Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://www.ilirias.com/jiasf
Volume 8 Issue 3(2017), Pages 15-24

COMBINATORIAL INEQUALITIES AND SUMS INVOLVING BERNSTEIN POLYNOMIALS AND BASIS FUNCTIONS

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Abstract. By using the Bernstein basis functions and Bernstein operator with their integral, we derive some combinatorial sums and identities including the Bernoulli numbers and polynomials, the Stirling numbers of the second kind and also some relations. Moreover, we give some inequalities involving combinatorial numbers and binomial coefficients.

1. Introduction

Recently, the Bernstein basis functions and the Bernstein polynomials have been many applications in mathematics, in statistics and in computer geometric design and other areas (*cf.* [1], [2], [3], [4], [6], [9]-[13]) and see also the references cited in each of these earlier works.

Let $x \in [0,1]$. Let n and k be nonnegative integers. The Bernstein basis functions $B_k^n(x)$ are defined by (*cf.* [4], [6]):

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.1}$$

where $k = 0, 1, \cdots, n$ and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Generating functions of these basis functions are given by

$$f_{\mathbb{B},k}(x,t) = \frac{t^k x^k e^{(1-x)t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}, \tag{1.2}$$

where $k = 0, 1, \ldots, n$ and $t \in \mathbb{C}$ and $x \in [0,1]$ (*cf.* [2], [9], [10]; and see also the references cited in each of these earlier works).

2010 *Mathematics Subject Classification.* 11B68; 11B83; 05A15; 05A19; 05A20; 12D10; 26C05; 30C15.

Key words and phrases. Bernstein basis functions, Bernstein polynomials, Bernoulli numbers, Bernoulli polynomials, Stirling numbers, Combinatorial sum, Combinatorial inequalities, Generating functions.

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Submitted December 22, 2016. Published March 20, 2017.
The authors are supported by the research fund of Akdeniz University.
Communicated by N.Braha.

Let $f:[0,1]\rightarrow\mathbb{C}$ is a continuous function. The sequence of the Bernstein polynomials is defined by

$$B_n\left(f:x\right)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_k^n(x). \tag{1.3}$$

The above sum converges uniformly to f on $[0,1]$ (*cf.* [6, p. 249], [16]).

Some properties of the Bernstein polynomials are given as follows:

B_n is a monotonic operator.

Let $x\in[0,1]$.

$$a\leq f(x)\leq b$$

this imply that

$$a\leq B_n\left(f:x\right)\leq b.$$

It is clear that if $a=0$, then

$$f(x)\geq 0$$

and also

$$B_n\left(f:x\right)\geq 0.$$

A continuous function f is convex if and only if

$$f(x)\leq B_n\left(f:x\right)$$

for every $x\in[0,1]$ (*cf.* [6, p. 249], [16]).

In order to give our results including the Bernoulli polynomials, the Bernstein polynomials, the Bernoulli polynomials and numbers, Stirling numbers, combinatorial sums and some combinatorial inequalities, we need the following identities and formulas for the beta function and the Euler gamma function.

The Bernoulli polynomials $B_n(x)$ are defined by:

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.4}$$

where $|t|<2\pi$. From the above generating functions, it is clear that

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}B_k, \tag{1.5}$$

where $B_n=B_n(0)$ denotes the Bernoulli numbers, which are given by the following formula:

$$\begin{aligned} B_0 &= 1 \\ \sum_{k=0}^{n-1}\binom{n}{k}B_k &= 0, \end{aligned}$$

(*cf.* [14], [15]; see also the references cited in each of these earlier works).

The beta function $B(\alpha,\beta)$ is defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=B(\beta,\alpha),\quad (Re(\alpha)>0, Re(\beta)>0) \tag{1.6}$$

(*cf.* [14, p. 9, Eq-(60)], [7]). A relation between the beta function and the gamma function is given by

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

or

$$B(n,m)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}=\frac{(n-1)!(m-1)!}{(n+m-1)!}, \tag{1.7}$$

where $n,m\in\mathbb{Z}^{+}$ (cf. [14, p. 9, Eq-(62)], [7]).

The results of these paper are summarized as follows:

In Section 2, we give modify generating functions for the Bernstein basis functions. Using these functions, we derive a relation between the Bernstein basis functions and the beta polynomials. In Section 3, by applying integral to the identities of the Bernoulli polynomials, we derive new identities including the Bernoulli polynomials and numbers, the Stirling numbers of the second kind, combinatorial sums including binomial coefficients. In Section 4, we give observation on inequalities including the Bernstein basis functions and combinatorial sums.

1.1. **Alternative forms of generating functions.** Here, we modify generating functions for the Bernstein basis functions. By using these functions, we give a relation between the Bernstein basis functions and the beta polynomials.

Let a be a positive integer. In [11], we gave the following alternative forms of the generating functions for the Bernstein basis functions:

$$F_{\mathbb{B},k}(t,x^{a})=\sum_{n=0}^{\infty}B_{k}^{n}(x^{a})\frac{t^{n}}{n!},$$

where

$$F_{\mathbb{B},k}(t,x^{a})=\frac{x^{ka}t^{k}}{k!}e^{t(1-x^{a})} \tag{1.8}$$

and

$$B_{k}^{n}(x^{a})=\left(\begin{array}{c}n\\k\end{array}\right)x^{ak}\left(1-x^{a}\right)^{n-k}.$$

We modify equation (1.8) as follows:

$$F_{\mathbb{B},k}(t,x^{2a})=\frac{x^{2ka}t^{k}}{k!}e^{t(1-x^{2a})}=\sum_{n=0}^{\infty}B_{k}^{n}(x^{2a})\frac{t^{n}}{n!}. \tag{1.9}$$

By using (1.9), we get

$$B_{k}^{n}(x^{2a})=\left(\begin{array}{c}n\\k\end{array}\right)x^{2ak}\left(1-x^{2a}\right)^{n-k}.$$

From the above equation, we have

$$B_{k}^{n}(x^{2a})=\left(\begin{array}{c}n\\k\end{array}\right)x^{2ak}\left(1-x^{a}\right)^{n-k}\left(1+x^{a}\right)^{n-k}.$$

Combining the above relation with the beta polynomials which are defined by

$$B_{n,k}\left(x\right)=x^{k}\left(1+x\right)^{n-k}$$

(*cf.* [12]), we get a relation between the Bernstein basis functions and the beta polynomials by the following theorem:

Theorem 1.1.

$$B_k^n(x^{2a}) = B_k^n(x^a) B_{n,k}(x^a). \tag{1.10}$$

Integrating both sides of equation (1.10) from 0 to 1, we get

$$\int_0^1 B_k^n(x^{2a})dx = \binom{n}{k} \frac{\Gamma\left(k+2-\frac{1}{2a}\right)\Gamma(n-k+1)}{\Gamma\left(n+3-\frac{1}{2a}\right)}.$$

Substituting $a=1$ into the above and using the following well-known identity for Euler-gamma function:

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!\sqrt{\pi}}{4^k k!},$$

(*cf.* [7], [11], [15]), we get the following corollary:

Corollary 1.2.

$$\int_0^1 B_k^n(x^2)dx = \frac{(2k+1)\,2^{2n-2k}}{(n+1)(2n+1)} \binom{2k}{k} \binom{2n}{n}^{-1}.$$

2. Combinatorial sums and identities via the Bernstein Polynomials

Combinatorial sums involving binomial coefficients and combinatorial identities play an important role in many branches of mathematics and statistics. In this section, by integrating the identities for the Bernstein basis functions, we derive many combinatorial sums including combinatorial identities and binomial coefficients.

In this section, integrations of the identities including the Bernstein Polynomials and the kth order forward difference with step size h, we derive some identities and combinatorial sums and combinatorial identities. We also give some new formulas including the Bernoulli polynomials and numbers, the Stirling numbers of the second kind.

Theorem 2.1. *Let $n \geq 0$. Then we have*

$$\sum_{j=k}^{n} \binom{j}{k} \binom{n}{k}^{-1} = \frac{n+1}{k+1},$$

where $k = 0, \ldots, n$.

Proof. Integrating both sides of the following equation from 0 to 1,

$$x^k = \sum_{j=k}^{n} \binom{j}{k} \binom{n}{k}^{-1} B_j^n(x)$$

for all $k = 0, \ldots, n$ (*cf.* [8]), we get

$$\frac{1}{n+1} \sum_{j=k}^{n} \binom{j}{k} \binom{n}{k}^{-1} = \frac{1}{k+1}.$$

Hence, the proof of the theorem is completed. □

Integrating both sides of equation (1.3) from 0 to 1, we arrive at the following lemma:

Lemma 2.2.

$$\int_0^1 B_n\left(f:x\right)dx = \frac{1}{n+1}\sum_{k=0}^{n} f\left(\frac{k}{n}\right). \tag{2.1}$$

Let $v \in \mathbb{N}_0$. Substituting $f(x) = x^v$ into (2.1), we get

$$\int_0^1 B_n\left(x^v:x\right)dx = \frac{1}{(n+1)\,n^v}\sum_{k=0}^{n} k^v.$$

Combining the left side of the above equation with the finite sums of powers including the Bernoulli polynomials and numbers, which is given by

$$\sum_{k=1}^{m} k^v = \frac{1}{v+1}\left(B_{v+1}(m+1) - B_{v+1}\right) \tag{2.2}$$

where m and v are positive integers (*cf.* [15, p. 81, Eq-(17)]), we arrive at the following theorem:

Theorem 2.3.

$$\int_0^1 B_n\left(x^v:x\right)dx = \frac{B_{v+1}(n+1) - B_{v+1}}{(n+1)\,(v+1)\,n^v}. \tag{2.3}$$

Remark. *In work of Zabandan* [16, Lemma 2.1-(2)]*, we see that*

$$\sum_{k=0}^{n} kB_k^n(x) = nx.$$

Integrating both sides of the above equation from 0 *to* 1*, we also obtain*

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

A forward difference with step size h is defined by

$$\Delta_h f(x) = f(x+h) - f(x).$$

Generally, the kth order forward difference with step size h is given by

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+(k-j)h)$$

where

$$\Delta_h^k f(x) = \Delta_h f(x)\left(\Delta_h^{k-1} f(x)\right).$$

Theorem 2.4.

$$\sum_{r=0}^{n} \binom{n}{r} \frac{\Delta_{\frac{1}{n}}^r f\left(0\right)}{r+1} = \sum_{r=0}^{n}\sum_{j=0}^{n-r} (-1)^j \binom{n}{r}\binom{n-r}{j} \frac{f\left(\frac{r}{n}\right)}{j+r+1}. \tag{2.4}$$

Proof. Phillips [6, p. 249, Theorem 7.1.1] gave the following formulas:

$$B_n\left(f:x\right) = \sum_{r=0}^{n} \binom{n}{r} \Delta^r f\left(0\right) x^r,$$

where Δ is the forward difference operator with step size $h=\frac{1}{n}$ and

$$B_n\left(f:x\right)=\sum_{r=0}^{n}\sum_{j=0}^{n-r}\left(-1\right)^{j}\left(\begin{array}{c} n \\ r \end{array}\right)\left(\begin{array}{c} n-r \\ j \end{array}\right)x^{r+j}f\left(\frac{r}{n}\right).$$

Integrating both sides of the above both equations from 0 to 1, after some elementary calculations and comparing left hand sides these equations, we arrive at the desired result. □

Theorem 2.5.

$$\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{r!}{r+1}S_2(m,r)=\sum_{r=0}^{n}\sum_{j=0}^{n-r}\left(-1\right)^{j}\left(\begin{array}{c} n \\ r \end{array}\right)\left(\begin{array}{c} n-r \\ j \end{array}\right)\frac{r^m}{j+r+1}. \tag{2.5}$$

Proof. We assume that $m\in\mathbb{N}_0$. Substituting $f(x)=x^m$ into (2.4), we get

$$\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{\Delta_{\frac{1}{n}}^{r}0^m}{r+1}=\frac{1}{n^m}\sum_{r=0}^{n}\sum_{j=0}^{n-r}\left(-1\right)^{j}\left(\begin{array}{c} n \\ r \end{array}\right)\left(\begin{array}{c} n-r \\ j \end{array}\right)\frac{r^m}{j+r+1} \tag{2.6}$$

where

$$\Delta_{\frac{1}{n}}^{r}x^m\,|_{x=0}=\Delta_{\frac{1}{n}}^{r}0^m.$$

A relation between $\Delta_{\frac{1}{n}}^{r}0^m$ and the Stirling numbers of the second kind is given as follows:

$$\Delta_{\frac{1}{n}}^{r}0^m=\frac{r!}{n^m}S_2(m,r),$$

By substituting the above identity into (2.6), we arrive at the desired result. □

We proved the following identity

$$\sum_{j=0}^{n}\left(-1\right)^{j}\left(\begin{array}{c} n \\ j \end{array}\right)\frac{1}{j+k+1}=\frac{1}{n+k+1}\left(\begin{array}{c} n+k \\ k \end{array}\right)^{-1} \tag{2.7}$$

(*cf.* [13]).

Substituting (2.7) into (2.5), we get

$$\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{r!}{r+1}S_2(m,r)=\frac{1}{n+1}\sum_{r=0}^{n}r^m. \tag{2.8}$$

Combining (2.8) with (2.2), we arrive at the following theorem:

Corollary 2.6. *Let m and n be positive integers. Then we have*

$$\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{r!}{r+1}S_2(m,r)=\frac{B_{m+1}(n+1)-B_{m+1}}{\left(n+1\right)\left(m+1\right)}. \tag{2.9}$$

Combining (2.3) with (2.9), we get the following corollary:

Corollary 2.7.

$$\int_0^1 B_n\left(x^m:x\right)dx=\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{r!}{r+1}S_2(m,r).$$

Theorem 2.8. *Let $k \in \mathbb{N}_0$. The kth derivative of $B_{n+k}(f:x)$ may be expressed in terms of kth differences of f as*

$$\frac{d^k}{dx^k} B_{n+k}(f:x) = \frac{(n+k)!}{n!} \sum_{r=0}^{n} \Delta^k_{\frac{1}{n+k}} f\left(\frac{r}{n+k}\right) B_r^n(x). \tag{2.10}$$

for all $n \geq 0$ and the operator Δ is applied with step size $h = \frac{1}{n+k}$.

Proof of this theorem is given by Phillips [6, p. 231, Theorem 7.1.3].

Lemma 2.9. *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \frac{d^k}{dx^k} B_{n+k}(f:x)\, dx = \frac{(n+k)!}{(n+1)!} \sum_{r=0}^{n} \Delta^k f\left(\frac{r}{n+k}\right), \tag{2.11}$$

where the operator Δ is applied with step size $h = \frac{1}{n+k}$.

Proof. Integrating both sides of equation (2.10) from 0 to 1, we get

$$\int_0^1 \frac{d^k}{dx^k} B_{n+k}(f:x)\, dx = \frac{(n+k)!}{n!} \sum_{r=0}^{n} \Delta^k f\left(\frac{r}{n+k}\right) \int_0^1 B_r^n(x) dx.$$

By using (1.6), one can easily has

$$\int_0^1 B_r^n(x) dx = \frac{1}{n+1}$$

(*cf.* [4]). Combining the above two equations, we arrive at the desired result. □

Lemma 2.10.

$$\int_0^1 \frac{d^k}{dx^k} B_{n+k}(f:x)\, dx = \frac{(n+k)!}{(n+1)!} \sum_{r=0}^{n+k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\left(\frac{r+j}{n+k}\right). \tag{2.12}$$

Proof. The higher-order derivatives of the Bernstein basis functions was given by author [9, Theorem 3.8] as follows:

$$\frac{d^k}{dx^k} B_r^n(x) = \frac{n!}{(n-k)!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_{r-j}^{n-k}(x)$$

By using the above derivative formula, we derive the following higher-order derivatives for the Bernstein polynomials in equation (1.3):

$$\frac{d^k}{dx^k} B_{n+k}(f:x) = \sum_{r=0}^{n+k} f\left(\frac{r}{n+k}\right) \frac{d^k}{dx^k} B_r^{n+k}(x).$$

After some elementary calculation, we have

$$\frac{d^k}{dx^k} B_{n+k}(f:x) = \frac{(n+k)!}{n!} \sum_{r=0}^{n+k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f\left(\frac{r+j}{n+k}\right) B_{r-j}^{n}(x)$$

see also [6, p. 253]. Integrating both sides of the above equation 0 to 1, we arrive at the desired result. □

Combining (2.11) with (2.12), we obtain the following theorem:

Theorem 2.11.

$$\sum_{j=0}^{n}\Delta^{k}f\left(\frac{r}{n+k}\right)=\sum_{r=0}^{n+k}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c}k\\ j\end{array}\right)f\left(\frac{r+j}{n+k}\right),$$

where the operator Δ *is applied with step size* $h=\frac{1}{n+k}$.

Remark. *Observe that*

$$\Delta^{k}f\left(\frac{t}{n+k}\right)=\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c}k\\ j\end{array}\right)f\left(\frac{t+j}{n+k}\right)$$

(cf. [6, p. 253]*).*

3. Observation on inequalities including the Bernstein basis functions

In this section, by using inequalities relations including convex functions and the Bernstein basis functions and combinatorial sums, we derive some inequalities involving binomial coefficients and sums of powers of the consecutive integers.

Theorem 3.1.

$$\frac{(n+1)^{2}}{2n+1}\sum_{k=0}^{n}\sum_{j=0}^{n}\left(\begin{array}{c}n\\ j\end{array}\right)\left(\begin{array}{c}n\\ k\end{array}\right)\left(\begin{array}{c}2n\\ j+k\end{array}\right)^{-1}f\left(\frac{j+k}{2n}\right)\geq\sum_{k=0}^{n}\sum_{j=0}^{n}f\left(\frac{j+k}{2n}\right).$$

Proof. In 2014, Rasa gave the following question which is related to the Bernstein basis functions and convex function:

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\left(B_{j}^{n}(x)B_{k}^{n}(x)-2B_{j}^{n}(x)B_{k}^{n}(y)+B_{j}^{n}(y)B_{k}^{n}(y)\right)f\left(\frac{j+k}{2n}\right)\geq 0, \quad (3.1)$$

where f is a convex function and $f\in C[0,1]$. This problem was proved by Mrowiec *et al.* [5] and also Abel [1]. Integrating both sides of equation (3.1) with respect to x and y from 0 to 1, we get

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\left(\int_{0}^{1}\int_{0}^{1}\left(B_{j}^{n}(x)B_{k}^{n}(x)-2B_{j}^{n}(x)B_{k}^{n}(y)+B_{j}^{n}(y)B_{k}^{n}(y)\right)dxdy\right)f\left(\frac{j+k}{2n}\right)\geq 0.$$

Combining the above equation with the following relations

$$\int_{0}^{1}\int_{0}^{1}B_{j}^{n}(x)B_{k}^{n}(x)\,dxdy=\frac{1}{(2n+1)}\left(\begin{array}{c}n\\ j\end{array}\right)\left(\begin{array}{c}n\\ k\end{array}\right)\left(\begin{array}{c}2n\\ j+k\end{array}\right)^{-1},$$

$$\int_{0}^{1}\int_{0}^{1}B_{j}^{n}(y)B_{k}^{n}(y)\,dxdy=\frac{1}{(2n+1)}\left(\begin{array}{c}n\\ j\end{array}\right)\left(\begin{array}{c}n\\ k\end{array}\right)\left(\begin{array}{c}2n\\ j+k\end{array}\right)^{-1}$$

and

$$\int_{0}^{1}\int_{0}^{1}B_{j}^{n}(x)B_{k}^{n}(y)\,dxdy=\frac{1}{(n+1)^{2}},$$

we get

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\left(\frac{2}{(2n+1)}\binom{n}{j}\binom{n}{k}\binom{2n}{j+k}^{-1}-\frac{2}{(n+1)^2}\right)f\left(\frac{j+k}{2n}\right)\geq 0.$$

Therefore, proof of the theorem is completed. □

Let

$$S_\delta=\left\{r:\left|\frac{r}{n}-x\right|\geq\delta\right\}.$$

That is, S_δ denotes the set of all values of r satisfying $\left|\frac{r}{n}-x\right|\geq\delta$ (*cf.* [6]).

Theorem 3.2.

$$\#S_\delta\leq\frac{n+1}{\delta^2}\sum_{r\in S_\delta}\binom{n}{r}\sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}\binom{n+2-j}{r+2-j}^{-1}\frac{1}{(n+3-j)}\left(\frac{r}{n}\right)^j$$

where $\#S_\delta$ denotes the number of elements in the set S_δ.

Proof. In work of Phillips [6, p. 256], we have

$$\sum_{r\in S_\delta}\binom{n}{r}x^r(1-x)^{n-r}\leq\frac{1}{\delta^2}\sum_{r\in S_\delta}\binom{n}{r}\left(\frac{r}{n}-x\right)^2x^r(1-x)^{n-r}.$$

Integrating both sides of the above equation from 0 to 1, we arrive at the desired result. □

In work of Phillips [6, p. 256], we also note that the latter sum is not greater than the sum of the same expression over all r. Since $0\leq x(1-x)\leq\frac{1}{4}$ on $[0,1]$, we have

$$\sum_{r\in S_\delta}\binom{n}{r}x^r(1-x)^{n-r}\leq\frac{1}{4n\delta^2}.$$

Integrating both sides of the above equation from 0 to 1, we arrive at the following corollary:

Corollary 3.3.

$$\#S_\delta\leq\frac{1}{4n\delta^2}.$$

Remark. *In work of Zabandan* [16, Theorem 2.4-(1)]*, we have the following result: Let f be a convex function on $[a,b]$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{1}{n+1}\sum_{k=0}^{n}\left(a+\frac{k}{n}(b-a)\right)\leq\frac{f(a)+f(b)}{2}.$$

From the above equation, we obtain

$$\frac{n(n+1)}{(b-a)^2}\int_a^b f(x)dx-an\leq\sum_{k=0}^{n}k\leq n\frac{(f(a)+f(b))(n+1)-2a}{2(b-a)}.$$

Substituting $f(x)=x^2$ into the above equation, we get

$$\frac{n(n+1)\left(a^2+ab+b^2\right)}{3(b-a)}-an\leq\sum_{k=0}^{n}k\leq n\frac{\left(a^2+b^2\right)(n+1)-2a}{2(b-a)}.$$

Substituting $a = 0$ and $b = 1$ into the above equation, we get the following inequality:

$$\frac{n(n+1)}{3} \leq \sum_{k=0}^{n} k \leq \frac{n(n+1)}{2}.$$

3.1. **Acknowledgements.** The authors are supported by the research fund of Akdeniz University.

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