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ASYMPTOTICALLY f-LACUNARY STATISTICAL EQUIVALENT OF SET SEQUENCES IN WIJSMAN SENSE

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Abstract. In this work, we introduce a generalization of asymptotically lacunary statistical equivalence of set sequences in Wijsman sense by using modulus functions and we obtain some inclusion results related to asymptotically f-lacunary statistical equivalence of set sequences in Wijsman sense with the notion of strongly asymptotically f-lacunary equivalence.

1. Introduction

Marouf [8] presented the study of the relationships between the asymptotic equivalence of two sequences and asymptotic regular matrix in 1993. As the same year lacunary statistical convergence was introduced by Fridy and Orhan [6]. Patterson [11] defined asymptotically statistical convergence by combining of the definition for asymptotically equivalence and statistical limit in 2003. Patterson and Savaş [12] extended this definition to lacunary sequences in 2006. Nuray and Rhoades [10] introduced Kuratowski, Wijsman and Hausdorff statistical convergence of sequences of sets in 2012. In addition to, Ulusu and Nuray [16] defined Wijsman lacunary statistical convergence of sequences of sets and gave the relations between Wijsman statistical convergence and Wijsman lacunary statistical convergence of sequences of sets in 2012. Afterwards, Ulusu and Nuray [17] extended the definition

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of asymptotically lacunary statistical equivalent sequence to Wijsman statistically convergent sequences and Wijsman lacunary statistically convergent sequences in 2013. Related papers can be seen in [3, 4, 15]. The concept of a modulus function was introduced by Nakano [9] in 1953. Maddox [7] and Ruckle [13] used a modulus function to construct some sequence spaces.

In this work, we introduce a generalization of asymptotically lacunary statistical equivalence of set sequences in Wijsman sense by using modulus functions and we obtain some inclusion results related to asymptotically lacunary f-statistical equivalence of set sequences in Wijsman sense with the notion of strongly asymptotically f-lacunary equivalence.

2. Definitions and Notations

Before beginning of the presentation of the main results, we recall the following definitions. Throughout the paper, for brevity, by the notation $\lim_k x_k$, we mean $\lim_{k\to\infty} x_k$ and by $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural numbers and the set of all real numbers, respectively.

Definition 2.1. [8] Let $x = (x_k)$ and $y = (y_k)$ be two nonnegative sequences, then we say that $x = (x_k)$ and $y = (y_k)$ are asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$). If the limit is L, then it will be denoted by $x \overset{L}{\sim} y$.

Definition 2.2. [5] The sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. It is denoted by $st - lim_k x_k = L$.

The sequence $\theta = \{k_r\}$ is called lacunary if there exist an increasing sequence of nonnegative integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 2.3. [12] Let θ be a lacunary sequence and $x = (x_k)$ and $y = (y_k)$ be two nonnegative sequences, then we say that x and y are asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geqslant \varepsilon\}| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L = 1$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any nonempty subsets A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definition 2.4. [1] Let (X, ρ) be a metric space. We say that the sequence $\{A_k\}$ is Wijsman convergent to A for any nonempty closed subsets $A, A_k \subseteq X$ if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

Definition 2.5. [10] Let (X, ρ) be a metric space. We say that the sequence $\{A_k\}$ is bounded for any nonempty closed subset A_k of X if

$$\sup_k d(x, A_k) < \infty$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Definition 2.6. [17] Let (X, ρ) be a metric space. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense) for any nonempty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ if for each $x \in X$

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1$$

(denoted by $A_k \sim B_k$).

Definition 2.7. [9] We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leqslant f(x) + f(y)$, for all $x \geqslant 0$, $y \geqslant 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

A modulus may be unbounded or bounded. For example, $f(x) = x^p$ where $0 < p \leqslant 1$, is unbounded, but $f(x) = \frac{x}{1+x}$ is bounded. It is said that a modulus $f : [0, \infty) \to [0, \infty)$ is slowly varying if the limit relation $\lim_{x\to\infty} \frac{f(ax)}{f(x)} = 1$ holds for every $a > 0$. All bounded modulus are slowly varying. The function $f(x) = log(x + 1)$ is an example of unbounded, slowly varying modulus (see, chapter 1 in [14]).

Lemma 2.8. [7] Let $f : [0, \infty) \to [0, \infty)$ be a modulus. Then there is a finite $\lim_{n\to\infty} \frac{f(t)}{t}$ and equality $\lim_{n\to\infty} \frac{f(t)}{t}$=inf$\{\frac{f(t)}{t} : t \in (0, \infty)\}$ holds.

Definition 2.9. [2] Let $f : [0, \infty) \to [0, \infty)$ be an unbounded modulus. The f-density $d^f(K)$ of a set $K \subseteq \mathbb{N}$ is defined as

$$d^f(K) := \lim_{n\to\infty} \frac{f(|\{k \leqslant n : k \in K\}|)}{f(n)}$$

if this limit exists. A sequence $(x_k) \subset \mathbb{R}$ is said to be f-statistically convergent to $l \in \mathbb{R}$, if for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \geqslant \varepsilon\}$ has the zero f-density.

3. Main Results

Now, we present our main results. Throughout this section, let f be an unbounded modulus function from $[0,\infty)$ to $[0,\infty)$, unless otherwise stated.

Definition 3.1. Let (X,ρ) be a metric space, θ be a lacunary sequence. For any non-empty closed subsets $A_k, B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically f-lacunary statistical equivalent of multiple L in Wijsman sense if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_r \frac{1}{f(h_r)} f\left(\left|\left\{k \in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geqslant \epsilon\right\}\right|\right) = 0 \tag{3.1}$$

(denoted by $A_k \overset{WS_\theta^L(f)}{\sim} B_k$) and simply asymptotically f-lacunary statistical equivalent in Wijsman sense if $L = 1$.

In equation (3.1) if we take $\theta = \{k_r\} = 2^r$ then it reduces to definition of asymptotically f-statistical equivalance of set sequences $\{A_k\}$ and $\{B_k\}$ [Konca, Ş, Küçükaslan, M: On asymptotically f-statistical equivalent set sequences in Wijsman sense, 2017, submitted], that is;

$$\lim_n \frac{1}{f(n)} f\left(\left|\left\{k \leqslant n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geqslant \epsilon\right\}\right|\right) = 0$$

denoted by $A_k \overset{WS^L(f)}{\sim} B_k$.

For $f(x) = x$ and $\theta = \{k_r\} = 2^r$, the Definition 3.1 reduces to the definition of asymptotically statistical equivalence of multiple L given in [17] denoted by $A_k \overset{WS_L}{\sim} B_k$.

Definition 3.2. Let (X,ρ) be a metric space, θ be a lacunary sequence. For any non-empty closed subsets $A_k, B_k \subseteq X$, such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically f-lacunary equivalent of multiple L in Wijsman sense if for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = 0 \tag{3.2}$$

(denoted by $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$) and simply strongly asymptotically f-lacunary equivalent in Wijsman sense if $L = 1$.

If $f(x) = x$ in (3.2), then the limit reduces to $\lim_r \frac{1}{h_r} \sum_{k\in I_r} \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| = 0$ for each $x \in X$ (denoted by $A_k \overset{[WN]_\theta^L}{\sim} B_k$) and we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunary equivalent of multiple L in Wijsman sense (see [17]).

For $\theta = \{k_r\} = 2^r$, the equation (3.2) reduces to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = 0, \qquad n \in \{1, 2, ...\}$$

and denoted by $A_k \overset{[Ces_L^W(f)]}{\sim} B_k$ [Konca, Ş, Küçükaslan, M: On asymptotically f-statistical equivalent set sequences in Wijsman sense, 2017, submitted].

Theorem 3.3. Assume that (X, ρ) is a metric space and let A_k, B_k be non-empty closed subsets of X and $\theta = (k_r)$ be a lacunary sequence. If $A_k \overset{WS_\theta^L(f)}{\sim} B_k$ then $A_k \overset{WS_\theta^L}{\sim} B_k$.

Proof. Assume that $A_k \overset{WS_\theta^L(f)}{\sim} B_k$. For all $x \in X$, $\varepsilon > 0$ and $r \in \mathbb{N}$, let us write

$$K_{x,\varepsilon}^r := \left\{k \in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geqslant \varepsilon\right\}. \tag{3.3}$$

If $A_k \overset{WS_\theta^L}{\nsim} B_k$, that is $\{A_k\}$ and $\{B_k\}$ are not asymptotically lacunary statistical equivalent of multiple L in Wijsman sense (see [17]), then there are $x \in X$ and $\varepsilon > 0$ such that $\limsup_r \frac{|K_{x,\varepsilon}^r|}{h_r} > 0$. Hence, there exists $p \in \mathbb{N}$ and an increasing sequence $(r_j) \subset \mathbb{N}$ such that

$$\lim_{j\to\infty} h_{r_j} = \infty \tag{3.4}$$

and $\frac{|K^{r_j}_{x,\varepsilon}|}{h_{r_j}} \geqslant \frac{1}{p}$ for every $j \in \mathbb{N}$. From the last inequality we have

$$h_{r_j} \leqslant p|K^{r_j}_{x,\varepsilon}|. \tag{3.5}$$

Using the subadditivity of f and (3.5), we obtain $f(h_{r_j}) \leqslant pf(|K^{r_j}_{x,\varepsilon}|)$. Consequently, the inequality

$$\frac{f(|K^{r_j}_{x,\varepsilon}|)}{f(h_{r_j})} \geqslant \frac{1}{p} \tag{3.6}$$

holds for every $j \in \mathbb{N}$. Equations (3.4) and (3.6) imply $\limsup_r \frac{|K^{r_j}_{x,\varepsilon}|}{h_{r_j}} \geqslant \frac{1}{p}$, contrary to $A_k \overset{WS^L_\theta(f)}{\sim} B_k$. □

Theorem 3.4. Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and let A_k, B_k be non-empty closed subsets of X. Assume that f is an unbounded modulus satisfying the inequalities

$$\lim_{t\to\infty} \frac{f(t)}{t} > 0 \ \ and \ \ f(xy) \geqslant \ cf(x)f(y)$$

with some $c \in (0, \infty)$ for all $x, y \in [0, \infty)$. Then the following statements hold

(i) If $A_k \overset{[WN_\theta]^f_L}{\sim} B_k$ then $A_k \overset{WS^L_\theta(f)}{\sim} B_k$.
(ii) If $\{A_k\} \in L_\infty$ and $A_k \overset{WS^L_\theta(f)}{\sim} B_k$ then $A_k \overset{[WN_\theta]^f_L}{\sim} B_k$.

Proof. (i) Let $K^r_{x,\varepsilon}$ be defined as in (3.3). By subadditivity of moduli we have

$$\begin{aligned}\sum_{k\in I_r} f\Big(\Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big|\Big) &= \sum_{k_{r-1}+1}^{k_r} f\Big(\Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big|\Big) \\ &\geqslant f\left(\sum_{k\in I_r}\Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big|\right) \\ &\geqslant f(|K^r_{x,\epsilon}|.\varepsilon) \geqslant cf(|K^r_{x,\varepsilon}|)f(\varepsilon).\end{aligned}$$

Then, we obtain the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big|\right) \geqslant c\frac{f(|K^r_{x,\varepsilon}|)}{f(h_r)}\frac{f(h_r)}{h_r}f(\varepsilon),$$

which gives us the result by taking the limit for $r \to \infty$.

(ii) Let $\{A_k\}$ be Wijsman bounded and let $A_k \overset{WS_\theta^L(f)}{\sim} B_k$. Then the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically f-lacunary statistical equivalent of multiple L in Wijsman sense if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_r \frac{1}{f(h_r)} f\left(\left|\left\{k \in I_r : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geqslant \epsilon\right\}\right|\right) = 0.$$

Since $\{A_k\} \in L_\infty$ then we may assume that there exists an $M > 0$ such that

$$\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \leqslant M$$

for each $x \in X$ and for all k. For a given $\varepsilon > 0$ we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) = \frac{1}{h_r}\sum_{k\in I_r, k\in K_{x,\varepsilon}^r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) + \frac{1}{h_r}\sum_{k\in I_r, k\notin K_{x,\varepsilon}^r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right)$$

$$< \frac{|K_{x,\varepsilon}^r|}{h_r} f(M) + \frac{1}{h_r} h_r f(\varepsilon).$$

Letting $r \to \infty$ we obtain

$$0 \leqslant \lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) < \varepsilon.$$

□

Theorem 3.5. Let (X, ρ) be a metric space, θ be a lacunary sequence and f, g be unbounded modulus functions. Then, for all A_k, B_k non-empty closed subsets of X,

$$A_k \overset{WS_\theta^{L_1}(f)}{\sim} B_k \textit{ and } A_k \overset{WS_\theta^{L_2}(g)}{\sim} B_k \textit{ imply } L_1 = L_2.$$

Proof. From Theorem 3.3 we have

$$A_k \overset{WS_\theta^{L_1}(f)}{\sim} B_k \Rightarrow A_k \overset{WS_\theta^{L_1}}{\sim} B_k$$

$$A_k \overset{WS_\theta^{L_2}(g)}{\sim} B_k \Rightarrow A_k \overset{WS_\theta^{L_2}}{\sim} B_k$$

using the uniqueness of statistical limits of numerical sequences we get $L_1 = L_2$. □

Corollary 3.6. Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence and A_k, B_k non-empty closed subsets of X. Then for every unbounded modulus $f:[0,\infty)\to[0,\infty)$, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically f-lacunary statistical equivalent of an unique multiple L if it exists.

Theorem 3.7. Let (X,ρ) be a metric space, A_k, B_k be non-empty closed subsets of X. If $f:[0,\infty)\to[0,\infty)$ is a modulus such that $\beta=\lim_r \frac{f(h_r)}{h_r}>0$ and $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$ then $A_k \overset{[WN]_\theta^L}{\sim} B_k$.

Proof. Let a modulus f satisfy the condition $\beta=\lim_r \frac{f(h_r)}{h_r}>0$ and let $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$. By Lemma 2.16 [2], we have $\beta=\inf\{\frac{f(h_r)}{h_r}\}$. Consequently, $f(h_r)\geqslant h_r.\beta$ holds. Then

$$\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\leqslant \beta^{-1}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right).$$

Since $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$, then we have $A_k \overset{[WN]_\theta^L}{\sim} B_k$ by taking limit for $r\to\infty$. $\square$

Corollary 3.8. Let (X,ρ) be a metric space, A_k, B_k be non-empty closed subsets of X and $\theta=\{k_r\}$ be a lacunary sequence. Assume that f is an unbounded modulus satisfying the inequalities $\lim_r \frac{f(h_r)}{h_r}>0$ and $f(xy)\geqslant cf(x)f(y)$ with same $c\in(0,\infty)$ for all $x,y\in[0,\infty)$. If $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$ then $A_k \overset{WS_\theta^L}{\sim} B_k$.

Theorem 3.9. Let (X,ρ) be a metric space, A_k, B_k be non-empty closed subsets of X and $f:[0,\infty)\to[0,\infty)$ be an unbounded modulus function satisfying $f(xy)\geqslant cf(x)f(y)$. If $\theta=(k_r)$ is a lacunary sequence with $\liminf_r q_r>1$ then

$$A_k \overset{WS^L(f)}{\sim} B_k \Rightarrow A_k \overset{WS_\theta^L(f)}{\sim} B_k.$$

Proof. Suppose that $\liminf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\geqslant 1+\delta$ for sufficiently large values of r, which implies that $\frac{h_r}{k_r}\geqslant\frac{\delta}{1+\delta}$. From the subadditivity of f and since $\frac{\delta}{1+\delta}.k_r\leqslant h_r$ then $f(\frac{\delta}{1+\delta}.k_r)\leqslant f(h_r)$. Assume that f

is an unbounded modulus function such that $f(x.y) \geqslant c.f(x).f(y)$. Thus we have, $f(h_r) \geqslant f(\frac{\delta}{1+\delta}.k_r) \geqslant c.f(k_r).f(\frac{\delta}{1+\delta})$. Hence, $\frac{f(h_r)}{f(k_r)} \geqslant c.f(\frac{\delta}{1+\delta})$. If $A_k \overset{WS^L(f)}{\sim} B_k$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, we have

$$\begin{aligned}
&\frac{1}{f(k_r)} f\Big(\Big|\{k \leqslant k_r : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\Big) \\
&\geqslant \frac{1}{f(k_r)} f\Big(\Big|\{k \in I_r : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\Big) \\
&= \frac{f(h_r)}{f(k_r)} \frac{1}{f(h_r)} f\Big(\Big|\{k \in I_r : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\Big) \\
&\geqslant c.f(\frac{\delta}{1+\delta}) \frac{1}{f(h_r)} f\Big(\Big|\{k \in I_r : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\Big).
\end{aligned}$$

This completes the proof. □

Theorem 3.10. Let (X, ρ) be a metric space, A_k, B_k be non-empty closed subsets of X, $\theta = \{k_r\}$ be a lacunary sequence and let f be an unbounded modulus such that $\limsup_r \frac{f(k_r)}{f(k_{r-1})} < \infty$. Then

$$A_k \overset{WS_\theta^L(f)}{\sim} B_k \Rightarrow A_k \overset{WS^L(f)}{\sim} B_k.$$

Proof. Let f be an unbounded modulus such that $\limsup_r \frac{f(k_r)}{f(k_{r-1})} < \infty$. Then there exists $M > 0$ such that $\frac{f(k_r)}{f(k_{r-1})} \leqslant M$. Suppose that $A_k \overset{WS_\theta^L(f)}{\sim} B_k$ and $\varepsilon > 0$. Then there exists $R > 0$ such that for every $j \geqslant R$

$$A_j = \frac{1}{f(h_j)} f\left(\Big|\{k \in I_j : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\right) < \varepsilon.$$

We can also find $H > 0$ such that $A_j < H$ for all $j \in \mathbb{N}$. Let n be any integer satisfying $k_{r-1} < n \leqslant k_r$ where $r > R$. Since $\theta = \{k_r\}$ is an increasing integer sequence and f is increasing unbounded modulus, then we have

$$\begin{aligned}
&\frac{1}{f(n)} f\left(\Big|\{k \leqslant n : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\right) \\
&< \frac{1}{f(k_{r-1})} f\left(\Big|\{k \leqslant k_r : \Big|\frac{d(x,A_k)}{d(x,B_k)} - L\Big| \geqslant \varepsilon\}\Big|\right)
\end{aligned}$$

$$= \frac{1}{f(k_{r-1})} f\left(\left| \left\{ 0 < k \leqslant k_1 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| + \left| \left\{ k_1 < k \leqslant k_2 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| + \right.$$

$$\left. + \cdots + \left| \left\{ k_{r-1} < k \leqslant k_r : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| \right)$$

$$= \frac{1}{f(k_{r-1})} f\left(\left| \left\{ k \in I_1 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| + \left| \left\{ k \in I_2 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| + \right.$$

$$\left. + \cdots + \left| \left\{ k \in I_r : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| \right)$$

$$\leqslant \frac{1}{f(k_{r-1})} f\left(\left| \left\{ k \in I_1 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \epsilon \right\} \right| \right) + \frac{1}{f(k_{r-1})} f\left(\left| \left\{ k \in I_2 : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| \right) +$$

$$+ \cdots + \frac{1}{f(k_{r-1})} f\left(\left| \left\{ k \in I_r : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \geqslant \varepsilon \right\} \right| \right)$$

$$= \frac{f(h_1).A_1}{f(k_{r-1})} + \frac{f(h_2).A_2}{f(k_{r-1})} + \cdots + \frac{f(h_R).A_R}{f(k_{r-1})} + \frac{f(h_{R+1}).A_{R+1}}{f(k_{r-1})} + \cdots + \frac{f(h_r).A_r}{f(k_{r-1})}$$

$$= \frac{f(k_1)}{f(k_{r-1})} A_1 + \frac{f(k_2-k_1)}{f(k_{r-1})} A_2 + \cdots + \frac{f(k_R-k_{R-1})}{f(k_{r-1})} A_R + \frac{f(k_{R+1}-k_R)}{f(k_{r-1})} A_{R+1} + \cdots + \frac{f(k_r-k_{r-1})}{f(k_{r-1})} A_r$$

$$\leqslant \{sup_{j\geqslant 1} A_j\} \left(\frac{f(k_1)}{f(k_{r-1})} + \frac{f(k_2-k_1)}{f(k_{r-1})} + \cdots + \frac{f(k_R-k_{R-1})}{f(k_{r-1})} \right) +$$

$$+ \{sup_{j\geqslant R} A_j\} \left(\frac{f(k_{R+1}-k_R)}{f(k_{r-1})} + \cdots + \frac{f(k_r-k_{r-1})}{f(k_{r-1})} \right)$$

$$\leqslant H \left(\frac{f(k_1)}{f(k_{r-1})} + \frac{f(k_2-k_1)}{f(k_{r-1})} + \cdots + \frac{f(k_R-k_{R-1})}{f(k_{r-1})} \right) + \varepsilon \left(\frac{f(k_{R+1}-k_R)}{f(k_{r-1})} + \cdots + \frac{f(k_r-k_{r-1})}{f(k_{r-1})} \right)$$

$$< H. \left(\frac{f(k_R)+f(k_R)+\cdots+f(k_R)}{f(k_{r-1})} \right) + \varepsilon \left(\frac{f(k_r)+f(k_r)+\cdots+f(k_r)}{f(k_{r-1})} \right)$$

$$= H.R.\frac{f(k_R)}{f(k_{r-1})} + \varepsilon.(r-R).\frac{f(k_r)}{f(k_{r-1})}$$

$$< (H.R + \varepsilon.(r-R)).M$$

This completes the proof.

□

Theorem 3.11. Let f be any unbounded modulus.

(i) If $\liminf_r q_r > 1$ then $A_k \overset{[Ces_L^W(f)]}{\sim} B_k \Rightarrow A_k \overset{[WN_\theta]_L^f}{\sim} B_k$.

(ii) If $\limsup_r q_r < \infty$ then $A_k \overset{[WN_\theta]_L^f}{\sim} B_k \Rightarrow A_k \overset{[Ces_L^W(f)]}{\sim} B_k$.

Proof. (i) Let $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geqslant 1+\delta$ for sufficiently large values of r, which implies that $\frac{h_r}{k_r} \geqslant \frac{\delta}{1+\delta}$. Let $A_k \overset{[Ces_L^W(f)]}{\sim} B_k$, then we have

$$\frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) \geqslant \frac{1}{k_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)$$

$$= \frac{h_r}{k_r}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)$$

$$\geqslant \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)$$

which yields that $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$.

(ii) If $\limsup_r q_r < \infty$, then there exists $K>0$, $q_r < K$ for every r. Now suppose that $A_k \overset{[WN_\theta]_L^f}{\sim} B_k$. Then for a given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for every $m \geqslant m_0$,

$$H_m = \frac{1}{h_m}\sum_{k\in I_m} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) < \varepsilon.$$

We can also find $R>0$ such that $H_m \leqslant R$ for all m. Let n be any integer with $k_r \geqslant n > k_{r-1}$ where $r > m_0$. Then we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) \leqslant \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)$$

$$= \frac{1}{k_{r-1}}\left[\sum_{k\in I_1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) + \sum_{k\in I_2} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) + \cdots + \sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\right]$$

$$= \frac{1}{k_{r-1}}\left[\frac{k_1}{k_1}\sum_{k\in I_1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) + \frac{k_2-k_1}{k_2-k_1}\sum_{k\in I_2} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\right.$$

$$\left. + \cdots + \frac{k_{m_0}-k_{m_0-1}}{k_{m_0}-k_{m_0-1}}\sum_{k\in I_{m_0}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right) + \cdots + \frac{k_r-k_{r-1}}{k_r-k_{r-1}}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\right)\right]$$

$$= \frac{k_1}{k_{r-1}}H_1 + \frac{k_2-k_1}{k_{r-1}}H_2 + \cdots + \frac{k_{m_0}-k_{m_0-1}}{k_{r-1}}H_{m_0} + \cdots + \frac{k_r-k_{r-1}}{k_{r-1}}H_r$$

$$\leqslant \frac{k_{m_0}}{k_{r-1}}\sup_{1<k<m_0} H_k + \sup_{k\geqslant m_0} H_k \frac{(k_r-k_{m_0})}{k_{r-1}}$$

$$< R\frac{k_{m_0}}{k_{r-1}} + \varepsilon K$$

which yields that $A_k \overset{[Ces_L^W(f)]}{\sim} B_k$. □

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References

[1] M. Baronti, P. L. Papini, *Convergence of sequences of sets*, Methods of Functional Analysis in Approximation Theory, **76** (1986) Birkhäuser, Basel, Switzerland, 135–155.

[2] V. K. Bhardwaj, S. Dhawan, O. A. Dovgoshey, *Density by moduli and Wijsman statistical convergence*, (2016), unpublished paper, 28 pages. arxiv:1611.08683.

[3] M. Et, H. Şengül, *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α*, Filomat, **28** (2014), no.8, 1593–1602.

[4] M. Et, H. Şengül, *On pointwise lacunary statistical convergence of order α of sequences of function*, Proc. Nat. Acad. Sci. India Sect. A, **85** (2015), no.2, 253–258.

[5] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), no.4, 301–313.

[6] J. A. Fridy, C. Orhan, *Lacunary Statistical Convergence*, Pacific Journal of Mathematics, **160** (1993), no.1, 43–51.

[7] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc., **100** (1986), 161–166.

[8] M. S. Marouf, *Asymptotic equivalence and summability*, International Journal of Mathematics and Mathematical Sciences, **16** (1993), no.4, 755–762.

[9] N. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5** (1953), no.1, 29–49.

[10] F. Nuray, B. E. Rhoades, *Statistical convergence of sequences of sets*, Fasciculi Mathematici, **49** (2012), 87–99.

[11] R. F. Patterson, *On asymptotically statistical equivalent sequences*, Demonstratio Mathematica, **36** (2003), no.1,149–153.

[12] R. F. Patterson, E. Savaş, *On asymptotically lacunary statistically equivalent sequences*, Thai Journal of Mathematics, **4** (2006), 267–272.

[13] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded* , Canad. J. Math., **25** (1973), 973–978.

[14] E. Seneta, *Regularly varying functions*, Lecture Notes in Mathematics 508 Springer-Verlag, Berlin-Heidelberg-Newyork, (1976).

[15] H. Şengül, M. Et, *On I-lacunary statistical convergence of order α of sequences of sets*, Filomat, **31** (2017), no.8, 2403–2412.

[16] U. Ulusu, F. Nuray, *Lacunary statistical convergence of sequence of sets*, Progress in Applied Mathematics, **4** (2012), no.2, 99–109.

[17] U. Ulusu, F. Nuray, *On asymptotically lacunary statistical equivalent set sequences*, Journal of Mathematics, (2013). Article ID 310438.

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