and let the mapping $f: X^2 \to X$ satisfying*

$$\begin{aligned} &A(f(x,y),\ldots,f(x,y),f(u,v)) \leq \beta_1 \frac{A(x,\ldots,x,u)+A(y,\ldots,y,v)}{2} \\ &+\beta_2 \frac{A(x,\ldots,x,f(x,y))A(u,\ldots,u,f(u,v))}{1+b[A(x,\ldots,x,f(u,v))+A(u,\ldots,u,f(x,y))+A(x,\ldots,x,u)+A(y,\ldots,y,v)]} \end{aligned}$$

for all $x,y,u,v \in X$ and β_1,β_2 are non-negative real numbers with $\beta_1+\beta_2<1$ and $s<\frac{1-\beta_2}{\beta_1}$. Then f has a unique common coupled fixed point.

3. Conclusion

In closing, we would like ask the following: is possible to prove the same results of Theorem 2.1 without the hypothesis of the symmetry of our space? if not, do we still have existence and uniqueness?

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