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STANCU VARIANT OF (p,q)-SZáSZ-MIRAKYAN OPERATORS

ANKITA R DEVDHARA, VISHNU NARAYAN MISHRA*

Abstract. In this paper, we investigate Stancu variant of generalized (p,q)-Szász-Mirakyan operators. Initially, we estimate moments of the operators. Using Korovkin's result, we have evaluated the uniform convergence of the operators. Direct results of the operators have been appraised. For better approximation we have calculated the rate of convergence of the operators. Last, we have proved Voronovskaya type theorem for the operators. Particularly, if $\alpha = 0$ then we get the generalized (p,q)-Szász-Mirakyan operators.

1. Preliminaries

In approximation theory, many authors introduced Stancu type generalization of various linear positive operator [13, 15]. The use of Bernstein polynomial in q-calculus was first introduced by Lupas[1]. The constant development in q-calculus has led us towards the new generalized approximating operators depending on q-integers[4, 12, 14]. In recent times Mursaleen et al. [7] introduced new way of approximating linear positive operators in $(p,q)-$ calculus. Mishra et al. and Khan and Lobiyal have also given some results in $(p,q)-$Calculus [5, 16]. The more research is going on in this area [3, 6, 8, 13].
We initiate by revising some definition from (p,q)-calculus [7]. Suppose, $q,p \in (0,1], q < p$.

Definition 1.1. The (p,q)-integer $[m]_{p,q}$ and (p,q)-factorial $[m]_{p,q}!$ are described by

$$[m]_{p,q} = \frac{p^m - q^m}{p-q}, \quad m = 0,1,2,\cdots, \ [0]_{p,q} = 0$$

and

$$[m]_{p,q}! = [m]_{p,q}[m-1]_{p,q}\cdots 1 \quad , \quad m \geq 1 \text{ and } [0]_{p,q}! = 1.$$

Definition 1.2. The (p,q)-Binomial expansion is

$$(y+z)^m_{p,q} = (y+z)(py+qz)(p^2y+q^2z)\cdots(p^{m-1}y+q^{m-1}z),$$

and the (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} m \\ j \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[j]_{p,q}![m-j]_{p,q}!}, \qquad 0 \leq j \leq m.$$

Definition 1.3. Let ψ be a function defined on the complex numbers. The (p,q)-derivative of ψ is:

$$D_{p,q}\psi(x) = \tfrac{\psi(px)-\psi(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q}\psi)(0) = \psi(0)$, on the condition that differentiability exist for ψ at 0.

Definition 1.4. Suppose ψ is any function and a is real number,

$$\int_0^a \psi(x)d_{p,q}x = (q-p)a\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\psi\left(\frac{p^j}{q^{j+1}}a\right) \ \ if \ \ \left|\frac{p}{q}\right| < 1.$$

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$$\int_0^a \psi(x)d_{p,q}x = (p-q)a\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\psi\left(\frac{q^j}{p^{j+1}}a\right) \quad if \quad \left|\frac{p}{q}\right| > 1.$$

Definition 1.5. A $(p,q)-$analogue of classical exponential function e^x [10] is

$$e_{p,q}(y) = \sum_{j=0}^{\infty}\frac{p^{\frac{j(j-1)}{2}}y^j}{[j]_{p,q}!},$$

$$E_{p,q}(y) = \sum_{j=0}^{\infty}\frac{q^{\frac{j(j-1)}{2}}y^j}{[j]_{p,q}!}.$$

The $(p,q)-$exponential function satisfy following property

$$e_{p,q}(y)E_{p,q}(-y) = E_{p,q}(y)e_{p,q}(-y) = 1.$$

2. Primary estimate

Acar [11] derived a (p,q)-analogue of Szász-Mirakyan operators as:

$$S_{m,p,q}(\psi;x) = \sum_{j=0}^{\infty}s_m(p,q;x)\psi\left(\frac{[j]_{p,q}}{q^{j-2}[m]_{p,q}}\right),$$

where

$$s_m(p,q;x) = \frac{1}{E_{p,q}([m]_{p,q}x)}q^{\frac{j(j-1)}{2}}\frac{[m]_{p,q}^j x^j}{[j]_{p,q}!}; j = 0,1,2,... \ \& \ m \in \mathbb{N}.$$

Lemma 2.1 ([11])**.** *Let $q,p \in (0,1], q < p$ and $m \in \mathbb{N}$, we obtain*

$$S_{m,p,q}(1;x) = 1,$$
$$S_{m,p,q}(t;x) = qx,$$
$$S_{m,p,q}(t^2;x) = pqx^2 + \frac{q^2x}{[m]_{p,q}}.$$

With the help of (p,q)-analogue of Szász-Mirakyan operators we present, $q,p \in (0,1], q < p$ & $m \in \mathbb{N}$ the Stancu variant of generalized (p,q)-Szász-Mirakyan operators given below as:

$$S^{\alpha}_{m,p,q}(\psi;x) = \sum_{j=0}^{\infty}s_m(p,q;x)\psi\left(\frac{[j]_{p,q}}{q^{j-2}[m]_{p,q}} + \frac{\alpha}{[m]_{p,q}}\right),$$

$$s_m(p,q;x) = \frac{1}{E_{p,q}([m]_{p,q}x)}q^{\frac{j(j-1)}{2}}\frac{[m]_{p,q}^j x^j}{[j]_{p,q}!}.$$

Lemma 2.2. *For $q,p \in (0,1], q < p$ and $m \in \mathbb{N}$, we obtain*

$$S^{\alpha}_{m,p,q}(1;x) = 1 \tag{2.1}$$

$$S^{\alpha}_{m,p,q}(t;x) = qx + \frac{\alpha}{[m]_{p,q}} \tag{2.2}$$

$$S^{\alpha}_{m,p,q}(t^2;x) = pqx^2 + \frac{x}{[m]_{p,q}}(q^2+2q\alpha) + \frac{\alpha^2}{[m]^2_{p,q}}. \tag{2.3}$$

Corollary. *With the help of above lemma (2.2), we obtain the formulas of the central moments as follow:*

$$S^{\alpha}_{m,p,q}(t-x;x) = (q-1)x + \frac{\alpha}{[m]_{p,q}}; \tag{2.4}$$

$$S^{\alpha}_{m,p,q}((t-x)^2;x) = (pq-2q+1)x^2 + (q^2+2q\alpha+\alpha)\frac{x}{[m]_{p,q}} + \frac{\alpha^2}{[m]^2_{p,q}}. \tag{2.5}$$

Remark. Here, $0 < q < 1$ & $q < p \leq 1$. To get the results, we consider the sequences (q_m), (p_m) in such a way that (q_m) tends to 1; (p_m) tends to 1; $(q_m)^m$ tends to a' & $(p_m)^m$ tends to b' as m tends to ∞. We can construct above mentioned sequences. For example, let $p_m = 1 + \frac{1}{2m}$ and $q_m = 1 + \frac{1}{3m}$. Then $p_m \to 1, q_m \to 1$; $p_m^m \to e^{1/2}, q_m^m \to e^{1/3}$ as $m \to \infty$, and $\lim_{m\to\infty} 1/[m]_{p_m,q_m} = 0$.

Consider, the sup norm: $\|f\|_2 = \|f\|_{C_2[0,\infty)} = \sup\{\,|f(x)| : x \in C_2[0,\infty)\,\}$.

Theorem 2.3. *Let the sequence (p_m) and (q_m) are given as in above remark (2). Then for each function $\psi \in C_2[0,\infty)$ we get,*

$$\lim_{m\to\infty} \|S^{\alpha}_{m,p,q}(\psi) - \psi\|_2 = 0;\ i = 0,1,2.$$

Proof. From Korovkin's theorem it is sufficient to verify the following three conditions

$$\lim_{m\to\infty} \|S^{\alpha}_{m,p,q}(t^i;x) - x^i\|_2 = 0;\ i = 0,1,2. \tag{2.6}$$

The uniform convergence of the above equation is trivial for the case $i = 0$.
Now, for $i = 1$;

$$\begin{aligned}\lim_{m\to\infty} \|S^{\alpha}_{m,p,q}(t;x) - x\|_2 &= \lim_{m\to\infty} \|q_m x + \frac{\alpha}{[m]_{p_m,q_m}} - x\|_2 \\ &\le \lim_{m\to\infty} \|q_m - 1\|_2 x + \lim_{m\to\infty} \left|\frac{\alpha}{[m]_{p_m,q_m}}\right| = 0.\end{aligned}$$

And, for $i = 2$;

$$\begin{aligned}\lim_{m\to\infty} \|S^{\alpha}_{m,p,q}(t^2;x) - x^2\|_2 &= \lim_{m\to\infty} \|p_m q_m x^2 + \frac{x}{[m]_{p_m,q_m}}(q_m^2 + 2q_m\alpha) + \frac{\alpha^2}{[m]^2_{p_m,q_m}}. - x^2\|_2 \\ &\le \lim_{m\to\infty} \|p_m q_m - 1\|_2 x^2 + \lim_{m\to\infty} \|(q_m^2 + 2q_m\alpha)\|_2 \frac{x}{[m]_{p_m,q_m}} = 0.\end{aligned}$$

Hence, we obtain the result. □

3. Direct result

In this segment of the paper, we demonstrate the theorem of local approximation of the operators $S^{\alpha}_{m,p,q}$. Consider $C_B[0,\infty)$ be the set of real valued bounded continuous functions ψ defined on $[0,\infty)$. Obviously, it is a linear space and the norm $\|.\|$ is define by,

$$\|\psi\| = \sup_{0\le x<\infty} |\psi(x)|.$$

Also, we consider the Peetre's $\mathcal{K}$-functional:

$$\mathcal{K}_2(\psi,\delta) = \inf_{g\in W^2}\{\|\psi - g\| + \delta\|g''\|\}, \tag{3.1}$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.
By [9], $\exists$ an absolute constant $C > 0\ \ni$

$$\mathcal{K}_2(\psi,\delta) \le C\omega_2(\psi,\sqrt{\delta}), \tag{3.2}$$

here

$$\omega_2(\psi,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}, x\in[0,\infty)} |\psi(x+2h) - 2\psi(x+h) + \psi(x)|$$

is the second order modulus of continuity of given function $\psi \in C_B[0,\infty)$. Moreover, the first order modulus of continuity of given function $\psi \in C_B[0,\infty)$ is given by;

$$\omega(\psi,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}, x\in[0,\infty)} |\psi(x+h) - \psi(x)|.$$

Theorem 3.1. *Suppose the sequences (p_m) and (q_m) are given as in above remark (2). Then we get*

$$|S^{\alpha}_{m,p_m,q_m}(\psi;x) - \psi(x)| \le C\omega_2(\psi;\delta_m(x)) + \omega\left(\psi, \left|(1-q)x - \frac{\alpha}{[m]_{p_m,q_m}}\right|\right),$$

$\forall x \in [0,\infty)$; *where* $\delta_m(x) = \left(S^{\alpha}_{m,p_m,q_m}(|t-x|^2;x) + (S^{\alpha}_{m,p_m,q_m}(|t-x|;x))^2\right)^{1/2}$

Proof. First, we consider the operator:

$$\hat{S}^{\alpha}_{m,p_m,q_m}(\psi;x) = S^{\alpha}_{m,p_m,q_m}(\psi;x) + \psi(x) - \psi(q_m x + \frac{\alpha}{[m]_{p_m,q_m}}). \tag{3.3}$$

Then, $\hat{S}^{\alpha}_{m,p_m,q_m}$ are in linear form. Also, by lemma:(2.2), we have

$$\hat{S}^{\alpha}_{m,p_m,q_m}(1;x) = 1, \tag{3.4}$$

$$\hat{S}^{\alpha}_{m,p_m,q_m}(t;x) = S^{\alpha}_{m,p_m,q_m}(t;x) + x - qx + \frac{\alpha}{[m]_{p_m,q_m}} = x. \tag{3.5}$$

Now, we consider $g \in W^2$. The Taylor's formula for $t \in \mathbb{R}^+$ of g is:

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{3.6}$$

Using above equation (3.5) and (3.6); we obtain

$$\hat{S}^{\alpha}_{m,p_m,q_m}(g;x) - g(x) = \hat{S}^{\alpha}_{m,p_m,q_m}\bigg(\int_x^t (t-u)g''(u)du;x\bigg).$$

Hence,

$$\begin{aligned}|\hat{S}^{\alpha}_{m,p_m,q_m}(g;x) - g(x)| \leq &\bigg|S^{\alpha}_{m,p_m,q_m}\bigg(\int_x^t (t-u)g''(u)du;x\bigg)\bigg| \\ &+ \bigg|\int_x^{qx+\frac{\alpha}{[m]_{p_m,q_m}}} (qx + \frac{\alpha}{[m]_{p_m,q_m}} - u)g''(u)du\bigg|.\end{aligned}$$

Here;

$$\bigg|\int_x^t (t-u)g''(u)du\bigg| \leq |t-x|^2\|g''\|.$$

Also;

$$\bigg|\int_x^{qx+\frac{\alpha}{[m]_{p_m,q_m}}} (qx + \frac{\alpha}{[m]_{p_m,q_m}} - u)g''(u)du\bigg| \leq \bigg|qx + \frac{\alpha}{[m]_{p_m,q_m}} - x\bigg|^2\|g''\|.$$

Hence,

$$\begin{aligned}|\hat{S}^{\alpha}_{m,p_m,q_m}(g;x) - g(x)| &\leq S^{\alpha}_{m,p_m,q_m}(|t-x|^2;x)\|g''\| + \bigg|qx + \frac{\alpha}{[m]_{p_m,q_m}} - x\bigg|^2\|g''\| \\ &= \|g''\|\delta_m^2(x).\end{aligned}$$

where,

$$\delta_m(x) = \bigg(S^{\alpha}_{m,p_m,q_m}(|t-x|^2;x) + \bigg(S^{\alpha}_{m,p_m,q_m}(|t-x|;x)\bigg)^2\bigg)^{1/2}.$$

Now,

$$|\hat{S}^{\alpha}_{m,p_m,q_m}(g;x) - g(x)| \leq \|g''\|\bigg(S^{\alpha}_{m,p_m,q_m}(|t-x|^2;x) + (S^{\alpha}_{m,p_m,q_m}(|t-x|;x))^2\bigg).$$

And

$$|\hat{S}^{\alpha}_{m,p_m,q_m}(\psi;x)| \leq |S^{\alpha}_{m,p_m,q_m}(\psi;x) + 2\|\psi\| \leq 3\|\psi\| \tag{3.7}$$

Hence;

$$\begin{aligned}|S^{\alpha}_{m,p_m,q_m}(\psi;x) - \psi(x)| \leq &|\hat{S}^{\alpha}_{m,p_m,q_m}(\psi-g;x) - (\psi-g)(x)| + |\hat{S}^{\alpha}_{m,p_m,q_m}(g;x) - g(x)| \\ &+ \bigg|\psi(x) - \psi(q_m x + \frac{\alpha}{[m]_{p_m,q_m}})\bigg| \\ &\leq 4\|\psi - g\| + \|g''\|\delta_m^2(x) + \omega\bigg(\psi, \bigg|(1-q)x - \frac{\alpha}{[m]_{p_m,q_m}}\bigg|\bigg).\end{aligned}$$

Considering infimum over all $g \in W^2$ on R.H.S.; we get

$$|S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)| \leq 4\mathcal{K}_2(\psi;\delta^2_m(x))+\omega\left(\psi,\left|(1-q)x-\frac{\alpha}{[m]_{p_m,q_m}}\right|\right).$$

For all $0<q_m<p_m\leq 1$ & with the help of (3.2); we obtain

$$|S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)| \leq C\omega_2(\psi;\delta_m(x))+\omega\left(\psi,\left|(1-q)x-\frac{\alpha}{[m]_{p_m,q_m}}\right|\right).$$

Hence proved. □

4. Rate of convergence

Suppose $A=[0,\infty)$.
Let $B_{x^2}(A)=\{\psi : A\to\mathbb{R} : |\psi(x)|<M(1+x^2); M>0$ & is constant depends on $\psi\}$.
And $C_M(A)=\{\psi\in B_{x^2}(A) : \psi$ is continuous$\}$;
$C^*_M(A)=\{\psi\in C_M(A) : \frac{\psi(x)}{1+x^2}\to$ finite as $x\to\infty\}$.
The norm of $C^*_M(A)$ is given as,

$$\|\psi\|_M=\sup_{x\geq 0}\frac{|\psi(x)|}{1+x^2}.$$

We define the local modulus of continuity of ψ as:

$$\omega_a(\psi,\delta)=\sup_{t,x\in[0,a];|t-x|\leq\delta}|\psi(t)-\psi(x)|.$$

Theorem 4.1. *Suppose $\psi\in C^*_M(A);(p_m)$ & (q_m) are define as in above remark (2) and $\omega_a(\psi,\delta)$ is modulus of continuity of ψ on $[0,a+1]\subset[0,\infty)$. Then we obtain*

$$\|S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)\|_{C_{[0,a]}}\leq 6M_\psi(1+a^2)\varsigma_m+2\omega_{a+1}(\psi;\sqrt{\varsigma_m}),$$

where

$$\varsigma_m=(1-p_mq_m)a^2+(1+3\alpha)\tfrac{a}{[m]_{p_m,q_m}}+\tfrac{\alpha^2}{[m]^2_{p_m,q_m}}.$$

Proof. Here $x\in[0,a]$ & $t\in\mathbb{R}^+\cup\{0\}$ [12]; we get

$$|\psi(t)-\psi(x)|\leq 6M_\psi(1+a^2)(t-x)^2+\omega_{a+1}(\psi;\delta_m)\left(\frac{|t-x|}{\delta_m}+1\right)$$

Now, making use of above inequality & Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)\|_{C_{[0,a]}} &\leq S^{\alpha}_{m,p_m,q_m}(|(\psi;x)-\psi(x)|;x)\\ &\leq 6M_\psi(1+a^2)S^{\alpha}_{m,p_m,q_m}((t-x)^2;x)\\ &\quad+\omega_{a+1}(\psi;\delta_m)\left(1+\frac{1}{\delta^2_m}S^{\alpha}_{m,p_m,q_m}(t-x)^2;x\right)^{1/2}.\end{aligned}$$

Hence, with the help of corollary: 2.1; $x\in[0,a]$, we have

$$\begin{aligned}S^{\alpha}_{m,p_m,q_m}((t-x)^2;x) &= (p_mq_m-2q_m+1)x^2+(q_m^2+2q_m\alpha+\alpha)\frac{x}{[m]_{p_m,q_m}}+\frac{\alpha^2}{[m]^2_{p_m,q_m}}\\ &\leq (p_mq_m-2q_m+1)a^2+(q_m^2+2q_m\alpha+\alpha)\frac{a}{[m]_{p_m,q_m}}+\frac{\alpha^2}{[m]^2_{p_m,q_m}}\\ &\leq (1-p_mq_m)a^2+(1+3\alpha)\frac{a}{[m]_{p_m,q_m}}+\frac{\alpha^2}{[m]^2_{p_m,q_m}}\\ &= \varsigma_m(x)\end{aligned}$$

Considering $\delta_m(x)=\sqrt{\varsigma_m(x)}$, we get the result. □

Some more results can be seen in [2].

5. Voronovskaya theorem for $S^{\alpha}_{m,p,q}$

Theorem 5.1. *Suppose (p_m) and (q_m) are as in above remark (2). For $\psi \in C^*_M(A) \ni \psi', \psi'' \in C^*_M(A)$; we get*

$$\lim_{m\to\infty}[m]_{p_m,q_m}|S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)| = (\alpha' x+\alpha)\psi'(x)+(\gamma x^2+(1+3\alpha)x)\psi''(x)/2$$

uniformly on $[0,B]$ for any $B>0$, where

$$\alpha' = \lim_{m\to\infty}[m]_{p_m,q_m}(q_m-1);$$
$$\gamma = \lim_{m\to\infty}[m]_{p_m,q_m}(p_m q_m - 2q_m+1).$$

Proof. Applying Taylor's formula on $\psi \in C^*_M(A)$; we get

$$\psi(t) = \psi(x)+(t-x)\psi'(x)+\tfrac{1}{2}\psi''(x)(t-x)^2+R(t,x)(t-x)^2,$$

where, $R(t,x)$ is Peano form of the remainder & $R(t,x)\to 0$ as $t\to x$.
Hence;

$$\begin{aligned}[m]_{p_m,q_m}(S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)) &= [m]_{p_m,q_m}\psi'(x)S^{\alpha}_{m,p_m,q_m}(t-x;x)\\ &+ [m]_{p_m,q_m}\frac{\psi''(x)}{2}S^{\alpha}_{m,p_m,q_m}((t-x)^2;x)\\ &+ [m]_{p_m,q_m}S^{\alpha}_{m,p_m,q_m}(R(t,x)(t-x)^2;x).\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$S^{\alpha}_{m,p_m,q_m}(R(t,x)(t-x)^2;x) \le \sqrt{S^{\alpha}_{m,p_m,q_m}(R^2(t,x);x)}\sqrt{S^{\alpha}_{m,p_m,q_m}((t-x)^4;x)}$$

Since, $R(t,x)\in C^*_M(A)$, by theorem: (2.3) and fact that $R(x,x)=0$; we have

$$\lim_{m\to\infty} S^{\alpha}_{m,p_m,q_m}(R^2(t,x);x) = R^2(x,x)=0$$

uniformly for $x\in[0,B]$. Therefore, we have

$$\lim_{m\to\infty}[m]_{p_m,q_m}S^{\alpha}_{m,p_m,q_m}(R(t,x)(t-x)^2;x)=0.$$

Hence,

$$\begin{aligned}\lim_{m\to\infty}(S^{\alpha}_{m,p_m,q_m}(\psi;x)-\psi(x)) &= \lim_{m\to\infty}[m]_{p_m,q_m}\psi'(x)S^{\alpha}_{m,p_m,q_m}(t-x;x)\\ &+ \lim_{m\to\infty}[m]_{p_m,q_m}\frac{\psi''(x)}{2}S^{\alpha}_{m,p_m,q_m}((t-x)^2;x). \qquad (5.1)\end{aligned}$$

Here;

$$\begin{aligned}\lim_{m\to\infty}[m]_{p_m,q_m}S^{\alpha}_{m,p_m,q_m}(t-x;x) &= \lim_{m\to\infty}[m]_{p_m,q_m}(q_m-1)x+\alpha\\ &= \alpha' x+\alpha; \qquad (5.2)\end{aligned}$$

$$\alpha' = \lim_{m\to\infty}[m]_{p_m,q_m}(q_m-1),$$

and

$$\begin{aligned}\lim_{m\to\infty}[m]_{p_m,q_m}S^{\alpha}_{m,p_m,q_m}((t-x)^2;x) &= \lim_{m\to\infty}[m]_{p_m,q_m}[(p_m q_m-2q_m+1)x^2+(q_m^2+2q_m\alpha+\alpha)x\\ &= \lim_{m\to\infty}[m]_{p_m,q_m}(p_m q_m-2q_m+1)x^2+(1+3\alpha)x\\ &= \gamma x^2+(1+3\alpha)x; \qquad (5.3)\end{aligned}$$

$$\gamma = \lim_{m\to\infty}[m]_{p_m,q_m}(p_m q_m-2q_m+1).$$

Using equations (5.2) and (5.3); we have proved the result. □

References

[1] A. Lupas, *A $q-$analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus 9, University of Cluj-Napoca, (1987) 85–92.

[2] AR Gairola, Deepmala, LN Mishra, *Rate of approximation by finite iterates of q-Durrmeyer operators*,Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **86(2)** (2016), 229–234.

[3] A. Wafi, N. Rao, Deepmala Rai, *Approximation properties by generalized-Baskakov-Kantorovich-Stancu type operators*, Appl. Math. Inf. Sci. Lett. **4(3)** (2016) 111–118.

[4] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inform. Sci. **5** (2011) 390–444.

[5] K. Khan, D.K. Lobiyal, *Bézier curves based on Lupas (p,q)-analogue of Bernstein functions in CAGD*, Journal of Computational and Applied Mathematics, **317, (2017), 458–477**

[6] M. Mursaleen, A. Alotaibi, *Statistical summability and approximation by de la Vallée-Poussin mean*,Applied Mathematics Letters 24(3) (2011) 320–324.

[7] M. Mursaleen, K.J. Ansari, A. Khan, *On (p, q)-analogue of Bernstein operators*, Applied Mathematics and Computation 266 (2015) 874–882.

[8] M. Mursaleen, K.J. Ansari, A. Khan, *Some approximation results by (p, q)-analogue of Bernstein-Stancu operators*, Applied Mathematics and Computation 264 (2015), 392–402.

[9] R.A. Devore, G.G. Lorentz *Constructive approximation.* Springer, Berlin, 1993.

[10] R. Jagannathan, K.S. Rao., *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, Proceedings of the International Conference on Number Theory and Mathematical Physics, (2005).

[11] T. Acar, *(p, q)-generalization of Szász-Mirakyan operators*, Mathematical Methods in the Applied Sciences, 39(10), (2016), 2685–2695.

[12] V. Gupta, A. Aral, *Convergence of the q-analogue of Szász-Beta operators*, Applied Mathematics and Computation 216, (2010), 374-–380.

[13] V.N. Mishra, S. Pandey *On Chlodowsky variant of (p; q) Kantorovich-Stancu-Schurer operators*, International Journal of Analysis and Applications 11(1) (2016) 28–39.

[14] V.N. Mishra, P. Sharma, L.N. Mishra *On statistical approximation properties of q-Baskakov-Szász-Stancu operators,*, Journal of Egyptian Mathematical Society, 24(3) (2016), 396–401.

[15] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications, 2013(1), (2013), 2013:586. doi:10.1186/1029-242X-2013-586..

[16] V.N. Mishra, S. Pandey, *On (p, q) Baskakov–Durrmeyer–Stancu Operators*, Advances in Applied Clifford Algebras, 27(2) (2017) 1633–1646.

Ankita R Devdhara

Applied Mathematics and Humanities Department, S.V. National Institute of Technology, Surat-395 007, Gujarat, India

E-mail address: krishna.devdhara@gmail.com

Vishnu Narayan Mishra

Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Anuppur, Madhya Pradesh, India

L. 1627 Awadh Puri Colony Beniganj, Phase-III, Opposite - I.T.I., Ayodhya Main Road, Faizabad, U.P. 224 001, India

E-mail address: vishnunarayanmishra@gmail.com, vishnu_narayanmishra@yahoo.co.in