SYMMETRY IDENTITIES FOR THE 2-VARIABLE UNIFIED
APOSTOL-TYPE POLYNOMIALS

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Abstract. In this paper, we introduce and investigate 2-variable unified Apostol-
type polynomials. We obtain some symmetry identities between these polyno-
mials and the generalized sum of integer powers. We give explicit relation for
this unified family. Also, we prove some recurrence relation for these polyno-
mials.

1. Introduction

Throughout this paper, we always make use of the following notations: \( \mathbb{N} \) denotes
the set of natural numbers, \( \mathbb{N}_0 \) denotes the set of nonnegative integers, \( \mathbb{R} \) denotes
the set of real numbers, \( \mathbb{C} \) denotes the set of complex numbers.

The generalized Apostol-Bernoulli polynomials \( B_{n}^{(\alpha)}(x, \lambda) \) of order \( \lambda \in \mathbb{N}_0 \), the
generalized Apostol-Euler polynomials \( E_{n}^{(\alpha)}(x, \lambda) \) of order \( \alpha \in \mathbb{N}_0 \) and the general-
ized Apostol-Genocchi polynomials \( G_{n}^{(\alpha)}(x, \lambda) \) of order \( \alpha \in \mathbb{N}_0 \) are defined by the
following generating functions ([5], [15]-[18], [20]) respectively,

\[
\sum_{n=0}^{\infty} B_{n}^{(\alpha)}(x, \lambda) \frac{t^n}{n!} = \left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt}, \quad |t| < 2\pi, \quad (\text{when } \lambda = 1, \quad |t| < |\log \lambda| \quad \text{when } \lambda \neq 1),
\]

(1)

\[
\sum_{n=0}^{\infty} E_{n}^{(\alpha)}(x, \lambda) \frac{t^n}{n!} = \left( \frac{2}{\lambda e^t + 1} \right)^\alpha e^{xt}, \quad (|t| < \pi, \quad \text{when } \lambda = 1, \quad |t| < |\log(-\lambda)|, \quad \text{when } \lambda \neq 1)
\]

(2)

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\[ \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, \lambda) \frac{t^n}{n!} = \left( \frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt}, \quad (|t| < \pi \text{ when } \lambda = 1, \quad |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1). \]  

(3)

The multiplier power sums are defined by Luo [16] as follows
\[ S_k^{(l)}(m; \lambda) = \sum_{0 \leq m_1 \leq \ldots \leq m_{vl}} \binom{n}{l} (-l)^{n-p} S_{k}^{(l)}(m; \lambda) \frac{t^n}{n!} \]  

(4)

From (4), we have
\[ \left( 1 - \lambda^m e^{mt} \right)^l = \frac{1}{l!} \sum_{n=0}^{\infty} \binom{n}{l} (-l)^{n-p} S_k^{(l)}(m; \lambda) \frac{t^n}{n!} \]  

(5)

(see detail in [16]).

Ozden et al in [19] defined the generalized Stirling numbers of the second kind as
\[ \sum_{n=0}^{\infty} S(n,v,a,b,\beta) \frac{t^n}{n!} = \left( \frac{\beta b e^t - a^b}{e^t} \right)^v. \]  

(6)

Ozarslan in [2] defined the unified Apostol-Bernoulli, Euler and Genocchi polynomials as
\[ \left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} P_n^{(\alpha)}(x, k, a, b) \frac{t^n}{n!}, \]  

(7)

where \( k \in \mathbb{N}_0, \ a, b \in \mathbb{R}\setminus\{0\}, \ \alpha, \beta \in \mathbb{C}. \)

Khan et al in ([9]-[11]) defined the 2-variable general polynomial family as
\[ e^{xt} \phi(y, t) = \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{n!}, \quad P_0(x, y) = 1 \]  

(8)

where
\[ \phi(y, t) = \sum_{n=0}^{\infty} \phi_n(y) \frac{t^n}{n!}, \quad \phi_0(y) \neq 0. \]

Khan et al in ([9]-[11]) gave some basic relations and explicit relations for this polynomial.

We define the 2-variable unified Apostol-Bernoulli, Euler and Genocchi polynomials \( P_n^{(\alpha)}(x, y; k, a, b) \) of order \( \alpha \) as
\[ \sum_{n=0}^{\infty} P_n^{(\alpha)}(x, y; k, a, b) \frac{t^n}{n!} = \left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^\alpha e^{xt} \phi(y, t) \]  

(9)

where
\[ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{n!} = e^{xt} \phi(y, t) \quad , \quad \phi(y, t) = \sum_{n=0}^{\infty} \phi_n(y) \frac{t^n}{n!}, \quad \phi_0(y) \neq 0. \]

Luo in ([15]-[18]) introduced and investigated for the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and Apostol-Genocchi polynomials. He
proved the multiplication formulas for Apostol-type polynomials. He proved some recurrence relation and explicit relationships for these polynomials. Srivastava et al in (20-23) gave some generalizations, proved some theorems, recurrence relations for the Apostol-Bernoulli, Euler, Genocchi polynomials.

Kurt in (13-14) proved some identities and symmetric relations for these polynomials.

Ozden et al in [19] gave unified representation for these polynomials. Ozarslan (1, 2) proved some relation and symmetric relations for the unified Apostol-type polynomials and Hermite-based Apostol-Bernoulli, Euler and Genocchi polynomials.

Khan et al ([9]-[11]) introduced and proved some relations for the 2-variable Apostol-type polynomials.

In this work, we define the 2-variable unified Apostol-type polynomials \( p_{\alpha}^{(n, \beta)}(x, y; k, a, b) \) of order \( \alpha \in \mathbb{N}_0 \). We give some basic relationships for these polynomials. Also, we prove some symmetric relations between for these polynomials.

2. Explicit Relation For The 2-Variable Unified Apostol-Type Polynomials

In this section, we aim to obtain the explicit relation of the polynomials \( p_{\alpha}^{(n, \beta)}(x, y; k, a, b) \). We prove some relations for these polynomials and give the relations between the 2-variable unified family of generalized Apostol-type polynomials and the Stirling numbers of second kind \( S(n, v, a, b, \beta) \) of order \( v \).

For \( \alpha = 1 \), we write again the equation (9) as

\[
F(x, y; k, a, b, \beta, t) = \sum_{n=0}^{\infty} p_{n, \beta}(x, y; k, a, b) \frac{t^n}{n!} = \left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right) e^{xt} \phi(y, t). \tag{10}
\]

We can obtain the following equations easily from (9)

\[
F^{(\alpha)}(x+1, y; k, a, b, \beta, t) = e^t F^{(\alpha)}(x, y; k, a, b, \beta, t), \tag{11}
\]

\[
F(x, k, a, b, \beta, t) F(w, k, a, b, \beta, t) = F^{(2)}(0, k, a, b, \beta, t) e^{(x+w)t}, \tag{12}
\]

\[
(\beta^b e^t + a^b) F(x, y; k, a^2, b, \beta^2, 2t) = 2^k F(2x, y, k, a, b, \beta, 2t) \tag{13}
\]

and

\[
F(x, y; k, a, b, \beta, t) F(u, 0; k, a, b, \beta, t) = F(k, a, b, \beta, t) F(x + u, y; k, a, b, \beta, t). \tag{14}
\]

**Theorem 1.** The 2-variable unified Apostol-type polynomials satisfy the following equations

\[
p_{\alpha}^{(n, \beta)}(x+1, y; k, a, b) = \sum_{l=0}^{n} \binom{n}{l} p_{\alpha}^{(n-l, \beta)}(x, y; k, a, b), \tag{15}
\]
There is the following relation between the Stirling numbers of the

Theorem 2. There is the following relation between the Stirling numbers of the
Second kind $S(n, v, a, b)$ and the 2-variable unified Apostol-type polynomials

$$pP_{n-k, \beta}(x, y; k, a, b) = \frac{(n-k)!}{2^k k!} \sum_{l=0}^{n} \binom{n}{l} P_{n-l, \beta}(x, y; k, a, b) S(l, 1, a, b)$$

Proof. From (6) and (9), we write as

$$\sum_{n=0}^{\infty} pP_{n, \beta}(x, y; k, a, b) \frac{t^n}{n!} = \frac{1}{2^{1-k}} \left( \frac{2^{1-k} t^k}{\beta^{k} a^{k} - a^{b}} \right)^{(\alpha)} e^{x t} \phi(y, t) \frac{\beta^{b} e^{t} - a^{b}}{t^{k}}$$

$$\sum_{n=0}^{\infty} pP_{n, \beta}(x, y; k, a, b) \frac{t^{n+k}}{n!} = \sum_{n=0}^{\infty} pP_{n, \beta}(x, y; k, a, b) \frac{t^{m}}{m!} \sum_{l=0}^{\infty} S(l, 1, a, b) \frac{t^{l}}{l!}.$$ 

By using Cauchy product and comprising the coefficient of $\frac{t^n}{m!}$, we have (19).}

Theorem 3. The following statements of the 2-variable unified Apostol-type polynomials hold:

1. $\frac{d}{dx} pP_{n, \beta}(x, y; k, a, b) = n pP_{n-1, \beta}(x, y; k, a, b)$
2. $\sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^{(\alpha)} pP_{n, \beta}(x, y; k, a, b) P_1(x, y)$
3. $\sum_{n=0}^{\infty} \binom{n}{m} pP_{m, \beta}(x, y; k, a, b) pP_{n-m, \beta}(u, y; k, a, b)$
4. $\sum_{n=0}^{\infty} \binom{n}{m} pP_{m, \beta}(x, y; k, a, b) pP_{n-m, \beta}(u, y; k, a, b)$
5. $\int_{x_0}^{x_1} pP_{n, \beta}(x; y; k, a, b) dx = \frac{1}{n} \left\{ pP_{n, \beta}(x_1; y; k, a, b) - pP_{n, \beta}(x_0; y; k, a, b) \right\}$
Kurt in \((13, 14)\) proved some symmetry identities for the unified Apostol-type polynomials. Ozarslan \([2]\) proved some relation for the Unified Apostol-Bernoulli, Euler, Genocchi polynomials.

In this section, we give new symmetry identities for the 2-variable unified Apostol-type polynomials.

**Theorem 4.** The following symmetry relations for the 2-variable unified Apostol-type polynomials hold true:

\[
\sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{c-1} \left( \frac{\beta}{a} \right)^i P_{m,\beta}^{(\alpha)} \left( x + \frac{di}{c}; dy; k, a, b \right) c^m
\]

\[
\sum_{j=0}^{d-1} \binom{\beta}{a}^j b j P_{n-m,\beta}^{(\alpha)} \left( X + \frac{cj}{d}; cY; k, a, b \right) d^{n-m}
\]

\[
= \sum_{m=0}^{n} \binom{n}{m} \sum_{i=0}^{d-1} \left( \frac{\beta}{a} \right)^i P_{m,\beta}^{(\alpha)} \left( x + \frac{ci}{d}; cy; k, a, b \right) d^m
\]

\[
\sum_{j=0}^{c-1} \binom{\beta}{a}^j b j P_{n-m,\beta}^{(\alpha)} \left( X + \frac{dj}{d}; dY; k, a, b \right) c^{n-m}
\]. (20)

**Proof.** Let \( f(t) \)

\[
f(t) = \frac{2^{1-k} \alpha e^{k\alpha} e^{k\beta} \phi(y, cdt) \left( \beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha \left( \beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha \right) e^{k\beta} \phi(Y, cdt) \right)}{(\beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha)^{k\alpha}}
\]

\[
= \frac{1}{(cd)^{k\alpha}} \left\{ \left( \frac{2^{1-k} (ct)^k}{\beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha} \right)^{(\alpha)} e^{k\alpha} \phi(y, cdt) a^{bc-b} \left( \frac{\beta}{a} \right)^bc e^{k\beta} - 1 \right\}
\]

\[
\left( \frac{2^{1-k} (dt)^k}{\beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha} \right)^{(\alpha)} e^{k\alpha} \phi(Y, cdt) a^{bd-b} \left( \frac{\beta}{a} \right)^bd e^{k\beta} - 1 \right\}
\]

since the expression for \( f(t) \) is symmetric in \( c \) and \( d \). We can expand \( f(t) \) into series in

\[
\phi(eY, dt) a^{bd-b} \sum_{j=0}^{d-1} \binom{\beta}{a}^j b j e^{k\beta} \left( \frac{2^{1-k} (dt)^k}{\beta \beta \beta \beta e^{k\beta} e^{k\beta} e^{k\beta} e^{k\beta} - \alpha} \right)^{(\alpha)} e^{k\alpha}
\]
By using the Cauchy product, we have

\[
\begin{align*}
&= \frac{1}{(cd)^{ka}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left( \sum_{i=0}^{d-1} \left( \frac{\beta}{a} \right)^i \right) p_{m}^{(\alpha)} \left( x + \frac{di}{c}, dy; k, a, b \right) c^m a^{bc+bd-2b} \\
&= \sum_{j=0}^{d-1} \left( \frac{\beta}{a} \right)^j p_{n-m}^{(\alpha)} \left( X + \frac{c}{d} j, cY; k, a, b \right) \frac{t^n}{n!} \quad (21)
\end{align*}
\]

in a similar manner,

\[
\begin{align*}
f(t) &= \frac{1}{(cd)^{ka}} \left( \frac{2^{1-k} dt}{\beta^b e^{dt} - a^b} \right)^{(\alpha)} e^{dx t} \phi (cy, dt) a^{bd-b} \sum_{j=0}^{c-1} \left( \frac{\beta}{a} \right)^j e^{dt j} \\
&= \frac{1}{(cd)^{ka}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \left( \sum_{i=0}^{d-1} \left( \frac{\beta}{a} \right)^i \right) p_{m}^{(\alpha)} \left( x + \frac{ic}{d}, cy; k, a, b \right) d^m a^{bd+bc-2b} \\
&= \sum_{j=0}^{d-1} \left( \frac{\beta}{a} \right)^j p_{n-m}^{(\alpha)} \left( X + \frac{dj}{c}, dY; k, a, b \right) c^n a^{bd} \frac{t^n}{n!} \quad (22)
\end{align*}
\]

From (21) and (22), we have (20).

**Theorem 5.** This is the following relations holds true.

\[
\sum_{m=0}^{n} \binom{n}{m} p_{m, \alpha}^{(\alpha)} (x, dy; k, a, b) c^m a^{bd} = \sum_{n=0}^{m} \binom{n}{m} p_{n-m, \alpha}^{(\alpha)} (x, cY; k, a, b) d^m a^{bd} \quad (23)
\]

**Theorem 6.** There is the following relation between the multiplier power sums and the 2-variable unified Apostol-type polynomials

\[
\begin{align*}
d^{kl} \sum_{l=0}^{n} \binom{n}{l} \left\{ \binom{p_{(\alpha+1)}^{(\alpha)}}{n-l, \alpha} (dx, dy; k, a, b) c^{n-l} \sum_{s=0}^{l} \binom{l}{s} \sum_{p=0}^{s} \binom{s}{p} \left( -\alpha \right)^{s-p} \\
S_{k}^{(\alpha)} \left( c, \frac{\beta}{a} \right) p_{l-s, \alpha}^{(\alpha)} (cX, cY; k, a, b) \right\} \\
&= c^{kl} \sum_{l=0}^{n} \binom{n}{l} \left\{ \binom{p_{(\alpha+1)}^{(\alpha)}}{n-l, \alpha} (cX, cY; k, a, b) d^{n-l} \sum_{s=0}^{l} \binom{l}{s} \sum_{p=0}^{s} \binom{s}{p} \left( -\alpha \right)^{s-p} \\
S_{k}^{(\alpha)} \left( d, \frac{\beta}{a} \right) p_{l-s, \alpha}^{(\alpha)} (dx, dy; k, a, b) \right\} \quad (24)
\end{align*}
\]
Proof. Let

\[ g(t) = \frac{t^{(\alpha+1)}e^{\alpha \int_0^t x \, dx} \left( \frac{2^{1-k}}{\beta^k e^t - a^k} \right)^{\alpha+1} e^{\alpha \int_0^t x \, dx} \phi(y, \int_0^t x \, dx)}{e^{\alpha \int_0^t x \, dx} \left( \frac{2^{1-k}}{\beta^k e^t - a^k} \right)^{\alpha+1} e^{\alpha \int_0^t x \, dx} \phi(y, \int_0^t x \, dx)}. \]

From the \( g(t) \) is symmetric in \( c \) and \( d \), we write as

\[ = \left\{ \frac{1}{c^k(a+1) \alpha} \left( \frac{(ct)^k}{\beta^k e^t - a^k} \right)^{\alpha+1} e^{\alpha \int_0^t x \, dx} \phi(y, \int_0^t x \, dx) \right\}. \]

By using the equation (5) in above equality, we write as

\[ S_k^{(\alpha)} \left( c, \left( \frac{\beta}{a} \right)^b \right) e^{\alpha \int_0^t x \, dx} \left( \frac{\sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \left( P_{n+1, \alpha} (dx, dy; \kappa, a, b) \right) c^n a^{b^n} \beta^{-b^n} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \right) \right) \]

Using cauchy product, we get

\[ = \frac{1}{c^k(a+1) \alpha} \left( \frac{\sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \left( P_{n+1, \alpha} (dx, dy; \kappa, a, b) \right) c^n a^{b^n} \beta^{-b^n} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \right) \]

In a similar manner

\[ g(t) = \frac{1}{d^k(a+1) \alpha} \left( \frac{\sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \left( P_{n+1, \alpha} (dx, dy; \kappa, a, b) \right) c^n a^{b^n} \beta^{-b^n} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} (s/p)(-\alpha)^s \cdot \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l)}{n!} \right) \]

Comparing the coefficients \( \frac{t^n}{n!} \) both sides above equations. We have (24).

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