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ON A CLASS OF ANALYTIC P-VALENT FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

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Abstract. In this work, we study a new class of p-valent analytic functions defined by Ruscheweyh derivatives. This class provides a transition from the class of bounded boundary rotations to the class of bounded radius rotations. We investigate some interesting properties of this class such as inclusion results, coefficient estimates and some integral preserving properties. A radius of convexity problem is also a focus of our investigations.

1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n \, z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. Also let $A(1) = A$. We denote by $S^*(\beta)$ and $C(\beta)$ the well known classes of starlike and convex of order β respectively. For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n \, z^n \quad \text{and} \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n \, z^n, \quad \text{for} \quad z \in E,$$

the convolution (Hadamard product) is given by

$$(f \star g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n \, b_n \, z^n, \quad \text{for} \quad z \in E.$$

Let $f(z) \in A(p)$. Denote by $D^{\delta+p-1} : A(p) \to A(p)$ the operator defined by

$$D^{\delta+p-1} f(z) = \frac{z}{(1-z)^{\delta+p}} * f(z) = z^p + \sum_{n=p+1}^{\infty} \varphi_n(\delta) \, a_n \, z^n, \quad (\delta > -p), \tag{1.2}$$

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where

$$\varphi_n(\delta) = \frac{(\delta+p)_{n-p}}{(n-p)!},$$

and $(\rho)_n$ is a Pochhammer symbol given as

$$(\rho)_n = \begin{cases} 1, \ n=0, \\ \rho(\rho+1)(\rho+2)\ldots(\rho+n-1), \ n \in \mathbb{N}. \end{cases}$$

It is obvious that when δ is any integer greater than p

$$D^{\delta+p-1} f(z) = \frac{z^p \left(z^{\delta-1} f(z)\right)^{(\delta+p-1)}}{(\delta+p-1)!}.$$

The following identity can easily be established.

$$(\delta+p) D^{\delta+p} f(z) = \delta D^{\delta+p-1} f(z) + z \left(D^{\delta+p-1} f(z)\right)'. \tag{1.3}$$

The operator $D^{\delta+p-1} f(z)$ is called the $(\delta+p-1)$th order Ruscheweyh derivative of $f(z)$. The symbol $D^{\delta+p-1}$ was introduced by Goel and Sohi [2] and for $p=1$, it was studied by Ruscheweyh [14].

Let $P_k(\alpha)$ be the class of analytic functions $h(z)$ defined in E satisfying the properties $h(0)=1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} h(z) - \alpha}{1-\alpha} \right| d\theta \le k\pi, \tag{1.4}$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \alpha < p$. When $\alpha = 0$, we obtain the class P_k defined in [13] and for $k=2, \alpha=0$, we have the class P of functions with positive real part. We can write (1.4) as

$$h(z) = \frac{1}{2} \int_0^{2\pi} \frac{1+(1-2\alpha) z e^{-i\theta}}{1-ze^{-i\theta}} \, d\mu(\theta),$$

where $\mu(\theta)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(\theta) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |\, d\mu(\theta)| \le k\pi.$$

Also, for $h(z) \in P_k(\alpha)$, we can write from (1.4)

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \ z \in E, \tag{1.5}$$

where $h_1(z), \ h_2(z) \in P(\alpha)$, $P(\alpha)$ is the class of functions with positive real part greater than α.

We now consider the following class.

Definition 1.1. A function $f(z) \in A(p)$ of the form (1.1) is in the class $V_k^p(\alpha, b, \delta)$ if and only if

$$\left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)}\right) \in P_k(\alpha), \quad z \in E,$$

where $k \ge 2$, $\delta > -p$, $0 \le \alpha < 1$ and $b \ne 0$.

For $p=1$ this class was introduced by Latha and Nanjunda Rao [3]. It contains several well known classes of analytic and univalent functions studied earlier.

We note the following special cases.
(**i**) $V_2^1(\alpha,1,1)=C(\alpha)$, $V_2(\alpha,2,0)=S^*(\alpha)$,
(**ii**) $V_k^1(\alpha,1,1)=V_k(\alpha)$, $V_k(\alpha,2,0)=R_k(\alpha)$,
where $V_k(\alpha)$ and $R_k(\alpha)$ denote the classses of bounded boundary and bounded radius rotation of order α, see for details [8, 9, 10].

2. Preliminary Results

We need the following results to obtain our results.

Lemma 2.1 [**5**]. Let $u=u_1+iu_2$, $v=v_1+iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions:
(i) $\Psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,
(ii) $(1,0)\in D$ and $\mathrm{Re}\Psi(1,0)>0$,
(iii) $\mathrm{Re}\Psi(iu_2,v_1)\le 0$, whenever $(iu_2,v_1)\in D$ and $v_1\le -\frac{1}{2}\left(1+u_2^2\right)$.
If $h(z)=1+c_1z+c_2z^2+\cdots$ is a function that is analytic in E such that $(h(z),zh'(z))\in D$ and $\mathrm{Re}\Psi(h(z),zh'(z))>0$ hold for all $z\in E$, then $\mathrm{Re}h(z)>0$ in E.

Lemma 2.2 [**10**]. Let $h(z)\in P_k$. Then, for $|z|=r<1$, we have

$$\frac{1-kr+r^2}{1-r^2}\le \mathrm{Re}h(z)\le |h(z)|\le \frac{1+kr+r^2}{1-r^2}.$$

Lemma 2.3. Let $h(z)\in P_k$. Then, for $|z|=r<1$, we have

$$|zh'(z)|\le \frac{r\left(k+4r+kr^2\right)\mathrm{Re}h(z)}{(1-r^2)(1+kr+r^2)}.$$

The result follows directly by using Lemma 2.2 and (1.5).

3. Main Results

Theorem 3.1. Let $f(z)\in V_k^p(\alpha,b,\delta)$ with $b>0$, $0\le\alpha<p$, $\delta>-p$. Then

$$|a_{n+p-1}|\le \frac{(\sigma)_{n+p-2}}{(n+p-2)!\,\varphi_{n+p-1}(\delta)}, \tag{3.1}$$

where $\sigma=\dfrac{kb(1-\alpha)(\delta+p)}{2}$.

This result is sharp.

Proof. Set

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+p}f(z)}{D^{\delta+p-1}f(z)}=h(z), \tag{3.2}$$

so that $p(z)\in P_k(\alpha)$. Let $h(z)=1+\sum\limits_{n=1}^{\infty}b_n\,z^n$. Then (3.2) can be written as

$$2\left(D^{\delta+p}f(z)-D^{\delta+p-1}f(z)\right)=bD^{\delta+p-1}f(z)\sum_{n=1}^{\infty}b_n\,z^n,$$

which implies that

$$\frac{2\varphi_{n+p-1}(\delta)(n-1)a_{n+p-1}}{(\delta+p)}=b\left(b_{n-1}+\varphi_2(\delta)\;a_{p+1}\;b_{n-2}+\cdots+\varphi_{n+p-2}(\delta)\;a_{n+p-2}\;b_1\right).$$

Using the coefficient estimates $|b_n|\le k(1-\alpha)$ for the class $P_k(\alpha)$, we obtain

$$|a_{n+p-1}|\le \frac{kb(1-\alpha)(\delta+p)}{2(n-1)\varphi_{n+p-1}(\delta)}\left(1+\varphi_2(\delta)\;|a_{p+1}|+\cdots+\varphi_{n+p-2}(\delta)\;|a_{n+p-2}|\right).$$

For $n=2$, $|a_{p+1}| \leq \dfrac{kb\,(1-\alpha)}{2}$.

Therefore (3.1) holds for $n=2$.

Assume that (3.1) is true for $n=m$ and consider

$$\begin{aligned}
|a_{m+p}| &\leq \frac{kb\,(1-\alpha)\,(\delta+p)}{2\ m\ \varphi_{m+p}\,(\delta)}\left(1+\varphi_2\,(\delta)\ \ |a_{p+1}|+\cdots+\varphi_{m+p-1}\,(\delta)\ \ |a_{m+p-1}|\right) \\
&\leq \frac{kb\,(1-\alpha)\,(\delta+p)}{2\ m\ \varphi_{m+p}\,(\delta)}\left\{1+\frac{kb\,(1-\alpha)\,(\delta+p)}{2!}\left(1+\frac{kb\,(1-\alpha)\,(\delta+p)}{2}\right)\right. \\
&\qquad \left.+\cdots+\frac{kb\,(1-\alpha)\,(\delta+p)}{(m-1)!}\prod_{j=1}^{m-2}\left(1+\frac{kb\,(1-\alpha)\,(\delta+p)}{2\ j}\right)\right\} \\
&= \frac{kb\,(1-\alpha)\,(\delta+p)}{2\ m\ \varphi_{m+p}\,(\delta)}\prod_{j=1}^{m-1}\left(1+\frac{kb\,(1-\alpha)\,(\delta+p)}{2\ j}\right) \\
&= \frac{(\sigma)_{m+p-1}}{(m)!\ \varphi_{m+p}\,(\delta)}\ .
\end{aligned}$$

Therefore, the result is true for $n=m+1$. Using mathematical induction, (3.1) holds true for all $n \geq p+1$.

This result is sharp for $\delta > -p$, $0 \leq \alpha < p$, $b>0$ and $k \geq 2$ as can be seen from the functions $f_0\,(z)$ which are given as

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+p}f_0\,(z)}{D^{\delta+p-1}f_0\,(z)}=(1-\alpha)\left[\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z}\right]+\alpha.$$

For different values of α, b, δ, we obtain the following corollaries proved by Noor [10].

Corollary 3.2. If $f\,(z) \in V_k^1\,(\alpha,2,0)=R_k\,(\alpha)$, then

$$|a_n| \leq \frac{(k\,(1-\alpha))_{n-1}}{(n-1)!}, \quad \text{for } n \geq 2.$$

This result is sharp.

Corollary 3.3. If $f\,(z) \in V_k^1\,(\alpha,1,1)=V_k\,(\alpha)$, then

$$|a_n| \leq \frac{(k\,(1-\alpha))_{n-1}}{n!}, \quad \text{for } n \geq 2.$$

This result is sharp.

Theorem 3.4. For $b>0$, $V_2^p\,(\alpha,b,\delta+p) \subseteq V_2^p\,(\beta,b+1,\delta+p-1)$, $z \in E$, where

$$\beta=\frac{-\eta+\sqrt{\eta^2+4\,(\delta+p)\,[b+1-(1-b)((b\alpha+1)(\delta+p)-b(1-\alpha))]}}{2\,(\delta+p)\,(b+1)}, \tag{3.3}$$

with $\eta=\eta\,(b,\alpha,\delta,p)=1-b\,(1-(\delta+p)(1-\alpha)+\alpha)$. The value of β is sharp.

Proof. Suppose $f\,(z) \in V_2^p\,(\alpha,b,\delta)$ and set

$$h\,(z)=1-\frac{2}{b+1}+\frac{2}{b+1}\frac{D^{\delta+p}f(z)}{D^{\delta+p-1}f(z)}, \tag{3.4}$$

where $h(z)$ is analytic in E with $h(0)=1$. Then simple computations, together with (3.4) and (1.3), yield

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\delta+p+1}f(z)}{D^{\delta+p}f(z)}=(1-\mu_1)+\mu_1\left[h(z)+\frac{\mu_2\ zh'(z)}{h(z)+\mu_3}\right], \tag{3.5}$$

with $\mu_1=\frac{\delta+p}{\delta+p+1}\frac{b+1}{b}$, $\mu_2=\frac{2}{(\delta+p)(b+1)}$, $\mu_3=\frac{2}{b+1}-1$. Since $f(z)\in V_2^p(\alpha,b,\delta)$, it follows that

$$\left[(1-\mu_1)+\mu_1\left(h(z)+\frac{\mu_2\ zh'(z)}{h(z)+\mu_3}\right)\right]\in P(\alpha),$$

or, equivalently,

$$\left[\frac{(1-\alpha-\mu_1)}{(1-\alpha)}+\frac{\mu_1}{(1-\alpha)}\left(h(z)+\frac{\mu_2\ zh'(z)}{h(z)+\mu_3}\right)\right]\in P. \tag{3.6}$$

We want to show that $h(z)\in P(\beta)$, where β is given by (3.3).

Let

$$h(z)=(1-\beta)h_0(z)+\beta,\ \ i=1,2.$$

Then, for $z\in E$

$$\left[\frac{(1-\alpha-\mu_1)}{(1-\alpha)}+\frac{\mu_1}{(1-\alpha)}\left((1-\beta)h_0(z)+\beta+\frac{\mu_2(1-\beta)\ zh_0'(z)}{(1-\beta)h_i(z)+\mu_3+\beta}\right)\right]\in P.$$

We now form the function $\Psi(u,v)$ by taking $u=h_0(z)$, $v=zh_0'(z)$ as

$$\Psi(u,v)=\frac{(1-\alpha-\mu_1)}{(1-\alpha)}+\frac{\mu_1}{(1-\alpha)}\left[(1-\beta)u+\beta+\frac{\mu_2(1-\beta)v}{(1-\beta)u+\mu_3+\beta}\right].$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify condition (iii) as.

$$\begin{aligned}\mathrm{Re}\Psi(iu_2,v_1) &= \frac{(1-\alpha-\mu_1)}{(1-\alpha)}+\frac{\mu_1}{(1-\alpha)}\left[\beta_1+\frac{\mu_2(1-\beta)(\mu_3+\beta)v_1}{(\mu_3+\beta)^2+(1-\beta)^2u_2^2}\right]\\ &\le \frac{(1-\alpha-\mu_1)}{(1-\alpha)}+\frac{\mu_1}{(1-\alpha)}\left[\beta_1-\frac{\mu_2(1-\beta)(\mu_3+\beta)\left(1+u_2^2\right)}{2\left[(\mu_3+\beta)^2+(1-\beta)^2u_2^2\right]}\right]\\ &= \frac{A+Bu_2^2}{C},\end{aligned}$$

where

$$\begin{aligned}A&=(\mu_3+\beta)\left[2(\mu_3+\beta)(1-\alpha-\mu_1+\mu_1\beta)-\mu_1\mu_2(1-\beta)\right],\\ B&=(1-\beta)\left[2(1-\beta)(1-\alpha-\mu_1+\mu_1\beta)-\mu_1\mu_2(\mu_3+\beta)\right],\\ C&=2(1-\alpha)\left[(\mu_3+\beta)^2+(1-\beta)^2u_2^2\right]>0.\end{aligned}$$

We notice that $\mathrm{Re}\Psi(iu_2,v_1)\le 0$ if and only if $A\le 0$ and $B\le 0$. From $A\le 0$, we obtain β as defined by (3.3) and $B\le 0$ gives us $0<\beta<1$. This proves that $h_0(z)\in P$, and hence $h(z)\in P_k(\beta)$.

By choosing the parameters $p=1$, $b=1$ and $\delta=0$, we obtain the following.

Corollary 3.5 [12]**.** Let $f(z)\in C(\alpha)$. Then $f(z)\in S^*(\beta)$, where

$$\beta=\frac{1}{4}\left[-(1-2\alpha)+\sqrt{(1-2\alpha)^2+8}\ \right].$$

This result is sharp.

For $\alpha = 0$ and $k = 2$ in Corollary 3.5, we have the following well known result

$$V_2(0) = C \subseteq R_2\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right), \text{ for } z \in E.$$

For a function $f(z) \in A(p)$, we consider the integral operator

$$F(z) = I_{\gamma,p}(f(z)) = \frac{(\gamma+p)}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)\, dt, \quad \gamma > -p, \text{ see [2].} \tag{3.7}$$

The operator $I_{\gamma,1}$, when $\gamma \in \mathbb{N}$ was introduced by Bernardi [1]. In particular, the operator $I_{1,1}$ was studied earlier by Libera [4] and Livingston [5].

Theorem 3.6. Let $f(z) \in V_k^p(\alpha, b, \delta)$ and let $F(z)$ be defined by (3.7). Then $F(z) \in V_k(\beta_2, b, \delta)$, where $0 < \beta_2 < 1$, $b > 0$ and

$$\beta_2 = \frac{1}{4}\left[-(2\mu_7 - 2\alpha + \mu_6) + \sqrt{(2\mu_7 - 2\alpha + \mu_6)^2 + 8(2\alpha\mu_7 + \mu_6)}\right]. \tag{2.4.1}$$

with $\mu_6 = 2$ and $\mu_7 = \frac{2(c-\delta)}{(\delta+p)} - 1 + \frac{2}{b}$.

The proof follows by using the same technique as in Theorem 3.4.

Theorem 3.7. If $f(z)$ is of the form (1.1) belongs to $V_k^p(\beta, b, \delta)$ and $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$, where $F(z)$ is the integral operator defined by (3.7), then

$$|b_{n+p-1}| \leq \frac{(\gamma+p)}{(\gamma+n+p-1)} \frac{(\sigma)_{n+p-2}}{(n-1)!\ \varphi_{n+p-1}(\delta)}, \quad \text{for } n \geq 2.$$

Proof. From (3.7), we obtain

$$1 + \gamma\, z + \sum_{n=p+1}^{\infty} (\gamma+p)\ a_n\ z^n = \gamma z + \sum_{n=p+1}^{\infty} \gamma\ b_n\ z^n + z + \sum_{n=p+1}^{\infty} n\ b_n\ z^n,$$

and thus

$$(n+p-1+\gamma)\, b_{n+p-1} = (\gamma+p)\ a_{n+p-1},\ n \geq 2.$$

From the above we have

$$|b_{n+p-1}| \leq \frac{(\gamma+p)}{(\gamma+n+p-1)}\ |a_{n+p-1}|,\ n \geq 2.$$

Using the estimates from Theorem 3.1, we obtain the required result.

Theorem 3.8. Let $f(z) \in V_k^P(0, b, \delta)$, $\delta > -p$, $b > 0$, $k \geq 2$ and $a = \frac{b(\delta+p)}{2} > 0$. Then

$$\Re\left(1 + \frac{z(D^{\delta+p-1}f(z))''}{(D^{\delta+p-1}f(z))'}\right) > 0, \text{ for } |z| < r_0$$

where r_0 is the least positive root of the equation

$$\left(4a^2 - 4ap + p^2\right)x^4 - (ka)\,x^3 + \left(4a(p-1) - k^2a^2 - 2p^2\right)x^2 - (ka)\,x + p^2 = 0. \tag{3.8}$$

This result is sharp.

Proof. Since $f(z) \in V_k^p(0, b, \delta)$ then

$$\frac{D^{\delta+p}f(z)}{D^{\delta+p-1}f(z)} = \frac{b(h(z)-1)+2}{2}, \tag{3.9}$$

where $h(z) \in P_k(0)$. Using the identity (1.3), we have from (3.9)

$$\frac{z\left(D^{\delta+p-1}f(z)\right)'}{D^{\delta+p-1}f(z)} = \frac{b\left(h(z)-1\right)(\delta+p)+2p}{2}. \tag{3.10}$$

Logarithmic differentiation of (3.10) yields

$$\frac{\left(z\left(D^{\delta+p-1}f(z)\right)'\right)'}{\left(D^{\delta+p-1}f(z)\right)'} = ah(z) - a + p + \frac{zh'(z)}{h(z)-1+\frac{p}{a}},$$

where $a = \frac{b(\delta+p)}{2}$. Then we have

$$\operatorname{Re}\left(1+\frac{z\left(D^{\delta+p-1}f(z)\right)''}{\left(D^{\delta+p-1}f(z)\right)'}\right) \geq a\operatorname{Re}h(z) + (p-a) - \frac{|zh'(z)|}{\left|h(z)-1+\frac{p}{a}\right|},$$

and hence, by using Lemma 2.2 and Lemma 2.3,

$$\operatorname{Re}\left(1+\frac{z\left(D^{\delta+p-1}f(z)\right)''}{\left(D^{\delta+p-1}f(z)\right)'}\right)$$

$$\geq \operatorname{Re}h(z)\left\{a+\frac{(p-a)\left(1-r^2\right)}{r^2+kr+1} - \frac{r\left(kr^2+4r+k\right)a}{\left(r^2+kr+1\right)\left((2a-p)r^2-kar+p\right)}\right\}$$

$$= \operatorname{Re}h(z)\left\{\frac{\left(4a^2-4ap+p^2\right)r^4-(ka)r^3+\left(4a(p-1)-k^2a^2-2p^2\right)r^2-(ka)r+p^2}{\left(r^2+kr+1\right)\left((2a-p)r^2-kar+p\right)}\right\} > 0,$$

Hence we havethe desired result.

For $D^{\delta+p-1}f_1(z)$ such that

$$\frac{D^{\delta+p}f_1(z)}{D^{\delta+p-1}f_1(z)} = \frac{b\left(h_k(z)-1\right)+2}{2},$$

where

$$h_k(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z},$$

we have

$$\frac{\left(z\left(D^{\delta+p-1}f_1(z)\right)'\right)'}{\left(D^{\delta+p-1}f_1(z)\right)'} = \frac{\left(4a^2-4ap+p^2\right)z^4-(ka)z^3+\left(4a(p-1)-k^2a^2-2p^2\right)z^2-(ka)z+p^2}{\left(z^2+kz+1\right)\left((2a-p)z^2-kaz+p\right)} = 0,$$

for $z = r_0$. Hence this radius r_0 is sharp.

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