EQUIVALENT ASYMPTOTIC FORMULAS FOR R-STIRLING NUMBERS OF THE FIRST KIND

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Abstract. Two asymptotic formulas for the r-Stirling numbers of the first kind obtained using different methods will be shown to be asymptotically equivalent valid within certain range of a parameter.

1. Introduction

The r-Stirling numbers of the first kind count the number of permutations of the set \{1, 2, \ldots, n\} with m cycles such that the first r elements are in distinct cycles. These numbers were first introduced by Andrei Broder [2]. The r-Stirling numbers of the second kind were also studied in [2] but focus here will be on the first kind. This study is motivated by the work of Chelluri, Richmond and Temme [5].

Andrei Broder denoted the r-Stirling numbers of the first kind by \(\left[ \begin{array}{c} n \\ m_r \end{array} \right] \). Since \(\left[ \begin{array}{c} n \\ m \end{array} \right] = 0\) for \(m < r\), this study considers the r-Stirling numbers of the first kind \(\left[ \begin{array}{c} n + r \\ m + r \end{array} \right] \), where \(n, m, r\) are positive integers. These numbers satisfy the relation

\[
z(n + 1)(n + 2)\cdots(n + 1 - r) = \sum_{m=0}^{n} \left[ \begin{array}{c} n + r \\ m + r \end{array} \right] (z - r)^m.
\] (1.1)

The generalized Stirling numbers of the first kind as generalized by Hsu and Shuie [7] denoted by \(S_{n,m}^{\alpha,\gamma}\) satisfy the relation

\[
z(z - \alpha)(z - 2\alpha)\cdots(z - (n - 1)\alpha) = \sum_{m=0}^{n} S_{n,m}^{\alpha,\gamma}(z - \gamma)^m,
\] (1.2)

where \(\alpha, \gamma\) are complex numbers. Taking \(\alpha = -1\) and \(\gamma = r\), (1.2) becomes

\[
z(n + 1)(n + 2)\cdots(n + 1 - r) = \sum_{m=0}^{n} S_{n,m}^{\alpha,\gamma}(z - r)^m,
\]
which is exactly (1.1). Thus,
\[ \begin{bmatrix} n + r \\ m + r \end{bmatrix}_r = S_{n,m}^{-1,r}. \] (1.3)

In this paper two asymptotic formulas for the \( r \)-Stirling numbers of the first kind obtained using different methods will be discussed and will be shown to be asymptotically equivalent within certain range of the parameter \( m \).

2. Asymptotic Formulas for \( r \)-Stirling Numbers

Let \( C \) be any closed contour enclosing \( r \). Applying the Cauchy-Integral Formula to (1) gives
\[ \begin{bmatrix} n + r \\ m + r \end{bmatrix}_r = \frac{1}{2\pi i} \int_C \frac{z(z + 1)(z + 2)\ldots(z + n - 1)(z - r)^{m+1}}{(z - r)^{m+1}} \, dz. \] (2.1)

A modified saddle point method used in [12] was applied to the integral above to obtain the following asymptotic approximation:

**Theorem 2.1.** For positive integers \( m, n \) and \( r \), the asymptotic formula holds,
\[ \begin{bmatrix} n + r \\ m + r \end{bmatrix}_r \sim e^B g(s_0) (n - 1)_m^{n - m - 1}, \] (2.2)
as \( n \to \infty \) valid uniformly in the range \( 0 < m < n \), where
\[ s_0 = \frac{nr}{n - m}, \] (2.3)
\[ B = \phi(z_0) - n \log s_0 + m \log(s_0 - r), \] (2.4)
and
\[ g(s_0) = \frac{1}{(z_0 - r)} \sqrt{\frac{s_0(s_0 - r)(n - m)}{\phi''(z_0)}}. \] (2.5)

The number \( z_0 \) is the unique positive solution to the equation \( \phi'(z) = 0 \), the function \( \phi(z) \) is
\[ \phi(z) = \log[z(z + 1)(z + 2)\ldots(z + n - 1)] - m \log(z - r), \] (2.6)
and \((n - 1)_m = (n - 1)(n - 2)\ldots(n - 1 - m + 1)\).

Remark. The number \( z_0 \) may be computed using mathematica.

Using the method in [9], Vega and Corcino [13] obtained an asymptotic formula for the generalized Stirling numbers of the first kind which is given by
\[ S_{n,m}^{\alpha,\gamma} \sim \frac{(-\alpha)^{n-m} \Gamma(R - \nu + n)}{(2\pi H)^{1/2} R^n \Gamma(R - \nu)} \left\{ 1 + \frac{3C_4}{H^2} - \frac{15C_3^2}{2H^3} \right\}, \] (2.7)
as \( n \to \infty \) valid for \( m \) in the range \( h(n) < m < n - O(n^\delta) \), where \( h(n) \) is a function such that \( \lim_{n \to \infty} h(n) = \infty \) and \( 0 < \delta < 1 \), \( \Gamma(x) \) is the gamma function, \( \nu = \frac{\gamma}{\alpha} < 1 \). In this paper, \( h(n) = \log n \) and \( \delta = 1/2 \). The \( H \) that appears in (2.7) is
\[ H = \sum_{h=1}^{n-1} \frac{(h - \nu)R}{(R + h - \nu)^2}, \] (2.8)
and \( R \) is the unique positive solution to the equation
\[
\sum_{h=1}^{n-1} \frac{R}{R + h - \nu} = m - 1. \tag{2.9}
\]

The constants \( C_3 \) and \( C_4 \) are given by
\[
C_3 = \frac{1}{6} \left[ 3H - 2(m - 1) + 2 \sum_{h=1}^{n-1} \frac{R^3}{(R + h - \nu)^3} \right], \tag{2.10}
\]
and
\[
C_4 = \frac{1}{24} \left[ 36C_3 - 11H + 6(m - 1) - 6 \sum_{h=1}^{n-1} \frac{R^4}{(R + h - \nu)^4} \right]. \tag{2.11}
\]

With a little modification in the computations in [13], the same formula as (2.7) is obtained when
\[
H = \sum_{h=0}^{n-1} \frac{(h - \nu)R}{(R + h - \nu)^2}, \tag{2.12}
\]
and \( R \) is the unique positive solution to the equation
\[
\sum_{h=0}^{n-1} \frac{R}{R + h + r} = m. \tag{2.13}
\]

Since \( \left[ \begin{array}{c} n + r \\ m + r \end{array} \right]_{r}^{m} = S_{n,m}^{-1,r} \) [see (1.2)], taking \( \alpha = -1, \gamma = r \) in (2.7), the following asymptotic formula for the \( r \)-Stirling numbers of the first kind is obtained:

**Theorem 2.2.** For positive integers \( m, n, r \) and as \( n \to \infty \), the following asymptotic formula for the \( r \)-Stirling numbers of the first kind holds:
\[
\left[ \begin{array}{c} n + r \\ m + r \end{array} \right]_{r} = \frac{\Gamma(R + r + n)}{(2\pi H)^{1/2}R^m \Gamma(R + r)} \left\{ 1 + \frac{3C_4}{H^2} - \frac{15C_3^2}{2H^3} \right\}, \tag{2.14}
\]
valid for \( m \) in the range \( \log n < m < n - O(n^{1/2}) \), where \( R \) is the unique positive solution to the equation
\[
\sum_{h=0}^{n-1} \frac{R}{R + h + r} = m. \tag{2.15}
\]

and
\[
H = \sum_{h=0}^{n-1} \frac{(h + r)R}{(R + h + r)^2}. \tag{2.16}
\]

The corresponding constants \( C_3 \) and \( C_4 \) are as follows,
\[
C_3 = \frac{1}{6} \sum_{h=0}^{n-1} \frac{R(h + r)(3R + h + r)}{(R + h + r)^3}, \tag{2.17}
\]
\[
C_4 = \frac{1}{24} \sum_{h=0}^{n-1} \frac{R(h + r)[-3R^2 + 4R(h + r) + (h + r)^2]}{(R + h + r)^4}. \tag{2.18}
\]
3. APPROXIMATION FOR $z_0$

The goal in this section is to find the asymptotics of the unique positive solution $z_0$ of the equation $\phi'(z) = 0$, where $\phi(z)$ is given in (8).

By definition, $z_0$ is the solution of the algebraic equation

$$\frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} = \frac{m}{z-r}.$$  \hspace{1cm} (3.1)

We are going to prove the following theorem.

**Theorem 3.1.** If $m$ is a fixed positive integer, then, as $n \to \infty$

$$z_0 \sim n - m + \frac{1}{2} - \frac{4m^2 - 4m + 1}{8n} + O\left(\frac{1}{n^2}\right).$$

**Proof.** First note that (3.1) can be written by using the Digamma function $\psi = (\log \Gamma)'$ as

$$\psi(z+n) - \psi(z) = \frac{m}{z-r}.$$  

Now making use of the fact that

$$\psi(z) \sim (\log z) - \frac{1}{2z} + O\left(\frac{1}{z^2}\right),$$

it can be seen that $z$ must tend to infinity as $n$ tends to infinity, at least when $m$ is fixed. That what happens when $m$ grows together with $n$ will be discussed in the Remark after the proof.

We can leave the $O\left(\frac{1}{n^2}\right)$ term in the approximation of the Digamma function and get that, asymptotically, (3.1) is equivalent to

$$\log(z+n) - \frac{1}{2(z+n)} - \log z + \frac{1}{2z} = \frac{m}{z-r}.$$  

This is equivalent to

$$\log \left(1 + \frac{n}{z}\right) + \frac{1}{2} \frac{n}{nz + z^2} = \frac{m}{z-r}.$$  

As $n$ tends to infinity $\log \left(1 + \frac{n}{z}\right) \sim \log n - \log z$, and $\frac{n}{nz + z^2} \sim \frac{1}{z}$. Also, as $z \to \infty$, $\frac{m}{z-r} \sim \frac{m}{z}$. In this step lose some weak $r$ dependence, but get an equation exactly solvable. At this point the equation is asymptotically equivalent to

$$\log z + \frac{1}{z} \left(m - \frac{1}{2}\right) = \log n.$$  

This equation can be solved in terms of the Lambert $W$ function \[6\]:

$$z_0 \sim \frac{1 - 2m}{2W\left(\frac{1-2m}{2n}\right)}.  \hspace{1cm} (3.2)$$

If $m$ is fixed then the Lambert function asymptotics \[6\] around $x = 0$,

$$W(x) \sim x - x^2 + O(x^3)$$  \hspace{1cm} (3.3)

yields that

$$z_0 \sim \frac{1 - 2m}{2W\left(\frac{1-2m}{2n}\right)} \sim n - m + \frac{1}{2} - \frac{4m^2 - am + 1}{8n} + O\left(\frac{1}{n^2}\right),$$

as stated in the theorem.  \[\square\]
Remark 3.2. It is interesting to see what happens when $m$ is not fixed, but grows together with $n$. If we still want $z_0$ to tend to infinity, (3.2) can be used. The expression $\frac{1 - 2m}{2W\left(\frac{1 - 2m}{2n}\right)}$ is a positive real number for all $m > 0$ such that $m \leq \frac{n}{e} + \frac{1}{2}$ (this fact comes from the shape of the Lambert function). If $m = O(n^\delta)$ with $\delta < 1$, then $z_0 \to \infty$. Indeed, with such an $m$ the argument of the Lambert function tends to 0 and (3.3) can be applied. Hence, keeping only one term in (3.3), we have that
\[
 z_0 \sim \frac{1 - 2m}{2 \left(\frac{1 - 2m}{2n}\right)} = n.
\]

4. Equivalence of the formulas

First we compare the quantities $z_0$ and $R$. The following Lemma gives the connection formula between $z_0$ and $R$.

Lemma 4.1.
\[
 z_0 = R + r.
\]

Proof. Note that
\[
 \phi'(z) = \frac{1}{z} + \frac{1}{z + 1} + \frac{1}{z + 2} + \ldots + \frac{1}{z + n - 1} - \frac{m}{z - r},
\]
and
\[
 \phi'(z_0) = 0.
\]
Thus,
\[
 \sum_{h=0}^{n-1} \frac{z_0 - r}{z_0 + h} = m. \tag{4.1}
\]
On the other hand, let
\[
 P(R, n, r) = \sum_{h=0}^{n-1} \frac{R}{R + h + r}. \tag{4.2}
\]
By (15), $P(R, n) = m$. Let $w = R + r$, then
\[
 P(R, n, r) = \sum_{h=0}^{n-1} \frac{w - r}{w + h} = m. \tag{4.3}
\]
Comparing (24) and (26) and using the fact that $z_0$ is unique, we conclude that
\[
 z_0 = w = R + r. \tag{4.4}
\]

Lemma 4.2.
\[
 \frac{1}{H} = O\left(\frac{1}{m}\right).
\]
Proof. From (17)

\[ H = \sum_{h=0}^{n-1} \frac{(h + r)R}{(R + h + r)^2}, \]

and

\[ P(R, n, r) = \sum_{h=0}^{n-1} \frac{R}{R + h + r} = m. \]

By partial fractions, \( H \) can be written

\[ H = \sum_{h=0}^{n-1} \left( \frac{R}{R + h + r} - \frac{R^2}{(R + h + r)^2} \right), \]

\[ = \sum_{h=0}^{n-1} \frac{R}{R + h + r} - \sum_{h=0}^{n-1} \frac{R^2}{(R + h + r)^2} \]

\[ = m - \sum_{h=0}^{n-1} \frac{R^2}{(R + h + r)^2} \]

\[ = m - R^2 \sum_{h=0}^{n-1} \frac{1}{(R + h + r)^2}. \]

Note that by Integral Test, \( \sum_{h=0}^{n-1} \frac{1}{(R + h + r)^2} \) is convergent.

Let \( \mu = \sum_{h=0}^{n-1} \frac{1}{(R + h + r)^2} \).

Then

\[ H = m - R^2 \mu \]

\[ \frac{1}{H} = \frac{1}{m - R^2 \mu} \]

We will show that \( \frac{1}{m - R^2 \mu} = O \left( \frac{1}{m} \right) \).

\[ \frac{1}{m - R^2 \mu} = \frac{1}{m \left( \frac{1 - \mu}{m} \right)} \]

\[ = \frac{1}{1 - \frac{R^2 \mu}{m}} \]

From Theorem 5.1,

\[ \frac{R}{m} \sim \frac{-r + n - m + \frac{1}{2} - \frac{4m^2}{8n} + \frac{4m}{8n} - \frac{1}{8n} + O \left( \frac{1}{n^2} \right)}{m} \]

\[ \sim \frac{-r}{m} + \frac{n - m}{m} + \frac{1}{2m} - \frac{4m}{8n} + \frac{4}{8n} - \frac{1}{8nm} + O \left( \frac{1}{n^2m} \right) \]

\[ \sim \frac{4}{8} + O \left( \frac{1}{m} \right) \]
So,
\[
\frac{R^2 \mu}{m} = \frac{R}{m} \cdot R \mu
\]
\[
\sim \left[ -\frac{4}{8} + O \left( \frac{1}{m} \right) \right] R \mu
\]
Then
\[
1 - \frac{R^2 \mu}{m} \sim 1 + \left[ -\frac{4}{8} + O \left( \frac{1}{m} \right) \right] R \mu
\]
\[
= \frac{1}{1 + \frac{1}{2} R \mu + R \mu O \left( \frac{1}{m} \right)} < 1
\]
Thus,
\[
\frac{1}{H} = \frac{1}{m - R^2 \mu} = O \left( \frac{1}{m} \right)
\]
\[\Box\]

**Theorem 4.3.** The formula in (22) can be written in the form
\[
\left[ \frac{n + r}{m + r} \right]_r \sim \frac{\Gamma \left( R + r + n \right)}{(2\pi H)^{1/2} R^m \Gamma \left( R + r \right)} \left\{ 1 + O \left( \frac{1}{m} \right) \right\}.
\]  
(4.5)

**Proof.** This follows from Lemma 4.2 and the fact that for each \( k \geq 2, |c_k| \leq H \) (This is (3.6) in [13]).

The following lemma gives the connection formula between \( \phi''(z_0) \) and \( H \).

**Lemma 4.4.**
\[
\phi''(z_0) = \frac{H}{(z_0 - r)^2}.
\]  
(4.6)

**Proof.** Recall
\[
\phi''(z) = \frac{m}{(z - r)^2} - \frac{1}{z^2} - \frac{1}{(z + 1)^2} - \frac{1}{(z + 2)^2} - \cdots - \frac{1}{(z + n - 1)^2},
\]
which can be written
\[
(z - r)^2 \phi''(z) = m - \sum_{h=0}^{n-1} \frac{(z - r)^2}{(z + h)^2}.
\]  
(4.7)

At \( z = z_0 \),
\[
(z_0 - r)^2 \phi''(z_0) = m - \sum_{h=0}^{n-1} \frac{(z_0 - r)^2}{(z_0 + h)^2} = m - \sum_{h=0}^{n-1} \frac{R^2}{(R + r + h)^2}.
\]  
(4.8)

It remains to show that
\[
m - \sum_{h=0}^{n-1} \frac{R^2}{(R + r + h)^2} = H.
\]  
(4.9)

Note that
\[
m = P(R, n) = \sum_{h=0}^{n-1} \frac{R}{R + h + r}.
\]
Thus,
\[
m - \sum_{h=0}^{n-1} \frac{R^2}{(R + r + h)^2} = \sum_{h=0}^{n-1} \frac{R}{R + h + r} - \sum_{h=0}^{n-1} \frac{R^2}{(R + r + h)^2} = \sum_{h=0}^{n-1} \frac{(r + h)R}{(R + r + h)^2} = H.
\]

\[\Box\]

Lemma 4.5.
\[
e^B g(s_0) \frac{(n - 1)_m n^{n-m-1}}{m!} = \frac{\Gamma(R + r + n)}{\Gamma(R + r)(R)^m \sqrt{H}} D,
\]
where
\[
D = \frac{m^m m^{n-m}}{n^m (n - m)^n (n - 1)_m m!} \sqrt{\frac{nm}{n - m}}.
\]

Proof. It follows from (4.8) and Lemma 4.4 that
\[
\phi''(z_0) = \frac{H}{(z_0 - r)^2}.
\]
Thus, from (2.5) we have,
\[
g(s_0) = \frac{1}{z_0 - r} \sqrt{\frac{s_0(s_0 - r)(n - m)}{\phi''(z_0)}}
\]
\[
= \frac{1}{z_0 - r} \sqrt{\frac{s_0(s_0 - r)(n - m)}{H/(z_0 - r)^2}}
\]
\[
= \frac{1}{\sqrt{H}} \sqrt{s_0(s_0 - r)(n - m)}
\]
\[
= \frac{r}{\sqrt{H}} \sqrt{\frac{nm}{n - m}}.
\]

Note that \( z_0 = R + r > r \).

We turn to the factor \( e^B \), where
\[
B = \phi(z_0) - n \log s_0 + m \log(s_0 - r).
\]
Then,
\[
e^B = e^{\phi(z_0)} \frac{(s_0 - r)^m}{s_0^n}.
\]
With \( \phi(z) \) given in (2.6) we have
\[
e^{\phi(z_0)} = \frac{\Gamma(z_0 + n)}{\Gamma(z_0)(z_0 - r)^m},
\]
and
\[
e^B = \frac{\Gamma(z_0 + n)}{\Gamma(z_0)(z_0 - r)^m} \frac{m^m}{n^m} \frac{(n - m)}{r}^{n-m}.
\]
Thus,

\[ e^{Rg(s_0)} \frac{(n-1)_m r^{n-m-1}}{m!} = \frac{\Gamma(z_0 + n) \Gamma(z_0 - r) m^n}{\Gamma(z_0) m^n} \frac{m^n}{n-m} \frac{r^n}{r^{n-m-1}} \]

\[ \times \frac{\Gamma(R + r + n)}{\sqrt{H}} \sqrt{\frac{nm}{n-m}} (n-1)_m r^{n-m-1} \]

\[ \times \frac{\Gamma(R + r + n) m^m (n-m)^{n-m}}{\Gamma(R + r) R^m n^n r^{n-m}} \]

\[ \times \frac{r}{\sqrt{H}} \sqrt{\frac{nm}{n-m}} (n-1)_m r^{n-m} r^{-1} \]

\[ = \frac{\Gamma(R + r + n) m^m (n-m)^{n-m} (n-1)_m}{\Gamma(R + r) R^m \sqrt{H} n^n (n-m)^{n-m}} \frac{1}{m!} \sqrt{\frac{nm}{n-m}} \]

\[ \times \frac{1}{\sqrt{2\pi e^{-m+\theta_1/12m}(n-m)^{1/2}}} \frac{1}{(n-m)^{1/2}} \frac{1}{(n-m-1)!} \]

\[ \times \exp \left[ -\frac{\theta_1}{12m} + \frac{\theta_2}{12(n-1)} - \frac{\theta_3}{n-m-1} \right] \]

\[ \times \left( 1 - \frac{1}{n} \right)^{n-1/2} \]

\[ \Gamma(R + r + n) R^m \sqrt{H} D \]

\[ \square \]

\textbf{Lemma 4.6.} Let

\[ D = \frac{m^m (n-m)^{n-m} (n-1)_m}{(n-m)^{1/2} (n-1)_m} \]

\[ \text{Then} \]

\[ D = \frac{1}{\sqrt{2\pi}} \left[ 1 + O(1/m) \right] \]

\[ \text{as } n \to \infty \text{ such that } n-m = O(n^{1/2}). \]

\textbf{Proof.}

\[ D = \frac{m^m (n-m)^{n-m} (n-1)_m}{n^n (n-m)^{1/2} (n-1)_m} \]

\[ = \frac{m^m (n-m)^{n-m} (n-1)_m}{(n-m)^{1/2} (n-1)_m} \frac{1}{n^{1/2} (n-m)!} \]

\[ = \frac{(n-m)^{n-m}}{n^n} \frac{1}{\sqrt{2\pi e^{-m+\theta_1/12m}(n-m)^{1/2}}} \frac{1}{(n-m)^{1/2}} \frac{1}{(n-m-1)!} \]

\[ \times \exp \left[ -\frac{\theta_1}{12m} + \frac{\theta_2}{12(n-1)} - \frac{\theta_3}{n-m-1} \right] \]

\[ \times \left( 1 - \frac{1}{n} \right)^{n-1/2} \]

where \( 0 < \theta_i < 1, i = 1, 2, 3 \). The last equality in the array above follows from Stirling’s formula for \( n! \) (see [1]). Thus,
\[
D = \left(1 - \frac{1}{n-m}\right) n^{n-m} \left(1 - \frac{1}{n}ight) \left(1 - \frac{1}{n-m}\right)^{1/2} \times \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{\theta_1}{12m} + \frac{\theta_2}{12(n-1)} - \frac{\theta_3}{n - m - 1} \right] = \frac{1}{\sqrt{2\pi}} [1 + O(1/m)].
\]

The following theorem follows from Lemma 4.1, Lemma 4.5 and Lemma 4.6.

**Theorem 4.7.** Let \(r, m, \) and \(n\) be positive integers. Then
\[
\binom{n+r}{m+r} = e^{B\theta(s_0)} \frac{(n-1)_m r^{n-m} - 1}{m!} = \frac{\Gamma(R + r + n)}{\Gamma(R + r) R^m \sqrt{2\pi} H} [1 + O(1/m)],
\]
as \(n \to \infty\) such that \(m = n - O(n^{1/2})\), where \(s_0\) is defined in (2.3) and \(R\) is the unique solution to (2.15).

**Acknowledgments.** The authors would like to thank the anonymous referee for evaluating the paper. They would also like to thank CNU-Center for Research and Development for the financial support extended to this research project.

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