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FRACTIONAL OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE PREQUASIINVEX

LADAN AVAZPOUR

Abstract. In this paper, we introduce the fractional Ostrowski type inequality for preinvex function. We use the Holder inequality to find some new bound for this type of inequality. Then, we accommodating it with the work which done before. After that we have Ostrowski's inequality for prequasiinvex and logarithmic preinvex via fractional integral.we end up with conclusion.

1. Introduction

Ostrowski's inequality introduced by Ostrowski [22] in 1938:

Theorem 1.1. *Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to R$ is bounded on (a,b). i.e, $\| f' \|_\infty = \sup_{t\in(a,b)} | f'(t) | < \infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f(x) \right| \quad \leq \quad \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \| f' \|_\infty$$

for any $x \in [a,b]$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.
The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering. For an interesting history and its uses for wide range of applied scientists dealing with fractional calculus, see [23, 24, 30, 21, 25]. Due to understanding the importance of Account deficits, more attention has been attracted to it and a lot of quality researches in this branch of mathematical analysis has been carried out. (see [3, 16, 29] and the references therein).
In resent years, many author introduced Integral inequalities in fractional sense [6, 7, 8, 5]. In [27], Erhan Set proved the Ostrowski's inequality for the function whose derivatives are s-convex in the second sense. Matloka in [18] proved Ostrowski

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Type Inequalities for Functions whose Derivatives are H-convex Via Fractional Integrals and so on.
In [31, 2, 17] authors considered the class of preinvex functions in definition 2.2. Alomari in [1] and Iscan in [12] proved Ostrowski's inequality for preinvex functions. We mix these two concept Ostrowski's inequality for preinvex functions and fractional Ostrowski's inequality and prove the fractional Ostrowski's inequality for preinvex functions.
In this paper, we propose the preinvex, logarithmic preinvex and prequasiinvex functions in order to prove the Ostrowski's inequality whose derivatives are preinvex , logarithmic preinvex and prequasiinvex via fractional integral.
In section 2, we recall some basic concepts. In section 3, we prove the Fractional Ostrowski's inequalities for preinvex fuctions. In section 4, we prove the section 3's theorems logarithmic preinvex and prequasiinvex functions. Finally, we end up with conclusion.

2. Preliminaries

Let $f : K \to \mathbb{R}^n$ and $\eta(.,.) : K \times K \to \mathbb{R}^n$, where K is a nonempty closed set in $\mathbb{R}^n$ be continuous functions. We recall the following definition from [31, 2, 17]

Definition 2.1. *The set $K \in \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in K$ and $t \in [0,1]$,*

$$x + t\eta(y,x) \in K.$$

The invex set K is also called a $\eta-$connected set.

For $\eta(x,y) = x - y$, every convex set is invex.
In the following we have the definition of preinvex function which will be used in the rest of the paper.

Definition 2.2. *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \le (1-t)f(x) + tf(y), \ \forall x, y \in K, \ t \in [0,1].$$

The function f is said to be preconcave with respect to η if and only if $-f$ is preinvex. Obviously, every convex function is a preinvex function with respect to $\eta(y,x) = y - x$, but the converse is not true.

Definition 2.3. *The function f on the invex set K is said to be logarithmic preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \le (f(x))^{1-t}(f(y))^t, \ \forall x, y \in K, \ t \in [0,1].$$

where $f(.) > 0$.

We have the definition of prequasiinvex function as follow, [26]:

Definition 2.4. *The function f on the invex set K is said to be prequasiinvex with respect to η, if*

$$f(x + t\eta(y,x)) \le \max\{f(x), f(y)\}, \ \forall x, y \in K, \ t \in [0,1].$$

where $f(.) > 0$.

We also need the following condition due to Mohan and Neogy [17]:
Condition C: Let $K \subset \mathbb{R}^n$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}$. For any $x, y \in K$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) \leq -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) \leq (1-t)\eta(x,y).$$

Note that for every $x, y \in K$ and every $t \in [0,1]$ from condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) \leq (t_2 - t_1)\eta(x,y).$$

Definition 2.5. *Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ for order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x \in [a,b].$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x \in [a,b].$$

respectively where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

For $\alpha = 1$, the fractional integral reduce to the classical integral.[11, 20, 23]

3. Fractional Ostrowski's type inequality

In this section we recall the Ostrowski type inequality for functions whose derivatives are preinvex [12]. We introduce these inequalities via fractional integral.

Lemma 3.1. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. then we have the equality as follow:*

$$f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du =$$
$$\eta(b,a) \left[\int_0^{\frac{x-a}{\eta(b,a)}} tf'(a + t\eta(b,a))dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (t-1)f'(a+t\eta(b,a))dt \right], \tag{3.1}$$

for each $x \in [a, a+\eta(b,a)]$.

So we have:

Theorem 3.1. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $| f' |$ is preinvex function on K. If f' is integrable on $[a, a+\eta(b,a)]$,then the following inequality holds::*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du \right| \leq \frac{\eta(b,a)}{6} \times \tag{3.2}$$
$$\left\{ \left[3\left(\frac{x-a}{\eta(b,a)}\right)^2 - 2\left(\frac{x-a}{\eta(b,a)}\right)^3 + 2\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3 \right] |f'(a)| + \right.$$
$$\left. \left[1 - 3\left(\frac{x-a}{\eta(b,a)}\right)^2 + 4\left(\frac{x-a}{\eta(b,a)}\right)^3 \right] |f'(b)| \right\} =$$

$$\frac{\eta(b,a)}{6}\times\left\{\left[3\left(\frac{x-a}{\eta(b,a)}\right)^2-2\left(\frac{x-a}{\eta(b,a)}\right)^3+2\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3\right]|f'(a)|+\right.$$
$$\left.\left[3\left(\frac{\eta(b,a)+a-x}{\eta(b,a)}\right)^2-2\left(\frac{\eta(b,a)+a-x}{\eta(b,a)}\right)^3+2\left(\frac{x-a}{\eta(b,a)}\right)^3\right]|f'(b)|\right\}$$

for all $x\in[a,a+\eta(b,a)]$*. The constant* $\frac{1}{6}$ *is best possible in the sense that it cannot be replaced by a smaller value.*

Proof. [12]. □

In order to find the new bound for our inequalities we need the following identity.

Lemma 3.2. *Let* $K\subset\mathbb{R}$ *be an open invex subset with respect to* $\eta(.,.):K\times K\to\mathbb{R}^n$ *and* $a,b\in K$ *with* $a<a+\eta(b,a)$*. Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. Then the following equality for fractional integrals holds:*

$$\frac{[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]}{\eta^{\alpha+1}(b,a)}f(x)-\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]=$$
$$\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha f'(a+t\eta(b,a))dt,\tag{3.3}$$

for each $x\in[a,a+\eta(b,a)]$.

Proof. By integration by part, we have:

$$I=I_1-I_2=\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha f'(a+t\eta(b,a))dt,$$

$$I_1=\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha f'(a+t\eta(b,a))dt=t^\alpha\frac{f(a+t\eta(b,a))}{\eta(b,a)}\Bigg|_0^{\frac{x-a}{\eta(b,a)}}$$
$$-\frac{\alpha}{\eta(b,a)}\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha-1}f(a+t\eta(b,a))dt=\frac{(x-a)^\alpha f(x)}{\eta^{\alpha+1}(b,a)}-\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)}J^\alpha_{x-}f(a)$$

and

$$I_2=\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha f'(a+t\eta(b,a))dt=(1-t)^\alpha\frac{f(a+t\eta(b,a))}{\eta(b,a)}\Bigg|_{\frac{x-a}{\eta(b,a)}}^1$$
$$+\frac{\alpha}{\eta(b,a)}\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\alpha-1}f(a+t\eta(b,a))dt$$
$$=-\frac{(\eta(b,a)+a-x)^\alpha f(x)}{\eta^{\alpha+1}(b,a)}+\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)}J^\alpha_{x+}f(a+\eta(b,a))$$

so we have:

$$[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]f(x)-\Gamma(\alpha+1)\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]=$$
$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha f'(a+t\eta(b,a))dt\right],\tag{3.4}$$

which complete the proof. □

Theorem 3.2. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $|f'|$ is preinvex function on K. For $\alpha > 0$, the following fractional inequality holds::*

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq \tag{3.5}$$
$$\left[\frac{(x-a)^{\alpha+1}}{\alpha+1} + \frac{[(\eta(b,a)+a-x)^{\alpha+2} - (x-a)^{\alpha+2}]}{(\alpha+2)\eta(b,a)}\right]|f'(a)| +$$
$$\left[\frac{[(x-a)^{\alpha+2} - (\eta(b,a)+a-x)^{\alpha+2}]}{(\alpha+2)\eta(b,a)} + \frac{(\eta(b,a)+a-x)^{\alpha+1}}{\alpha+1}\right]|f'(b)|$$

for all $x \in [a, a+\eta(b,a)]$.

Proof. From (3.4) and since $|f'|$ is preinvex on $[a, a+\eta(b,a)]$, we have

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq$$
$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha|f'(a+t\eta(b,a))|dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha|f'(a+t\eta(b,a))|dt\right] \leq$$
$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha[(1-t)|f'(a)| + t|f'(b)|]dt\right.$$
$$\left. + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha[(1-t)|f'(a)| + t|f'(b)|]dt\right] \leq$$
$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha(1-t)|f'(a)|dt + \int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha+1}|f'(b)|dt\right.$$
$$\left. + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha+1}|f'(a)|dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha t|f'(b)|dt\right] \leq$$
$$\eta^{\alpha+1}(b,a)\left[\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(a)| + \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(b)|\right.$$
$$\left. + \frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(a)| + \left[\frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(b)|\right] \leq$$
$$\left[\frac{(x-a)^{\alpha+1}}{\alpha+1} + \frac{[(\eta(b,a)+a-x)^{\alpha+2} - (x-a)^{\alpha+2}]}{(\alpha+2)\eta(b,a)}\right]|f'(a)| +$$
$$\left[\frac{[(x-a)^{\alpha+2} - (\eta(b,a)+a-x)^{\alpha+2}]}{(\alpha+2)\eta(b,a)} + \frac{(\eta(b,a)+a-x)^{\alpha+1}}{\alpha+1}\right]|f'(b)|$$

where we have used the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha(1-t)|f'(a)|dt = \left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(a)|$$
$$\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha+1}|f'(b)|dt = \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(b)|$$
$$\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha+1}|f'(a)|dt = \frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(a)|$$

$$\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}t|f'(b)|dt=\left[\frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+1}}{\alpha+1}-\frac{(\frac{\eta(b,a)+a-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(b)|$$

which complete the proof. □

Remark 3.1. (1) *In theorem* (3.2)*, choose* $\alpha=1$*, then inequality* (3.5) *reduce to* (3.2).

(2) *In theorem* (3.2)*, suppose* $\eta(b,a)=b-a,\ |f'|<M$*, then inequality* (3.5) *reduce to*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b))\right]\right|\leq \quad (3.6)$$

$$\frac{M}{b-a}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right]$$

This inequality is the same inequality of corollary 1 in [27].

(3) *With all assumption of theorem* (3.2)*, let* $x=\frac{2a+\eta(b,a)}{2},\ \eta(b,a)=b-a,\ \alpha=1$*, we have*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-a)}{8}[|f'(a)|+|f'(b)|]$$

which is the inequality (1.3) in [12].

(4) *With all assumption of theorem* (3.2)*, let* $x=\frac{2a+\eta(b,a)}{2},\ \eta(b,a)=b-a,\ \alpha=1$ *and* $|f'|<M$ *we have*

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-a)M}{4}$$

which is the inequality (1.2) in [12].

Theorem 3.3. *Let* $K\subset\mathbb{R}$ *be an open invex subset with respect to* $\eta(.,.):K\times K\to\mathbb{R}^n$ *and* $a,b\in K$ *with* $a<a+\eta(b,a)$*. Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. If* f' *is integrable on* $[a,a+\eta(b,a)]$ *and* $|\ f'\ |^q$ *is preinvex function on* K *for some fixed* $q>1$*. If* η *satisfies condition C,for* $\alpha>0$*,then the following fractional inequality holds:*

$$\left|[(x-a)^{\alpha}+(\eta(b,a)+a-x)^{\alpha}]f(x)-\Gamma(\alpha+1)\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(a+\eta(b,a))\right]\right|\leq$$

$$\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\ (a+\eta(b,a)-x)^{\alpha+1}\left(\frac{|f'(a+\eta(b,a))|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right] \quad (3.7)$$

for all $x\in[a,a+\eta(b,a)]$*, where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (3.4) and since $|\ f'\ |^q$ is preinvex on $[a,a+\eta(b,a)]$, we have

$$\left|[(x-a)^{\alpha}+(\eta(b,a)+a-x)^{\alpha}]f(x)-\Gamma(\alpha+1)\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(a+\eta(b,a))\right]\right|\leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}|f'(a+t\eta(b,a))|dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}|f'(a+t\eta(b,a))|dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right]\leq$$

we have the following integrals and inequalities:

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{p\alpha}dt\right)^{\frac{1}{p}}=\left(\frac{t^{p\alpha+1}}{p\alpha+1}\Big|_0^{\frac{x-a}{\eta(b,a)}}\right)^{\frac{1}{p}}=\left(\frac{(\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}=\left(\frac{-(1-t)^{p\alpha+1}}{p\alpha+1}\Big|_{\frac{x-a}{\eta(b,a)}}^1\right)^{\frac{1}{p}}=\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}$$

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}}|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\leq\left(\frac{x-a}{\eta(b,a)}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)\right)^{\frac{1}{q}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\leq\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\left(\frac{|f'(a+\eta(b,a))|^q+|f'(x)|^q}{2}\right)\right)^{\frac{1}{q}}$$

so we have

$$\left|[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]f(x)-\Gamma(\alpha+1)\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]\right|\leq$$
$$\eta^{\alpha+1}(b,a)\left[\left(\frac{(\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{x-a}{\eta(b,a)}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\left(\frac{|f'(a+\eta(b,a))|^q+|f'(x)|^q}{2}\right)\right)^{\frac{1}{q}}\right]\leq$$
$$\eta^{\alpha+1}(b,a)\left[\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{1-\frac{x-a}{\eta(b,a)}}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{|f'(a+\eta(b,a))|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right]\leq$$
$$\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+(a+\eta(b,a)-x)^{\alpha+1}\left(\frac{|f'(a+\eta(b,a))|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right]$$

which complete the proof. □

Remark 3.2. (1) *In theorem* (3.3)*, choose* $\alpha=1$*, then inequality* (3.7) *reduce to equation (2.5) of theorem (2.4) in* [12].

(2) *In theorem (3.3), suppose* $\eta(b,a)=b-a$, $|f'|<M$, *then inequality (3.7) reduce to*

$$\left|\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(b)\right]\right|\leq \quad (3.8)$$
$$\frac{M}{(p\alpha+1)^{\frac{1}{p}}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

which is the same inequality of corollary 2 in [27] and so on.

In the following theorem we prove the previous theorem in a different way:

Theorem 3.4. *Let* $K\subset\mathbb{R}$ *be an open invex subset with respect to* $\eta(.,.):K\times K\to\mathbb{R}^n$ *and* $a,b\in K$ *with* $a<a+\eta(b,a)$. *Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. If* f' *is integrable on* $[a,a+\eta(b,a)]$ *and* $|\ f'\ |^q$ *is preinvex function on* K *for some fixed* $q\geq 1$. *For* $\alpha>0$, *the following fractional inequality holds:*

$$\left|[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]f(x)-\Gamma(\alpha+1)\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]\right|\leq \quad (3.9)$$
$$\eta^{\alpha+1}(b,a)\left[\left(\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}-\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(a)|^q+\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{(\frac{a+\eta(b,a)-x}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{(\frac{a+\eta(b,a)-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(a)|^q\right.$$
$$\left.\left.+\left[\frac{(\frac{a+\eta(b,a)-x}{\eta(b,a)})^{\alpha+1}}{\alpha+1}-\frac{(\frac{a+\eta(b,a)-x}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

for all $x\in[a,a+\eta(b,a)]$.

Proof. From (3.4) and since $|\ f'\ |^q$ is preinvex on $[a,a+\eta(b,a)]$, we have

$$\left|[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]f(x)-\Gamma(\alpha+1)\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]\right|\leq$$
$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha|f'(a+t\eta(b,a))|dt+\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha|f'(a+t\eta(b,a))|dt\right]\leq$$
$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right]\leq$$
$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha[(1-t)|f'(a)|^q+t|f'(b)|^q]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha[(1-t)|f'(a)|^q+t|f'(b)|^q]dt\right)^{\frac{1}{q}}\right]\leq$$

we have the following integrals and inequalities:

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha dt\right)^{1-\frac{1}{q}} = \left(\frac{t^{\alpha+1}}{\alpha+1}\Big|_0^{\frac{x-a}{\eta(b,a)}}\right)^{1-\frac{1}{q}} = \left(\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha dt\right)^{1-\frac{1}{q}} = \left(\frac{-(1-t)^{\alpha+1}}{\alpha+1}\Big|_{\frac{x-a}{\eta(b,a)}}^1\right)^{1-\frac{1}{q}} = \left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(a)|^q + \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(a)|^q + \left[\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(b)|^q\right)^{\frac{1}{q}}$$

so we have

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J_{x-}^\alpha f(a) + J_{x+}^\alpha f(a+\eta(b,a))\right]\right| \leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(a)|^q + \frac{(\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}|f'(a)|^q\right.$$

$$\left.\left.+\left[\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1} - \frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+2}}{\alpha+2}\right]|f'(b)|^q\right)^{\frac{1}{q}}\right]$$

which complete the proof. □

In theorem (3.4), choose $\alpha = 1$, then inequality (3.9) reduce to theorem (2.8) in [12].

4. Ostrowski's inequalities for logarithmic preinvex functions and prequasiinvex functions via fractional integral

In this section we prove the Ostrowski's type inequality for logarithmic preinvex functions via fractional integral. Then we prove it for prequasiinvex functions via fractional integral.

Theorem 4.1. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $|\ f'\ |$ is log-preinvex function on K. For $\alpha > 0$, the following fractional inequality holds:*

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J_{x-}^\alpha f(a) + J_{x+}^\alpha f(a+\eta(b,a))\right]\right| \leq \tag{4.1}$$

$$\eta^{\alpha+1}(b,a)|f'(a)|\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}-\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}\gamma(\alpha,\frac{x-a}{\eta(b,a)})\right.$$

$$-\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}+\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}$$

$$\left.\left(-(\frac{|f'(b)|}{|f'(a)|})(1-\frac{x-a}{\eta(b,a)})^{\alpha}\left((1-\frac{x-a}{\eta(b,a)})log(\frac{|f'(b)|}{|f'(a)|})\right)^{-\alpha}\Gamma(\alpha,-(\frac{x-a}{\eta(b,a)}-1)log(\frac{|f'(b)|}{|f'(a)|}))\right)\right]$$

for all $x \in [a, a+\eta(b,a)]$.

Proof. From (3.4) and since $|f'|$ is log-preinvex on $[a, a+\eta(b,a)]$, we have

$$\left|[(x-a)^{\alpha}+(\eta(b,a)+a-x)^{\alpha}]f(x)-\Gamma(\alpha+1)\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(a+\eta(b,a))\right]\right|\leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}|f'(a+t\eta(b,a))|dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}|f'(a+t\eta(b,a))|dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}[|f'(a)|^{(1-t)}|f'(b)|^{t}]dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}[|f'(a)|^{(1-t)}|f'(b)|^{t}]dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}[|f'(a)|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}]dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}[|f'(a)|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}]dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)|f'(a)|\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)|f'(a)|\left[\frac{(\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}-\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}\gamma(\alpha,\frac{x-a}{\eta(b,a)})\right.$$

$$-\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}+\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}$$

$$\left.\left(-(\frac{|f'(b)|}{|f'(a)|})(1-\frac{x-a}{\eta(b,a)})^{\alpha}\left((1-\frac{x-a}{\eta(b,a)})log(\frac{|f'(b)|}{|f'(a)|})\right)^{-\alpha}\Gamma(\alpha,-(\frac{x-a}{\eta(b,a)}-1)log(\frac{|f'(b)|}{|f'(a)|}))\right)\right]$$

where

$$\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}dt=\frac{(\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}-\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}\gamma(\alpha,\frac{x-a}{\eta(b,a)})$$

$$\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{t}dt=-\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha}(\frac{|f'(b)|}{|f'(a)|})^{(\frac{x-a}{\eta(b,a)})}}{log(\frac{|f'(b)|}{|f'(a)|})}$$

$$+\frac{\alpha}{log(\frac{|f'(b)|}{|f'(a)|})}\left(-(\frac{|f'(b)|}{|f'(a)|})(1-\frac{x-a}{\eta(b,a)})^{\alpha}\left((1-\frac{x-a}{\eta(b,a)})log(\frac{|f'(b)|}{|f'(a)|})\right)^{-\alpha}\right.$$

$$\left.\Gamma(\alpha,-(\frac{x-a}{\eta(b,a)}-1)log(\frac{|f'(b)|}{|f'(a)|}))\right)$$

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$$

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$$

□

Theorem 4.2. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $\mid f' \mid$ is prequasiinvex function on K. For $\alpha > 0$, the following fractional inequality holds:*

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq \tag{4.2}$$
$$\eta^{\alpha+1}(b,a)\max\{|f'(a)|,|f'(b)|\}\left[\frac{(x-a)^{\alpha+1}+(\eta(b,a)+a-x)^{\alpha+1}}{\alpha+1}\right]$$

for all $x \in [a, a+\eta(b,a)]$.

Proof. From (3.4) and since $\mid f' \mid$ is prequasiinvex on $[a, a+\eta(b,a)]$, we have

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha|f'(a+t\eta(b,a))|dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha|f'(a+t\eta(b,a))|dt\right] \leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha[\max\{|f'(a)|,|f'(b)|\}]dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha[\max\{|f'(a)|,|f'(b)|\}]dt\right] \leq$$

$$\eta^{\alpha+1}(b,a)\max\{|f'(a)|,|f'(b)|\}\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha dt\right] \leq$$

$$\eta^{\alpha+1}(b,a)\max\{|f'(a)|,|f'(b)|\}\left[\frac{(x-a)^{\alpha+1}}{\alpha+1} + \frac{(\eta(b,a)+a-x)^{\alpha+1}}{\alpha+1}\right]$$

which completes the proof. □

Theorem 4.3. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $\mid f' \mid^q$ is prequasiinvex function on K, for some fixed $q > 1$. If η satisfies condition C,for $\alpha > 0$,then the following fractional inequality holds:*

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq \tag{4.3}$$
$$(\max\{|f'(a)|^q,|f'(b)|^q\})^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1}+(\eta(b,a)+a-x)^{\alpha+1}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$, for all $x \in [a, a+\eta(b,a)]$.

Proof. From (3.4) and since $\mid f' \mid^q$ is prequasiinvex on $[a, a+\eta(b,a)]$, we have

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha|f'(a+t\eta(b,a))|dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha|f'(a+t\eta(b,a))|dt\right] \leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right] \leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}} \max\{|f'(a)|^q, |f'(b)|^q\}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1 \max\{|f'(a)|^q, |f'(b)|^q\}dt\right)^{\frac{1}{q}}\right] \leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\frac{(\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\max\{|f'(a)|^q, |f'(b)|^q\}\frac{x-a}{\eta(b,a)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}\left(\max\{|f'(a)|^q, |f'(b)|^q\}(1-\frac{x-a}{\eta(b,a)})\right)^{\frac{1}{q}}\right] \leq$$

$$\eta^{\alpha+1}(b,a)\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1} + \left(\frac{\eta(b,a)+a-x}{\eta(b,a)}\right)^{\alpha+1}\right]$$

$$\leq \left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1} + (\eta(b,a)+a-x)^{\alpha+1}\right]$$

where

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{p\alpha}dt\right)^{\frac{1}{p}} = \left(\frac{t^{p\alpha+1}}{p\alpha+1}\Big|_0^{\frac{x-a}{\eta(b,a)}}\right)^{\frac{1}{p}} = \left(\frac{(\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}} = \left(\frac{-(1-t)^{p\alpha+1}}{p\alpha+1}\Big|_{\frac{x-a}{\eta(b,a)}}^1\right)^{\frac{1}{p}} = \left(\frac{(1-\frac{x-a}{\eta(b,a)})^{p\alpha+1}}{p\alpha+1}\right)^{\frac{1}{p}}$$

the proof is complete. □

Theorem 4.4. *Let $K \subset \mathbb{R}$ be an open invex subset with respect to $\eta(.,.) : K \times K \to \mathbb{R}^n$ and $a, b \in K$ with $a < a+\eta(b,a)$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a+\eta(b,a)]$ and $\mid f' \mid^q$ is prequasiinvex function on K, for some fixed $q \geq 1$. If η satisfies condition C,for $\alpha > 0$,then the following fractional inequality holds:*

$$\left|[(x-a)^\alpha + (\eta(b,a)+a-x)^\alpha]f(x) - \Gamma(\alpha+1)\left[J^\alpha_{x-}f(a) + J^\alpha_{x+}f(a+\eta(b,a))\right]\right| \leq \tag{4.4}$$

$$\leq \left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)\left[(x-a)^{\alpha+1} + (\eta(b,a)+a-x)^{\alpha+1}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$, *for all* $x\in[a,a+\eta(b,a)]$.

Proof. From (3.4) and since $|f'|^q$ is prequasiinvex on $[a,a+\eta(b,a)]$, we have

$$\left|[(x-a)^\alpha+(\eta(b,a)+a-x)^\alpha]f(x)-\Gamma(\alpha+1)\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(a+\eta(b,a))\right]\right|\leq$$

$$\eta^{\alpha+1}(b,a)\left[\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha|f'(a+t\eta(b,a))|dt+\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha|f'(a+t\eta(b,a))|dt\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha|f'(a+t\eta(b,a))|^q dt\right)^{\frac{1}{q}}\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha\max\{|f'(a)|^q,|f'(b)|^q\}dt\right)^{\frac{1}{q}}\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left[\left(\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\max\{|f'(a)|^q,|f'(b)|^q\}\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\max\{|f'(a)|^q,|f'(b)|^q\}\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{\frac{1}{q}}\right]\leq$$

$$\eta^{\alpha+1}(b,a)\left(\max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)\left[\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+\left(\frac{\eta(b,a)+a-x}{\eta(b,a)}\right)^{\alpha+1}\right]$$

$$\leq\left(\max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)\left[(x-a)^{\alpha+1}+(\eta(b,a)+a-x)^{\alpha+1}\right]$$

where

$$\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^\alpha dt\right)^{1-\frac{1}{q}}=\left(\frac{t^{\alpha+1}}{\alpha+1}\Big|_0^{\frac{x-a}{\eta(b,a)}}\right)^{1-\frac{1}{q}}=\left(\frac{(\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}=\left(\frac{-(1-t)^{\alpha+1}}{\alpha+1}\Big|_{\frac{x-a}{\eta(b,a)}}^1\right)^{1-\frac{1}{q}}=\left(\frac{(1-\frac{x-a}{\eta(b,a)})^{\alpha+1}}{\alpha+1}\right)^{1-\frac{1}{q}}$$

the proof is complete . □

5. Conclusion

Recently, investigating some problems in fractional case have been considered widely. To do so, we make inquiries about fractional Ostrowski inequality for the functions whose derivatives are preinvex, prequasiinvex and logarithmic preinvex. An interesting line of work is the study of fuzzy fractional Ostrowski inequality for the functions whose derivatives are preinvex, prequasiinvex and logarithmic preinvex.

References

[1] M. Alomari, S. Hussain, *An Inequality of Ostrowski's Type for preinvex Functions with Applications*,Tamsui Oxford Journal of Information and Mathematical Sciences **29(1)** (2013) 29–37.

[2] M. Aslam Noor, *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure Appl. Math. **8** (2007) 1–6.

[3] A. Ashyralyev, *A note on fractional derivatives and fractional powers of operators,* Journal of Mathematical Analysis and Applications, **357** (2009) 232–236.

[4] M. Alomari, M. Darus, S.S. Dragomir, P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett. **23** (2010) 1071–1076.

[5] A. Barani, A.G. Ghazanfari, S.S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, RGMIA Research Report Collection,**14**(2011).

[6] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci. **9 (4)** (2010) 493–497.

[7] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal.**1 (1)** (2010) 51–58.

[8] Z. Dahmani, L. Tabharit, S. Taf, *Some fractional integral inequalities*, Nonl. Sci. Lett. A **1 (2)** (2010) 155–160.

[9] S.S. Dragomir, S. Fitzpatrik, *The Hadamards inequality for s-convex functions in the second sense*, Demonstratio Math. **32 (4)** (1999) 687–696.

[10] S.S. Dragomir and Th. M. Rassias, *(Eds) Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London. (2002).

[11] R. Gorenflo, F. Mainardi, *Fractional calculus; integral and differential equations of fractional order*, Springer Verlag, Wien. (1997) 223–276.

[12] I. Iscan, *Ostrowski type inequalities for functions whose derivatives are preinvex*, Bulletin of the Iranian Mathematical Society **40** (2014) 373–386.

[13] U.S. Kirmaci et al., *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comp., **193** (2007) 26–35.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amesterdam, (2006).

[15] Lakshmikantham, V. and A.S. Vatsala. *Theory of Fractional Differential Inequalities and Applications*, Communications is Applied Analysis. **11** (2007) 395–402.

[16] S. D. Lin and H. M. Srivastava, *Some miscellaneous properties and applications of certain operators of fractional calculus*, Taiwanese Journal of Mathematics, **14** (2010) 2469–2495.

[17] S.R.Mohan and S.K. Neogy, *On invex sets and preinvex functions*, J. Math. Anal. Appl. **189** (1995) 901–908.

[18] Marian Matloka, *Ostrowski Type Inequalities for Functions whose Derivatives are H-convex Via Fractional Integrals*, Journal of Scientific Research and Reports, **3(12)** (2014) 1633–1641.

[19] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Classical and New Inequalities in Analysis, Kluwer Academic Publishers*, Dordrecht, **106** (1993).

[20] S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA,**2** (1993).

[21] K.B. Oldham, J. Spainer, *The fractional calculus*, Academic Press, New York, (1974).

[22] A. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem intetegralmittelwert*, Comment. Math. Helv, **10** (1938) 226–227.

[23] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999).

[24] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calculus Appl. Anal.**5** (2002) 367–386 .

[25] G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Yverdon. (1993).

[26] Mehmet Zeki Sarikaya, Necmettin Alp, Hakan Bozkurt,*On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemporary Analysis and applied Mathematics, **1** (2013) 237–252.

[27] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Computers and Mathematics with Applications, **63** (2012) 1147–1154

[28] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011) 1140–1153.

[29] Z. Tomovski, R. Hilfer, and H. M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Lefler type functions*, Integral Transforms and Special Functions, **21** (2010) 797–814.

[30] J. Tenreiro Machado, V. Kiryakova, F. Mainardi, *Recent history of fractional calculus*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011) 1140–1153.

[31] X. M. Yang and D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl. **256** (2001) 229–241.

Ladan Avazpour
Department of Mathematics, Yasooj Branch, Islamic Azad University, Yasooj, Iran
E-mail address: avazpour.l@gmail.com