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ON SUBCLASS OF ALPHA-QUASI CONVEX FUNCTIONS ASSOCIATED WITH CONIC REGIONS

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Abstract. The core objective of this article is to introduce and study new classes of α-quasi-convex functions in conic regions. Various interesting properties such as integral representation, sufficiency criteria, inclusion results and the effect of certain integral operators on these classes has also been examined.

1. Introduction

Let $\mathcal{A}$ be the class of functions having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, we denote the class $\mathcal{S}$ of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Goodman [1] introduced the class $\mathcal{UCV}$ of uniformly convex functions. A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{UCV}$ if for every circular arc $\xi \subset \mathbb{U}$, with center in $\mathbb{U}$, the arc $f(\xi)$ is convex. An interesting characterization of class $\mathcal{UCV}$ was given in [2], see also [3] as:

$$f(z) \in \mathcal{UCV} \Leftrightarrow f(z) \in \mathcal{A} \quad \text{and} \quad \mathfrak{Re}\left(1 + \tfrac{zf''(z)}{f'(z)}\right) > \left|\tfrac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{U}).$$

Kanas and Wiśniowska [4, 5] introduced and studied the classes of k-uniformly convex functions denoted by $k - \mathcal{UCV}$ and the corresponding class $k - \mathcal{ST}$ related by the Alexander type relation i.e. $f(z) \in k - \mathcal{UCV} \Leftrightarrow zf'(z) \in k - \mathcal{ST}$. The classes $k - \mathcal{ST}$ and $k - \mathcal{UCV}$ are defined as:

$$\begin{aligned} k - \mathcal{ST} &= \left\{ f(z) \in \mathcal{A} : \mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U} \right\}, \\ k - \mathcal{UCV} &= \left\{ f(z) \in \mathcal{A} : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{U} \right\}. \end{aligned}$$

In [4] and [5], respectively, where their geometric definitions and connections with the conic domains were also considered. If $k \geq 0$, then the class $k - \mathcal{UCV}$ is defined

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purely geometrically as a subclass of univalent functions which map the intersection of $\mathbb{U}$ with any disk centered at $\zeta, |\zeta| \leq k$, onto a convex domain. Therefore, the notion of k-uniform convexity is a generalization of the notion of convexity. Observe that, if $k = 0$ then the center ζ is the origin and the class $k - \mathcal{UCV}$ reduces to the class $\mathcal{C}$. Moreover for $k = 1$ it coincides with the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [1] and studied extensively by Rønning [3] and independently by Ma and Minda [2]. We note that the class $k - \mathcal{UCV}$ started much earlier in papers [6, 7] with some additional conditions but without the geometric interpretation. Later the class k-uniformly close-to-convex functions denoted by $k - \mathcal{UK}$ and is defined as:

$$\begin{aligned} k - \mathcal{UK} \quad = \quad & \left\{ f(z) \in \mathcal{A} : \mathfrak{Re}\left(\frac{zf'(z)}{g(z)}\right) > k \left| \frac{zf'(z)}{g(z)} - 1 \right|, \right. \\ & \left. g(z) \in k - \mathcal{ST}, \quad z \in \mathbb{U} \right\} \end{aligned}$$

was considered by Acu [8], for detail study on these classes we refer [9, 10, 11].
If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence, see [12, 13].

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The class k-uniformly close-to-convex functions of complex order β denoted by $k - \mathcal{UK}_\beta$ and is defined as

$$\begin{aligned} k - \mathcal{UK}_\beta \quad = \quad & \left\{ f(z) \in \mathcal{A} : \ \mathfrak{Re}\left[1 + \frac{1}{\beta}\left(\frac{zf'(z)}{g(z)} - 1\right)\right] > k \left| \frac{1}{\beta}\left(\frac{zf'(z)}{g(z)} - 1\right) \right|, \right. \\ & \left. g(z) \in k - \mathcal{ST}, \quad z \in \mathbb{U} \right\}. \end{aligned}$$

This class was introduced and studied by Noor et al. [14].
We now define the following class.

Definition 1.1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$ if and only if for $\alpha \geq 0$ and β, $\gamma \in \mathbb{C} \setminus \{0\}$*

$$\mathfrak{Re}\left\{J(\alpha, \beta, \gamma; f(z), g(z))\right\} > k \left| J(\alpha, \beta, \gamma; f(z), g(z)) - 1 \right|, \quad (k \geq 0, \ z \in \mathbb{U}),$$

for some $g(z) \in k - \mathcal{ST}$, where

$$J(\alpha, \beta, \gamma; f(z), g(z)) = 1 + \frac{1-\alpha}{\beta}\left[\frac{zf'(z)}{g(z)} - 1\right] + \frac{\alpha}{\gamma}\left[\frac{(zf'(z))'}{g'(z)} - 1\right]. \tag{1.2}$$

Special Cases:

(i) $\mathcal{TQ}(k, 0, \beta, \gamma) = k - \mathcal{UK}_\beta$, the class of k-uniformly close-to-convex functions of complex order β, see [14].

(ii) $\mathcal{TQ}(k, 1, \beta, \gamma) = k - \mathcal{UQ}_\gamma$, the class of k-uniformly quasi-convex functions of complex order γ, see [14].

(iii) For $g(z) = z$, $k = 1$, $\alpha = 0, 1$, $\beta = 1 - \sigma$ $(0 \leq \sigma < 1)$ and $\gamma = 1 - \eta\,(0 \leq \eta < 1)$, we obtain the classes $\mathcal{UK}(\sigma)$ and $\mathcal{UQ}(\eta)$, the classes of uniformly close-to-convex of real order σ and uniformly quasi-convex functions of real order η introduced and investigated in [15].

(iv) $\mathcal{TQ}(0, \alpha, 1, 1) = \mathcal{Q}_\alpha$, the class of alpha quasi-convex functions, introduced and studied in [16].

(v) $\mathcal{TQ}(0,0,\beta,\gamma) = \mathcal{K}(\beta)$, the class of close-to-convex functions of complex order β, see [17, 18].

(vi) $\mathcal{TQ}(0,1,\beta,\gamma) = C^*(\gamma)$, the class of quasi-convex functions of complex order γ, see [17, 18].

(vii) When we take $k = 0$, $\alpha = 0, 1$ and $\beta = \gamma = 1$, we obtain the classes $\mathcal{K}$ and $\mathcal{Q}$, the classes of close-to-convex and quasi-convex functions introduced and investigated by Kaplan et al. [19] and Noor et al. [20] respectively.

Geometric Interpretation

It is known that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{TQ}(k,\alpha,\beta,\gamma)$ if and only if $J(\alpha,\beta,\gamma; f(z), g(z))$ takes all the values in conic domain Ω_k such that

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{1.3}$$

In particular cases, the boundary $\partial\Omega_k$ of the above set becomes the imaginary axis when $k = 0$, a hyperbola when $0 < k < 1$, a parabola when $k = 1$, and an ellipse when $k > 1$. All of these curves have the vertex at the point $k/(k+1)$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$; it is ever symmetric with respect to the real axis. This domain has been extensively studied by Noor et.al [14].

The function which plays the role of extremal functions for these conic regions are given as:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & k = 0; \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1; \\ 1 + \frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\operatorname{arc\,tanh}\sqrt{z}\right\}, & 0 < k < 1; \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right) + \frac{\gamma}{1-k^2}, & k > 1; \end{cases} \tag{1.4}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{t}z}$, $z \in \mathbb{U}$ and $t \in (0,1)$ is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t) = R(\sqrt{1-t^2})$ is complementary integral of $R(t)$, see [4, 5, 14]. Moreover, $p_k(\mathbb{U}) = \Omega_k$ and $p_k(\mathbb{U})$ is convex univalent in $\mathbb{U}$, see [4, 5]. Therefore, we have

$$\mathfrak{Re}\{p(z)\} > k\,|p(z) - 1| \quad \Leftrightarrow \quad p(z) \prec p_k(z). \tag{1.5}$$

Let $\mathcal{P}(p_k(z))$ denote the class of all functions p which are analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \prec p_k(z)$ for $z \in \mathbb{U}$. It is easy to verify that $\mathcal{P}(p_k(z))$ is a convex set. It is known [21] that

$$\mathcal{P}(p_k(z)) \subset \mathcal{P}\left(\frac{k}{k+1}\right) \subset \mathcal{P},$$

where $\mathcal{P}$ is the class of Carathéodory functions, see [12].

Taking this geometrical interpretation into consideration, we can rephrase the Definition 1.1 as:

Let $f(z) \in \mathcal{A}$. Then $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$ if and only if for given $\alpha \geq 0$, β and $\gamma \in \mathbb{C} \setminus \{0\}$

$$J(\alpha, \beta, \gamma; f(z), g(z)) \prec p_k(z), \quad (z \in \mathbb{U}),$$

or

$$J(\alpha, \beta, \gamma; f(z), g(z)) \in \mathcal{P}(p_k(z)), \quad (z \in \mathbb{U}),$$

for some $g(z) \in k - \mathcal{ST}$, where $J(\alpha, \beta, \gamma; f(z), g(z))$ is given by (1.2).

2. Lemmas

Lemma 2.1. [22]. *Let $\lambda, \rho \in \mathbb{C}$ be such that $\lambda \neq 0$ and let $\phi(z) \in \mathcal{A}$ be convex and univalent in $\mathbb{U}$ with $\mathfrak{Re}\{\lambda\phi(z) + \rho\} > 0$ $(z \in \mathbb{U})$. Also let $q(z) \in \mathcal{A}$ and $q(z) \prec \phi(z)$, if $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and if $p(z)$ satisfies*

$$\left(p(z) + \frac{zp'(z)}{\lambda q(z) + \rho}\right) \prec \phi(z), \tag{2.1}$$

then $p(z) \prec \phi(z)$.

The conditions $k, \alpha \geq 0$ and $\beta, \gamma \in \mathbb{C} \setminus \{0\}$ on the parameters are assumed throughout the entire paper unless otherwise mentioned.

3. Main Results

Theorem 3.1. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{TQ}(k, \alpha, \beta, \gamma)$ if and only if for there exists $p(z) \in \mathcal{P}(P_k(z))$ and $g(z) \in k - \mathcal{ST}$ such that*

$$f'(z) = \frac{\gamma}{\alpha z g^{\frac{\gamma}{\beta}\left(\frac{1}{\alpha}-1\right)}(z)} \int_0^z (p(\zeta) - \mu)\, g^{\frac{\gamma}{\beta}\left(\frac{1}{\alpha}-1\right)}(\zeta)\, g'(\zeta)\, \mathrm{d}\zeta, \tag{3.1}$$

for $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$ and $\mu = 1 - \frac{\alpha}{\gamma} - \frac{1-\alpha}{\beta}$.

Proof. From (3.1), we have

$$\frac{\alpha}{\gamma}(zf'(z)) = g^{-\frac{\gamma}{\beta}\left(\frac{1}{\alpha}-1\right)}(z) \int_0^z (p(\zeta) - \mu)\, g^{\frac{\gamma}{\beta}\left(\frac{1}{\alpha}-1\right)}(\zeta)\, g'(\zeta)\, \mathrm{d}\zeta. \tag{3.2}$$

After differentiation and simple computation of (3.2) we obtain

$$(1-\alpha)\left[1 + \frac{1}{\beta}\left[\frac{zf'(z)}{g(z)} - 1\right]\right] + \alpha\left[1 + \frac{1}{\gamma}\left[\frac{(zf'(z))'}{g'(z)} - 1\right]\right] = p(z).$$

If right hand side belongs to $\mathcal{P}(P_k(z))$, so does left hand side and conversely. □

Theorem 3.2. *Let $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$ where*

$$\mathfrak{Re}\left(\alpha + (1-\alpha)\frac{\gamma}{\beta}\right) > 0. \tag{3.3}$$

Then $f(z) \in k - \mathcal{UK}_\delta$, where

$$\delta = \frac{\beta\gamma}{(1-\alpha)\gamma + \alpha\beta}.$$

Proof. For $\alpha = 0$, we have $\delta = \beta$ and it immediately follows that $f(z) \in k - \mathcal{UK}_\delta$, see also special case (i), page 2. For $\alpha \neq 0$ let

$$1 + \frac{1}{\delta}\left(\frac{zf'(z)}{g(z)} - 1\right) = p(z),$$

where p is analytic in $\mathbb{U}$ and $p(0) = 1$. Then

$$\frac{(zf'(z))'}{g'(z)} = \delta p(z) + 1 - \delta + \frac{\delta z p'(z)}{q(z)},$$

where $\frac{zg'(z)}{g(z)} = q(z)$. Therefore, $J(\alpha, \beta, \gamma; f(z), g(z))$ becomes

$$\begin{aligned} J(\alpha, \beta, \gamma; f(z), g(z)) &= (1-\alpha)\left[1 + \frac{\delta}{\beta}[p(z) - 1]\right] + \alpha\left[1 + \frac{\delta}{\gamma}\left[p(z) + \frac{zp'(z)}{q(z)} - 1\right]\right] \\ &= \left[1 - \frac{\alpha\delta}{\gamma}\left(\frac{1-\alpha}{\beta} + \frac{\alpha}{\gamma}\right)\right] + \left(\frac{1-\alpha}{\beta} + \frac{\alpha}{\gamma}\right)\delta p(z) + \frac{\alpha\delta}{\gamma}\frac{zp'(z)}{q(z)}. \end{aligned}$$

Now for $\delta = \frac{\beta\gamma}{(1-\alpha)\gamma + \alpha\beta}$, we have

$$J(\alpha, \beta, \gamma; f(z), g(z)) = p(z) + \frac{zp'(z)}{\frac{\gamma}{\alpha\delta}q(z)}.$$

Since $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$, then

$$p(z) + \frac{zp'(z)}{\frac{\gamma}{\alpha\delta}q(z)} \prec p_k(z).$$

To apply Lemma 2.1 with $\lambda = \frac{\gamma}{\alpha\delta}$ and $\rho = 0$, we we need $\mathfrak{Re}\lambda > 0$. We have

$$\mathfrak{Re}\lambda = \mathfrak{Re}\left(\frac{\gamma}{\alpha\delta}\right) = \mathfrak{Re}\left(\alpha + (1-\alpha)\frac{\gamma}{\beta}\right) > 0$$

because of (3.3). Therefore, by Lemma 2.1, we obtain

$$1 + \frac{1}{\delta}\left(\frac{zf'(z)}{g(z)} - 1\right) = p(z) \prec p_k(z).$$

Consequently, by (1.5), $f(z) \in k - \mathcal{UK}_\delta$. This completes the proof. □

Corollary 3.3. *Let $f(z) \in \mathcal{TQ}(0, \alpha, 1, 1) = \mathcal{Q}_\alpha$. Then $f(z) \in \mathcal{K}$. That is, $\mathcal{Q}_\alpha \subset \mathcal{K}$, $\alpha \geq 0$.*

The above result is well-known inclusion proved in [16].

Theorem 3.4. *For $\alpha > \alpha_1 \geq 0$,*

$$\mathcal{TQ}(k, \alpha, \beta, \gamma) \subseteq \mathcal{TQ}(k, \alpha_1, \beta, \gamma).$$

Proof. Let $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$. Consider

$$\begin{aligned} J(\alpha_1, \beta, \gamma; f(z), g(z)) &= 1 + \frac{1-\alpha_1}{\beta}\left[\frac{zf'(z)}{g(z)} - 1\right] + \frac{\alpha_1}{\gamma}\left[\frac{(zf'(z))'}{g'(z)} - 1\right] \\ &= \left(1 - \frac{\alpha_1}{\alpha}\right)\left\{1 + \frac{1}{\beta}\left[\frac{zf'(z)}{g(z)} - 1\right]\right\} \\ &+ \frac{\alpha_1}{\alpha}\left\{1 + \frac{1-\alpha}{\beta}\left[\frac{zf'(z)}{g(z)} - 1\right] + \frac{\alpha}{\gamma}\left[\frac{(zf'(z))'}{g'(z)} - 1\right]\right\} \\ &= \left(1 - \frac{\alpha_1}{\alpha}\right) J(0, \beta, \gamma; f(z), g(z)) + \frac{\alpha_1}{\alpha} J(\alpha, \beta, \gamma; f(z), g(z)). \end{aligned}$$

Now since $f(z) \in \mathcal{TQ}(k,\alpha,\beta,\gamma)$, we have $J(\alpha,\beta,\gamma;f(z),g(z)) \in \mathcal{P}(p_k(z))$. Also Theorem 3.2 gives us that $J(0,\beta,\gamma;f(z),g(z)) \in \mathcal{P}(p_k(z))$. Since $\mathcal{P}(p_k(z))$ is convex set, therefore $J(\alpha_1,\beta,\gamma;f(z),g(z)) \in \mathcal{P}(p_k(z))$ and consequently $f(z) \in \mathcal{TQ}(k,\alpha_1,\beta,\gamma)$. This completes the proof. □

Theorem 3.5. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{TQ}(k,\alpha,\beta,\gamma)$, if*

$$\sum_{n=2}^{\infty} \Phi_n(k;\alpha,\beta,\gamma) < |\beta\gamma|, \tag{3.4}$$

where

$$\Phi_n(k;\alpha,\beta,\gamma) = \left\{ \begin{array}{c} (k+1)\left[n(1-\alpha)|\gamma| + n^2|\alpha\beta|\right]|a_n| + \left[\begin{array}{c}(k+1)\left[n(1-\alpha)|\gamma|+|\alpha\beta|\right]\\ +|\beta\gamma|(n+1)\end{array}\right]|b_n| \\ +\sum\limits_{j=2}^{n-1}\left\{\begin{array}{c}(k+1)\left[j(n+1-j)(1-\alpha)|\gamma|+j^2|\alpha\beta|\right]|a_jb_{n+1-j}| \\ +(n+1-j)\left[|\beta\gamma|+(k+1)\left[(1-\alpha)|\gamma|+|\alpha\beta|\right]\right]|b_jb_{n+1-j}|\end{array}\right\}. \end{array} \right\} \tag{3.5}$$

Proof. Let us assume that relation (3.4) holds. Now it is sufficient to show that

$$k\left|J(\alpha,\beta,\gamma;f(z),g(z))-1\right| - \mathfrak{Re}\left[J(\alpha,\beta,\gamma;f(z),g(z))-1\right] < 1.$$

First we consider

$$\begin{aligned} \left|J(\alpha,\beta,\gamma;f(z),g(z))-1\right| &= \left|(1-\alpha)\left[1+\frac{1}{\beta}\left[\frac{zf'(z)}{g(z)}-1\right]\right]+\alpha\left[1+\frac{1}{\gamma}\left[\frac{(zf'(z))'}{g'(z)}-1\right]\right]-1\right| \\ &= \left|\frac{(1-\alpha)\gamma zf'(z)g'(z)+\alpha\beta(zf'(z))'g(z)-\left[(1-\alpha)\gamma+\alpha\beta\right]g(z)g'(z)}{\beta\gamma g(z)g'(z)}\right|. \end{aligned} \tag{3.6}$$

Using (1.1) and the series $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ in (3.6), we have

$$\begin{aligned} (zf'(z))'g(z) &= \tfrac{1}{z}\left(z+\sum_{n=2}^{\infty}n^2a_nz^n\right)\left(z+\sum_{n=2}^{\infty}b_nz^n\right) \\ &= \tfrac{1}{z}\left(\sum_{n=0}^{\infty}n^2a_nz^n\right)\left(\sum_{n=0}^{\infty}b_nz^n\right) \\ &= \tfrac{1}{z}\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}j^2a_jb_{n-j}z^n\right) \\ &= z+\sum_{n=3}^{\infty}\sum_{j=0}^{n}j^2a_jb_{n-j}z^{n-1} \\ &= z+\sum_{n=2}^{\infty}\sum_{j=0}^{n+1}j^2a_jb_{n+1-j}z^{n} \\ &= z+\sum_{n=2}^{\infty}\left[n^2a_n+b_n+\sum_{j=2}^{n-1}j^2a_jb_{n+1-j}\right]z^n. \end{aligned}$$

Similarly, we have

$$zf'(z)g'(z) = z+\sum_{n=2}^{\infty}\left[n(a_n+b_n)+\sum_{j=2}^{n-1}j(n+1-j)a_jb_{n+1-j}\right]z^n,$$

and

$$g(z)g'(z) = z+\sum_{n=2}^{\infty}\left[(n+1)b_n+\sum_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]z^n.$$

Using the above equalities in (3.6), we obtain

$$
\begin{aligned}
&|J(\alpha,\beta,\gamma;f(z),g(z))-1| \\
&= \left|\frac{\left[\begin{array}{c}(1-\alpha)\gamma\left[z+\sum\limits_{n=2}^{\infty}\left[n(a_n+b_n)+\sum\limits_{j=2}^{n-1}j(n+1-j)a_jb_{n+1-j}\right]z^n\right] \\ +\alpha\beta\left[z+\sum\limits_{n=2}^{\infty}\left[n^2a_n+b_n+\sum\limits_{j=2}^{n-1}j^2a_jb_{n+1-j}\right]z^n\right] \\ -[(1-\alpha)\gamma+\alpha\beta]\left[z+\sum\limits_{n=2}^{\infty}\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]z^n\right]\end{array}\right]}{\beta\gamma\left[z+\sum\limits_{n=2}^{\infty}\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]z^n\right]}\right| \\
&= \left|\frac{\left[\begin{array}{c}(1-\alpha)\gamma\left(\sum\limits_{n=2}^{\infty}\left[n(a_n+b_n)+\sum\limits_{j=2}^{n-1}j(n+1-j)a_jb_{n+1-j}\right]z^n\right) \\ +\alpha\beta\left(\sum\limits_{n=2}^{\infty}\left[n^2a_n+b_n+\sum\limits_{j=2}^{n-1}j^2a_jb_{n+1-j}\right]z^n\right) \\ -[(1-\alpha)\gamma+\alpha\beta]\left(\sum\limits_{n=2}^{\infty}\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]z^n\right)\end{array}\right]}{\beta\gamma\left[z+\sum\limits_{n=2}^{\infty}\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]z^n\right]}\right| \\
&\leq \frac{\left[\begin{array}{c}|(1-\alpha)\gamma|\left(\sum\limits_{n=2}^{\infty}\left|\left[n(a_n+b_n)+\sum\limits_{j=2}^{n-1}j(n+1-j)a_jb_{n+1-j}\right]\right||z|^n\right) \\ +|\alpha\beta|\left(\sum\limits_{n=2}^{\infty}\left|\left[n^2a_n+b_n+\sum\limits_{j=2}^{n-1}j^2a_jb_{n+1-j}\right]\right||z|^n\right) \\ +[|(1-\alpha)\gamma+\alpha\beta|]\left(\sum\limits_{n=2}^{\infty}\left|\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]\right||z|^n\right)\end{array}\right]}{|\beta\gamma|\left[|z|-\sum\limits_{n=2}^{\infty}\left|\left[(n+1)b_n+\sum\limits_{j=2}^{n-1}(n+1-j)b_jb_{n+1-j}\right]\right||z|^n\right]}
\end{aligned}
$$

$$\leq \frac{\sum_{n=2}^{\infty}\left\{\begin{array}{c}\left[n(1-\alpha)|\gamma|(|a_n|+|b_n|)+\sum_{j=2}^{n-1} j(n+1-j)|(1-\alpha)\gamma||a_j b_{n+1-j}|\right]\\ +\left[n^2|\alpha\beta||a_n|+|\alpha\beta||b_n|+\sum_{j=2}^{n-1} j^2|\alpha\beta||a_j b_{n+1-j}|\right]\\ +\left[(n+1)[(1-\alpha)|\gamma|+|\alpha\beta]|b_n|+\sum_{j=2}^{n-1}(n+1-j)[(1-\alpha)|\gamma|+|\alpha\beta]|b_j b_{n+1-j}|\right]\end{array}\right\}}{|\beta\gamma|\left[1-\sum_{n=2}^{\infty}\left[(n+1)|b_n|+\sum_{j=2}^{n-1}(n+1-j)|b_j b_{n+1-j}|\right]\right]}, \tag{3.7}$$

because from (3.4) it follows that

$$|\beta\gamma|\left[1-\sum_{n=2}^{\infty}\left[(n+1)|b_n|+\sum_{j=2}^{n-1}(n+1-j)|b_j b_{n+1-j}|\right]\right]>0.$$

From (3.7), we have

$$k\left|J(\alpha,\beta,\gamma;f(z),g(z))(z)-1\right|-\mathfrak{Re}\left[J(\alpha,\beta,\gamma;f(z),g(z))(z)-1\right] < (k+1)\left|J(\alpha,\beta,\gamma;f(z),g(z))(z)-1\right|$$

$$\leq \frac{(k+1)\sum_{n=2}^{\infty}\left\{\begin{array}{c}\left[n(1-\alpha)|\gamma|(|a_n|+|b_n|)+\sum_{j=2}^{n-1} j(n+1-j)(1-\alpha)|\gamma||a_j b_{n+1-j}|\right]\\ +\left[n^2|\alpha\beta||a_n|+|\alpha\beta||b_n|+\sum_{j=2}^{n-1} j^2|\alpha\beta||a_j b_{n+1-j}|\right]+\\ \left[(n+1)[(1-\alpha)|\gamma|+|\alpha\beta]|b_n|+\sum_{j=2}^{n-1}(n+1-j)[(1-\alpha)|\gamma|+|\alpha\beta]|b_j b_{n+1-j}|\right]\end{array}\right\}}{|\beta\gamma|\left[1-\sum_{n=2}^{\infty}\left[(n+1)|b_n|+\sum_{j=2}^{n-1}(n+1-j)|b_j b_{n+1-j}|\right]\right]}.$$

The last inequality is bounded above by 1, if

$$\sum_{n=2}^{\infty}(k+1)\left\{\begin{array}{c}\left[n(1-\alpha)|\gamma|(|a_n|+|b_n|)+\sum_{j=2}^{n-1} j(n+1-j)(1-\alpha)|\gamma||a_j b_{n+1-j}|\right]\\ +\left[n^2|\alpha\beta||a_n|+|\alpha\beta||b_n|+\sum_{j=2}^{n-1} j^2|\alpha\beta||a_j b_{n+1-j}|\right]\\ +\left[(n+1)[(1-\alpha)|\gamma|+|\alpha\beta]|b_n|+\sum_{j=2}^{n-1}(n+1-j)[(1-\alpha)|\gamma|+|\alpha\beta]|b_j b_{n+1-j}|\right]\end{array}\right\}$$

$$<|\beta\gamma|\left[1-\sum_{n=2}^{\infty}\left[(n+1)|b_n|+\sum_{j=2}^{n-1}(n+1-j)|b_j b_{n+1-j}|\right]\right].$$

Hence

$$\sum_{n=2}^{\infty}\Phi_n(k;\alpha,\beta,\gamma)<|\beta\gamma|,$$

where $\Phi_n(k;\alpha,\beta,\gamma)$ is given by (3.5). This completes the proof. □

When we take $g(z) = z$, $k = 1$, $\alpha = 0, 1$, $\beta = 1 - \sigma$ $(0 \leq \sigma < 1)$ and $\gamma = 1 - \eta\,(0 \leq \eta < 1)$, we obtain the following sufficient conditions for the functions to be in the classes $\mathcal{UK}(\sigma)$ and $\mathcal{UQ}(\eta)$ which are proved in [15].

Corollary 3.6. [15]. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{UK}(\sigma)$ if*

$$\sum_{n=2}^{\infty} n\,|a_n| < \frac{(1-\sigma)}{2}.$$

Corollary 3.7. [15]. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{UQ}(\eta)$ if*

$$\sum_{n=2}^{\infty} n^2\,|a_n| < \frac{(1-\eta)}{2}.$$

For $f(z) \in A$, consider the following integral operator defined by

$$F(z) = I_m[f] = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t)\,\mathrm{d}t, \quad m = 1, 2, 3 \ldots. \tag{3.8}$$

This operator was introduced by Bernardi [24] in 1969. In particular, the operator I_1 was considered by Libera [25]. Now let us consider this integral operator and prove the following result.

Theorem 3.8. *Let $f(z) \in \mathcal{TQ}(k, \alpha, \beta, \gamma)$. Then $I_m[f](z) \in k - \mathcal{UK}_\delta$ for $\delta = \frac{\beta\gamma}{(1-\alpha)\gamma + \alpha\beta}$.*

Proof. Let the function $g(z)$ be such that (1.2) is satisfied. It can easily be seen that ([26]) the function $G(z) = I_m[g](z) \in k - \mathcal{ST}$ and from (3.8), one can deduce the following.

$$(1+m)zf'(z) = z\,(zF'(z))' + mzF'(z) \tag{3.9}$$

and

$$(1+m)g(z) = zG'(z) + mG(z). \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} \frac{zf'(z)}{g(z)} &= \frac{z\,(zF'(z))' + mzF'(z)}{zG'(z) + mG(z)} \\ &= \frac{\frac{z\left(zF'(z)\right)'}{G(z)} + \frac{mzF'(z)}{G(z)}}{\frac{zG'(z)}{G(z)} + m}. \end{aligned} \tag{3.11}$$

Let $1 + \frac{1}{\delta}\left(\frac{zF'(z)}{G(z)} - 1\right) = p(z)$ with $p(z)$ analytic in $\mathbb{U}$ and $p(0) = 1$. Then

$$\frac{zf'(z)}{g(z)} = \delta\left(p(z) + \frac{zp'(z)}{q(z)+m}\right) + 1 - \delta, \tag{3.12}$$

where $q(z) = \frac{zG'(z)}{G(z)}$. Let

$$h(z) = p(z) + \frac{zp'(z)}{q(z)+m}. \tag{3.13}$$

Then from (3.12), we have

$$\frac{zf'(z)}{g(z)} = \delta h(z) + 1 - \delta, \tag{3.14}$$

and hence

$$\frac{(zf'(z))'}{g'(z)} = \delta\left(h(z) + \frac{zh'(z)}{\phi(z)}\right) + 1 - \delta, \tag{3.15}$$

where $\phi(z) = \frac{zg'(z)}{g(z)}$. Using (3.14) and (3.15) in (1.2), we have

$$\begin{aligned} J(\alpha,\beta,\gamma;f(z),g(z))(z) &= (1-\alpha)\left[1+\frac{\delta}{\beta}[h-1]\right] + \alpha\left[1+\frac{\delta}{\gamma}\left[h(z)+\frac{zh'(z)}{\phi(z)}-1\right]\right] \\ &= \left[1-\frac{[(1-\alpha)\gamma+\alpha\beta]\delta}{\beta\gamma}\right] + \left[\frac{(1-\alpha)\gamma+\alpha\beta}{\beta\gamma}\right]\delta h(z) + \frac{zh'(z)}{\frac{\gamma}{\alpha\delta}\phi(z)}. \end{aligned}$$

Taking $\delta = \frac{\beta\gamma}{(1-\alpha)\gamma+\alpha\beta}$, we have

$$J(\alpha,\beta,\gamma;f(z),g(z)) = h(z) + \frac{zh'(z)}{\frac{\gamma}{\alpha\delta}\phi(z)}.$$

Using Lemma 2.1 with $\lambda = \frac{\gamma}{\alpha\delta}$ and $\rho = 0$, we obtain

$$h(z) \prec p_k(z).$$

From (3.13), we have

$$p(z) + \frac{zp'(z)}{q(z)+m} \prec p_k(z).$$

Again by employing Lemma 2.1 with $\lambda = 1$ and $\rho = m$, we immediately obtain

$$p(z) \prec p_k(z),$$

which implies that

$$1 + \frac{1}{\delta}\left(\frac{zF'(z)}{G(z)} - 1\right) \prec p_k(z),$$

and hence

$$F(z) \in k - \mathcal{UK}_\delta.$$

This completes the proof. □

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