Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: http://ilirias.com/jiasf
Volume 8 Issue 2(2017), Pages 18-30.

GENERALIZATIONS OF SHERMAN'S THEOREM BY TAYLOR'S FORMULA

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Abstract. Extensions of Sherman's theorem to convex functions of higher order and to real weights are obtained by using Taylor's formula. New upper bounds for Sherman's difference and generalized inequalities are established. Some related Cauchy-type means are discussed.

1. Introduction and preliminaries

S. Sherman [16] proved the following theorem.

Theorem 1.1. *Let* $[\alpha, \beta] \subset \mathbb{R}$ *and for fixed* $l, m \geq 2$, *let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in [0, \infty)^l$ *and* $\mathbf{b} \in [0, \infty)^m$ *be such that*

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \text{ and } \mathbf{a} = \mathbf{b}\mathbf{A} \tag{1.1}$$

for some row stochastic matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$. *Then for every convex function* $\phi : [\alpha, \beta] \to \mathbb{R}$ *we have*

$$\sum_{i=1}^{m} b_i \phi(y_i) \leq \sum_{j=1}^{l} a_j \phi(x_j). \tag{1.2}$$

Recall here that a matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ is called *row stochastic* if all of its entries are greater or equal to zero (nonnegative) and the sum of the entries in each row is equal to 1. With $\mathbf{A}^T$ we denote the transpose of $\mathbf{A}$. Sherman obtained this useful generalization of well known Majorization theorem replacing the classical concept of majorization between real vectors $\mathbf{x} = (x_1, ..., x_l)$ and $\mathbf{y} = (y_1, ..., y_m)$ with the notion of weighted majorization (1.1) for pairs $(\mathbf{x}, \mathbf{a})$ and $(\mathbf{y}, \mathbf{b})$, where corresponding weights $\mathbf{a} = (a_1, ..., a_l)$ and $\mathbf{b} = (b_1, ..., b_m)$ are nonnegative. In recent times, Sherman's result has attracted the interest of several mathematicians (see [1]-[7], [10]-[14]). We also refer the reader to [5]. The main purpose of this paper is to give new generalizations of Sherman's theorem to n-convex functions and to real, not necessary nonnegative choice of weights $\mathbf{a}$, $\mathbf{b}$ and matrix $\mathbf{A}$. The techniques we use are based on the classical real analysis and an application of

1991 *Mathematics Subject Classification.* 26A51, 52A40.
Key words and phrases. majorization, n-convexity, Sherman's theorem, Taylor interpolating polynomial, Čebyšev functional, Grüss type inequalities, Ostrowski type inequalities, exponentially convex functions, log-convex functions, means.

Submitted October 3, 2016. Published December 4, 2016.
Communicated by R. Agarwal.

Taylor's formula with the integral remainder which we introduce in the sequel. We always assume that $[\alpha, \beta] \subset \mathbb{R}$ without having to be emphasized.

Theorem 1.2. *Let $n \in \mathbb{N}$ and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$ and $c \in [\alpha, \beta]$. Then for all $x \in [\alpha, \beta]$ we have*

$$\phi(x) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(c)}{k!}(x-c)^k + \frac{1}{(n-1)!}\int_c^x \phi^{(n)}(t)(x-t)^{n-1}dt. \tag{1.3}$$

Remark 1.3. *If we apply Taylor's formula (1.3) to ϕ at the point $c = \alpha$, we have*

$$\phi(x) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(x-\alpha)^k + \frac{1}{(n-1)!}\int_\alpha^\beta ((x-t)_+)^{n-1}\phi^{(n)}(t)dt, \tag{1.4}$$

where

$$\int_\alpha^\beta ((x-t)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^x (x-t)^{n-1}\phi^{(n)}(t)dt + \int_x^\beta 0dt.$$

Similarly, applying Taylor's formula (1.3) to function ϕ at the point $c = \beta$, we have

$$\phi(x) = \sum_{k=0}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}(\beta-x)^k - \frac{(-1)^{n-1}}{(n-1)!}\int_\alpha^\beta ((t-x)_+)^{n-1}\phi^{(n)}(t)dt, \tag{1.5}$$

where

$$\int_\alpha^\beta ((t-x)_+)^{n-1}\phi^{(n)}(t)dt = \int_\alpha^x 0dt + \int_x^\beta (t-x)^{n-1}\phi^{(n)}(t)dt.$$

Here we involve the real valued function

$$x \mapsto (x-t)_+ = \begin{cases} x-t, & t \le x \\ 0, & t > x \end{cases}. \tag{1.6}$$

The notion of n-convexity was defined in terms of divided differences by T. Popoviciu. A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, $n \ge 0$, if its n-th order divided differences $[x_0, ..., x_n; \phi]$ are nonnegative for all choices of $(n+1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$. If ϕ is n-convex, then without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)} \ge 0$ (see [15]).

In continuation we give definition and some basic facts about exponential convexity. For more details see [9].

Definition 1.4. *A function $\phi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} p_i p_j \phi\left(\frac{x_i + x_j}{2}\right) \ge 0$$

holds for all choices $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, ..., n$.
A function ϕ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Definition 1.5. *A function $\phi : I \to (0, \infty)$ is log-convex if a function $\log \phi$ is convex, or equivalently, if*

$$\phi\left((1-\lambda)s + \lambda t\right) \le \phi(s)^{1-\lambda}\phi(t)^\lambda$$

for all $s, t \in I$, $\lambda \in [0, 1]$.

Remark 1.6. *By Definition 1.4, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$. Further, a function ϕ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. Moreover, exponential convexity implies log-convexity when ϕ is positive.*

At the end, we introduce two recently obtained results involving the Čebyšev functional defined for $f, g \in L_1[\alpha, \beta]$ by

$$T(f,g) := \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt.$$

Theorem 1.7. [8, Theorem 1] *Let $f : [\alpha, \beta] \to \mathbb{R}$ be Lebesgue integrable and $g : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(g')^2 \in L_1[\alpha, \beta]$. Then*

$$|T(f,g)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta-x)[g'(x)]^2 dx \right)^{\frac{1}{2}}. \tag{1.7}$$

The constant $\frac{1}{\sqrt{2}}$ in (1.7) is the best possible.

Theorem 1.8. [8, Theorem 2] *Let $g : [\alpha, \beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dg(x). \tag{1.8}$$

The constant $\frac{1}{2}$ in (1.8) is the best possible.

Here with $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the usual Lebesgue norms on space $L_p[a, b]$.

2. Main results

Theorem 2.1. *Let $n \in \mathbb{N}$ and $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$. Let $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ and $\mathbf{b} \in \mathbb{R}^m$ be such that (1.1) holds for some matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition $\sum_{j=1}^{l} a_{ij} = 1, i = 1, 2, ..., m$. The following identities hold:*
(i)

$$\begin{aligned}
&\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) \qquad (2.1)\\
&= \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left[\sum_{j=1}^{l} a_j (x_j - \alpha)^k - \sum_{i=1}^{m} b_i (y_i - \alpha)^k \right]\\
&+ \frac{1}{(n-1)!} \int_\alpha^\beta \left[\sum_{j=1}^{l} a_j \left((x_j - t)_+\right)^{n-1} - \sum_{i=1}^{m} b_i \left((y_i - t)_+\right)^{n-1} \right] \phi^{(n)}(t) dt
\end{aligned}$$

(ii)

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) \tag{2.2}$$
$$= \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}\left[\sum_{j=1}^{l} a_j(\beta - x_j)^k - \sum_{i=1}^{m} b_i(\beta - y_i)^k\right]$$
$$- \frac{(-1)^{n-1}}{(n-1)!}\int_{\alpha}^{\beta}\left[\sum_{j=1}^{l} a_j((t - x_j)_+)^{n-1} - \sum_{i=1}^{m} b_i\left((t - y_i)_+\right)^{n-1}\right]\phi^{(n)}(t)dt.$$

Proof. (i) Using (1.4) in the difference $\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i)$, we get

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i)$$
$$= \sum_{j=1}^{l} a_j\left[\sum_{k=0}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}(x_j - \alpha)^k + \frac{1}{(n-1)!}\int_{\alpha}^{\beta}((x_j - t)_+)^{n-1}\phi^{(n)}(t)dt\right]$$
$$- \sum_{i=1}^{m} b_i\left[\sum_{k=0}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}(y_i - \alpha)^k + \frac{1}{(n-1)!}\int_{\alpha}^{\beta}((y_i - t)_+)^{n-1}\phi^{(n)}(t)dt\right].$$

Now, by interchanging the order of summation and integration, we get

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) = \sum_{k=0}^{n-1}\frac{\phi^{(k)}(\alpha)}{k!}\left[\sum_{j=1}^{l} a_j(x_j - \alpha)^k - \sum_{i=1}^{m} b_i(y_i - \alpha)^k\right]$$
$$+ \frac{1}{(n-1)!}\int_{\alpha}^{\beta}\left[\sum_{j=1}^{l} a_j\left((x_j - t)_+\right)^{n-1} - \sum_{i=1}^{m} b_i\left((y_i - t)_+\right)^{n-1}\right]\phi^{(n)}(t)dt.$$

Since under assumption (1.1) we have

$$\sum_{j=1}^{l} a_j(x_j - \alpha)^k - \sum_{i=1}^{m} b_i(y_i - \alpha)^k = 0 \quad \text{for} \quad k = 0, 1,$$

the identity (2.1) immediately follows.
(ii) Analog, using (1.5), we obtain (2.2). □

Using obtained identities we extend Sherman's theorem to n-convex functions and to real, not necessary nonnegative values of vectors $\mathbf{a}$, $\mathbf{b}$ and matrix $\mathbf{A}$.

Theorem 2.2. *Suppose that all the assumptions of Theorem 2.1 hold.*
(i) If ϕ is n-convex on $[\alpha, \beta]$ and

$$\sum_{j=1}^{l} a_j\left((x_j - t)_+\right)^{n-1} - \sum_{i=1}^{m} b_i\left((y_i - t)_+\right)^{n-1} \geq 0, \quad t \in [\alpha, \beta], \tag{2.3}$$

then

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) \geq \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left[\sum_{j=1}^{l} a_j(x_j-\alpha)^k - \sum_{i=1}^{m} b_i(y_i-\alpha)^k \right]. \tag{2.4}$$

(ii) If ϕ is n-convex on $[\alpha,\beta]$ and

$$(-1)^{n-1} \left[\sum_{j=1}^{l} a_j((t-x_j)_+)^{n-1} - \sum_{i=1}^{m} b_i\left((t-y_i)_+\right)^{n-1} \right] \leq 0, \quad t\in[\alpha,\beta], \tag{2.5}$$

then

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) \geq \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!} \left[\sum_{j=1}^{l} a_j(\beta-x_j)^k - \sum_{i=1}^{m} b_i(\beta-y_i)^k \right]. \tag{2.6}$$

Proof. (i) By Theorem 2.1 the identity (2.1) holds. Since ϕ is n-convex, then $\phi^{(n)} \geq 0$ on $[\alpha,\beta]$. Moreover, if (2.3) holds, it follows that the last term in (2.1) is nonnegative, i.e. the inequality (2.4) holds.
(ii) Analog to (i). □

Remark 2.3. *If in addition we take into account Sherman's condition of nonnegativity of vectors* **a**, **b** *and matrix* **A**, *then the assumption (2.3) is equivalent to the requirement that $x \mapsto (x-t)_+^{n-1}$, $t\in[\alpha,\beta]$, is convex function on $[\alpha,\beta]$ what exactly is true. Therefore, by Theorem 2.2 the inequality (2.4) holds. Particularly, for $n=2$ the sum on the right hand side of (2.4) is zero and we get Sherman's inequality (1.2) as direct consequence.*

Preliminary consideration we extend to the following theorem.

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.1 hold. Additionally, let* **a**, **b** *and* **A** *be nonnegative.*
(i) If ϕ is n-convex on $[\alpha,\beta]$, then (2.4) holds. Moreover, if the function

$$\tilde{F}(\cdot) = \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(\cdot-\alpha)^k \tag{2.7}$$

is convex on $[\alpha,\beta]$, then

$$0 \leq \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left[\sum_{j=1}^{l} a_j(x_j-\alpha)^k - \sum_{i=1}^{m} b_i(y_i-\alpha)^k \right] \leq \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i). \tag{2.8}$$

(ii) If n is even and ϕ is n-convex on $[\alpha,\beta]$, then (2.6) holds. Moreover, if the function

$$\bar{F}(\cdot) = \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}(\beta-\cdot)^k \tag{2.9}$$

is convex on $[\alpha,\beta]$*, then*

$$0\le\sum_{k=2}^{n-1}\frac{(-1)^k\phi^{(k)}(\beta)}{k!}\left[\sum_{j=1}^{l}a_j(\beta-x_j)^k-\sum_{i=1}^{m}b_i(\beta-y_i)^k\right]\le\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i). \tag{2.10}$$

Proof. (i) Since the function $x\mapsto((x-t)_+)^{n-1}$, $t\in[\alpha,\beta]$, is convex for every $n\in\mathbb{N}$, then by Sherman's theorem we have

$$\sum_{j=1}^{l}a_j\left((x_j-t)_+\right)^{n-1}-\sum_{i=1}^{m}b_i\left((y_i-t)_+\right)^{n-1}\ge0,\quad t\in[\alpha,\beta],$$

i.e. the condition (2.3) is satisfied. Moreover, by Theorem 2.2 the inequality (2.4) holds. Further, the right hand side of (2.4) can be written in the form

$$\sum_{j=1}^{l}a_j\tilde{F}(x_j)-\sum_{i=1}^{m}b_i\tilde{F}(y_i),$$

where $\tilde{F}$ is defined as in (2.7). If $\tilde{F}$ is convex, then by Sherman's theorem we have

$$\sum_{j=1}^{l}a_j\tilde{F}(x_j)-\sum_{i=1}^{m}b_i\tilde{F}(y_i)\ge0,$$

i.e. the right hand side of the inequality (2.4) is nonnegative, so (2.8) holds.
(ii) Since the function

$$y\mapsto(-1)^n\left((t-y)_+\right)^n=\begin{cases}0, & t<y\\ (y-t)^n, & t\ge y\end{cases},\quad t\in[\alpha,\beta],$$

is convex for even n and concave for odd n, then the function $y\mapsto(-1)^{n-1}\left((t-y)_+\right)^{n-1}$ is concave for even n and applying Sherman's theorem we have

$$(-1)^{n-1}\left[\sum_{j=1}^{l}a_j((t-x_j)_+)^{n-1}-\sum_{i=1}^{m}b_i\left((t-y_i)_+\right)^{n-1}\right]\le0,\quad t\in[\alpha,\beta],$$

i.e. the condition (2.5) is satisfied. Now, applying Theorem 2.2 we get (2.6). Further, the right hand side of (2.6) can be written in the form

$$\sum_{j=1}^{l}a_j\bar{F}(x_j)-\sum_{i=1}^{m}b_i\bar{F}(y_i),$$

where $\bar{F}$ is defined as in (2.9). The rest of proof is analog to (i). □

Remark 2.5. *Note that the function* $x\mapsto(x-\alpha)^k$ *is convex on* $[\alpha,\beta]$*, for each* $k\in\mathbb{N}$*, i.e.*

$$\sum_{j=1}^{l}a_j(x_j-\alpha)^k-\sum_{i=1}^{m}b_i(y_i-\alpha)^k\ge0,$$

for each $k\in\mathbb{N}$*. If (2.4) holds and if* $\phi^{(k)}(\alpha)\ge0$, $k=2,...,n-1$*, then the right hand side of (2.4) is nonnegative and the double inequality (2.8) holds. Note that (2.8) presents new lower bounds for Sherman's difference* $\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)$*. Similar conclusion we have in the second case when the inequality (2.6) holds. Since*

$\sum_{j=1}^{l} a_j(\beta - x_j)^k - \sum_{i=1}^{m} b_i(\beta - y_i)^k \geq 0$ *for each* $k \in \mathbb{N}$, *then if* $\phi^{(k)}(\beta) \geq 0$ *for even* k *and* $\phi^{(k)}(\beta) \leq 0$ *for odd* k, *we get the lower bound in the form (2.10).*

3. Upper bound for generalized Sherman's inequality

In the previous section we obtained the lower bounds for Sherman's difference in the forms (2.8) and (2.10). Here we present the upper bounds. In order to obtain the desired results, we define two functions $\mathcal{B}_1, \mathcal{B}_2 : [\alpha, \beta] \to \mathbb{R}$ by

$$\mathcal{B}_1(t) = \sum_{j=1}^{l} a_j \left((x_j - t)_+\right)^{n-1} - \sum_{i=1}^{m} b_i \left((y_i - t)_+\right)^{n-1}, \tag{3.1}$$

$$\mathcal{B}_2(t) = (-1)^{n-1} \left[\sum_{j=1}^{l} a_j ((t - x_j)_+)^{n-1} - \sum_{i=1}^{m} b_i \left((t - y_i)_+\right)^{n-1} \right],$$

where $n \in \mathbb{N}$, $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ and $\mathbf{b} \in \mathbb{R}^m$ are such that (1.1) holds for some matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition $\sum_{j=1}^{l} a_{ij} = 1, i = 1, 2, ..., m$.

We also consider the Čebyšev functionals defined by

$$T(\mathcal{B}_p, \mathcal{B}_p) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{B}_p^2(t)dt - \left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{B}_p(t)dt \right)^2, \quad p = 1, 2.$$

Theorem 3.1. *Let* $n \in \mathbb{N}$ *and* $\phi : [\alpha, \beta] \to \mathbb{R}$ *be such that* $\phi^{(n)}$ *is absolutely continuous on* $[\alpha, \beta]$ *with* $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L_1[\alpha, \beta]$. *Let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ *and* $\mathbf{b} \in \mathbb{R}^m$ *be such that (1.1) holds for some matrix* $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ *satisfying the condition* $\sum_{j=1}^{l} a_{ij} = 1, i = 1, 2, ..., m$. *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be defined as in (3.1).*
(i) Then

$$|R_1(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(n-1)!} [T(\mathcal{B}_1, \mathcal{B}_1)]^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} (t - \alpha)(\beta - t)[\phi^{(n+1)}(t)]^2 dt \right)^{\frac{1}{2}}, \tag{3.2}$$

where

$$\begin{aligned} R_1(\phi; \alpha, \beta) = & \sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) \\ & - \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!} \left[\sum_{j=1}^{l} a_j (x_j - \alpha)^k - \sum_{i=1}^{m} b_i (y_i - \alpha)^k \right] \\ & - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(\beta - \alpha)(n-1)!} \int_{\alpha}^{\beta} \mathcal{B}_1(t)dt. \end{aligned} \tag{3.3}$$

(ii) Then

$$|R_2(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(n-1)!} [T(\mathcal{B}_2, \mathcal{B}_2)]^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} (t - \alpha)(\beta - t)[\phi^{(n+1)}(t)]^2 dt \right)^{\frac{1}{2}}, \tag{3.4}$$

where

$$R_2(\phi;\alpha,\beta) = \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) \tag{3.5}$$
$$- \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}\left[\sum_{j=1}^{l} a_j(\beta-x_j)^k - \sum_{i=1}^{m} b_i(\beta-y_i)^k\right]$$
$$- \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-1)!}\int_\alpha^\beta \mathcal{B}_2(t)dt.$$

Proof. (i) Comparing (2.1) and (3.3) we have

$$R_1(\phi;\alpha,\beta) = \frac{\beta-\alpha}{(n-1)!}T(\mathcal{B}_1,\phi^{(n)}).$$

Applying Theorem 1.7 on the functions $\mathcal{B}_1$ and $\phi^{(n)}$ we obtain (3.2).
(ii) Similarly, we can prove this part. □

Theorem 3.2. *Let $n \in \mathbb{N}$ and $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous and $\phi^{(n+1)} \geq 0$ on $[\alpha,\beta]$. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ and $\mathbf{b} \in \mathbb{R}^m$ be such that (1.1) holds for some matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition $\sum_{j=1}^{l} a_{ij} = 1, i = 1,2,...,m$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be defined as in (3.1). Then:*
(i)

$$|R_1(\phi;\alpha,\beta)| \leq \frac{\beta-\alpha}{(n-1)!}\|\mathcal{B}_1'\|_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right], \tag{3.6}$$

where $R_1(\phi;\alpha,\beta)$ is defined by (3.3).
(ii)

$$|R_2(\phi;\alpha,\beta)| \leq \frac{\beta-\alpha}{(n-1)!}\|\mathcal{B}_2'\|_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right], \tag{3.7}$$

where $R_2(\phi;\alpha,\beta)$ is defined by (3.5)

Proof. (i) Since $R_1(\phi;\alpha,\beta) = \frac{\beta-\alpha}{(n-1)!}T(\mathcal{B}_1,\phi^{(n)})$, then applying Theorem 1.8 on the functions $\mathcal{B}_1$ and $\phi^{(n)}$ we have

$$|R_1(\phi;\alpha,\beta)| = \frac{\beta-\alpha}{(n-1)!}\left|T(\mathcal{B}_1,\phi^{(n)})\right| \tag{3.8}$$
$$\leq \frac{1}{2(n-1)!}\|\mathcal{B}_1'\|_\infty\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt.$$

Since

$$\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt = \int_\alpha^\beta [2t-(\alpha+\beta)]\,\phi^{(n)}(t)dt \tag{3.9}$$
$$= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right],$$

then combining (3.8) and (3.9) we obtain (3.6).
(ii) Similarly, we can prove this part. □

Theorem 3.3. *Suppose that all the assumptions of Theorem 2.1 hold. Let (p,q) be a pair of conjugate exponents, i.e. $1 \le p,q \le \infty$, $1/p+1/q=1$ and $\phi^{(n)} \in L_p[\alpha,\beta]$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be defined as in (3.1). Then*
(i)

$$\left|\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left[\sum_{j=1}^{l} a_j(x_j-\alpha)^k - \sum_{i=1}^{m} b_i(y_i-\alpha)^k\right]\right| \le \frac{1}{(n-1)!}\left\|\phi^{(n)}\right\|_p \|\mathcal{B}_1\|_q, \tag{3.10}$$

(ii)

$$\left|\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}\left[\sum_{j=1}^{l} a_j(\beta-x_j)^k - \sum_{i=1}^{m} b_i(\beta-y_i)^k\right]\right| \le \frac{1}{(n-1)!}\left\|\phi^{(n)}\right\|_p \|\mathcal{B}_2\|_q. \tag{3.11}$$

The constants on the right-hand side of (3.10) and (3.11) are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. (i) Applying the well known Hölder's inequality to (2.1), we obtain

$$\left|\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left[\sum_{j=1}^{l} a_j(x_j-\alpha)^k - \sum_{i=1}^{m} b_i(y_i-\alpha)^k\right]\right|$$
$$= \left|\int_\alpha^\beta \mathcal{S}(t)\phi^{(n)}(t)dt\right| \le \left\|\phi^{(n)}\right\|_p \left(\int_\alpha^\beta |\mathcal{S}(t)|^q\, dt\right)^{\frac{1}{q}},$$

where we denote

$$\mathcal{S}(t) = \frac{1}{(n-1)!}\left(\sum_{j=1}^{l} a_j\left((x_j-t)_+\right)^{n-1} - \sum_{i=1}^{m} b_i\left((y_i-t)_+\right)^{n-1} dt\right).$$

The proof of the sharpness is analog to one in proof of [4, Theorem 11].
(ii) Similar to the first part. □

4. Mean value theorems and Exponential convexity

Note that the inequalities (2.4) and (2.6) are linear in ϕ. This motivates us to define, under the assumptions of Theorem 2.2, the following linear functionals:

$$A_1(\phi) = \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \sum_{k=2}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}\left[\sum_{j=1}^{l} a_j(x_j-\alpha)^k - \sum_{i=1}^{m} b_i(y_i-\alpha)^k\right], \tag{4.1}$$

$$A_2(\phi) = \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \sum_{k=2}^{n-1} \frac{(-1)^k\phi^{(k)}(\beta)}{k!}\left[\sum_{j=1}^{l} a_j(\beta-x_j)^k - \sum_{i=1}^{m} b_i(\beta-y_i)^k\right].$$

Theorem 2.2 guarantees that for every n-convex function $\phi : [\alpha,\beta] \to \mathbb{R}$ we have $A_p(\phi) \ge 0$, $p = 1,2$. Using constructed functionals we derive mean-value theorems.

Theorem 4.1. *Let $\phi \in C^n([\alpha,\beta])$ and A_p, $p=1,2$, be the linear functionals defined as in (4.1). Then there exist $\xi_p \in [\alpha,\beta]$, $p=1,2$, such that*

$$A_p(\phi) = \phi^{(n)}(\xi_p) A_p(\varphi_0), \quad p=1,2, \tag{4.2}$$

where $\varphi_0(x) = x^n/n!$.

Proof. Our proof proceeds similarly to the proof of [9, Theorem 4.1]. □

Theorem 4.2. *Let $\phi,\psi \in C^n([\alpha,\beta])$ and A_p, $p=1,2$, be the linear functionals defined as in (4.1). Then there exist $\xi_p \in [\alpha,\beta]$, $p=1,2$, such that*

$$\frac{\phi^{(n)}(\xi_p)}{\psi^{(n)}(\xi_p)} = \frac{A_p(\phi)}{A_p(\psi)}, \quad p=1,2, \tag{4.3}$$

assuming that both denominators are non-zero.

Proof. Our proof proceeds similarly to the proof of [9, Corollary 4.2]. □

Remark 4.3. *If $\frac{\phi^{(n)}}{\psi^{(n)}}$ is an invertible function, then from (4.3) we have*

$$\xi_p = \left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1} \left(\frac{A_p(\phi)}{A_p(\psi)}\right)$$

which presents a new mean of Cauchy's type of interval $[\alpha,\beta]$.

The following results will enable us to construct exponentially convex functions.

Theorem 4.4. *Let A_p, $p=1,2$, be the linear functionals defined by (4.1). Let $\mathcal{F}_1 = \{f_v : [\alpha,\beta] \to \mathbb{R} : v \in I\}$, where I is an interval in $\mathbb{R}$, be a family of functions such that for every choice of $t+1$ mutually different points $z_0, z_1, ..., z_t \in [\alpha,\beta]$ the mapping $v \mapsto [z_0, z_1, ..., z_t; f_v]$ is n-exponentially convex in the Jensen sense on I. Then the function $v \mapsto A_p(f_v)$ is n-exponentially convex in the Jensen sense on I. Moreover, if the function $v \mapsto A_p(f_v)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. Let $q_j \in \mathbb{R}, s_j \in I$, $j=1,...,n$, and $s_{jk} = \frac{s_j+s_k}{2}$, $1 \le j,k \le n$. We consider the function $h : [\alpha,\beta] \to \mathbb{R}$ defined by

$$h(x) = \sum_{j,k=1}^{n} q_j q_k f_{s_{jk}}(x),$$

where $f_{s_{jk}} \in \mathcal{F}_1$. Since the function $v \mapsto [z_0, z_1, ..., z_t; f_v]$ is n-exponentially convex in the Jensen sense on I, we have

$$[z_0, z_1, ..., z_t; h] = \sum_{j,k=1}^{n} q_j q_k \left[z_0, z_1, ..., z_t; f_{s_{jk}}\right] \ge 0.$$

Therefore, we have

$$A_p(h) = \sum_{j,k=1}^{n} q_j q_k A_p\left(f_{s_{jk}}\right) \ge 0,$$

for each $p=1,2$, i.e. the function $v \mapsto A_p(f_v)$ is n-exponentially convex in the Jensen sense on I. Further, if $v \mapsto A_p(f_v)$ is continuous on I, then it is n-exponentially convex on I by Definition 1.4. □

The next corollary is an easy consequence of the previous theorem.

Corollary 4.5. *Let A_p, $p = 1, 2$, be the linear functionals defined by (4.1). Let $\mathcal{F}_2 = \{f_v : [\alpha, \beta] \to \mathbb{R} : v \in I\}$, where I is an interval in $\mathbb{R}$, be a family of functions such that for every choice of $t+1$ mutually different points $z_0, z_1, ..., z_t \in [\alpha, \beta]$ the mapping $v \mapsto [z_0, z_1, ..., z_t; f_v]$ is exponentially convex in the Jensen sense on I. Then the function $v \mapsto A_p(f_v)$ is exponentially convex in the Jensen sense on I. Moreover, if $v \mapsto A_p(f_v)$ is continuous on I, then it is exponentially convex on I.*

Corollary 4.6. *Let A_p, $p = 1, 2$, be the linear functionals defined as in (4.1). Let $\mathcal{F}_3 = \{f_v : [\alpha, \beta] \to \mathbb{R} : v \in I\}$, where I is an interval in $\mathbb{R}$, be a family of functions such that for every choice of $t+1$ mutually different points $z_0, z_1, ..., z_t \in [\alpha, \beta]$ the mapping $v \mapsto [z_0, z_1, ..., z_t; f_v]$ is 2-exponentially convex in the Jensen sense on I. Then the following statements hold:*
(i) If the function $v \mapsto A_p(f_v)$ is continuous on I, then it is 2-exponentially convex on I. If $v \mapsto A_p(f_v)$ is additionally positive, then it is also log-convex on I.
(ii) If the function $v \mapsto A_p(f_v)$ is positive and differentiable on I, then for all $r, s, u, w \in I$ such that $r \leq u$, $s \leq w$, we have

$$M_{r,s}\left(A_p, \mathcal{F}_3\right) \leq M_{u,w}\left(A_p, \mathcal{F}_3\right),$$

where

$$M_{r,s}\left(A_p, \mathcal{F}_3\right) = \begin{cases} \left(\frac{A_p(f_r)}{A_p(f_s)}\right)^{\frac{1}{r-s}}, & r \neq s \\ \exp\left(\frac{\frac{d}{dr}(A_p(f_r))}{A_p(f_r)}\right), & r = s \end{cases}, \tag{4.4}$$

for $f_r, f_s \in \mathcal{F}_3$.

Proof. (i) The first part of statement is an easy consequence of Theorem 4.4 and the second one of Remark 1.6.
(ii) From (i) we have that the function $v \mapsto A_p(f_v)$ is log-convex on I, i.e. the function $v \mapsto \log A_p(f_v)$ is convex on I. Then by definition of convexity (see [15, page 2]) we have

$$\frac{\log A_p(f_r) - \log A_p(f_s)}{r - s} \leq \frac{\log A_p(f_u) - \log A_p(f_w)}{u - w} \tag{4.5}$$

for $r \leq u$, $s \leq w$, $r \neq s$, $u \neq w$, i.e.

$$M_{r,s}\left(A_p, \mathcal{F}_3\right) \leq M_{u,w}\left(A_p, \mathcal{F}_3\right).$$

Case $r = s$, $u = w$ follows from (4.5) as limiting case. □

Remark 4.7. *Note that the results from Theorem 4.4, Corollary 4.5 and Corollary 4.6 still hold when two of the points $z_0, z_1, ..., z_t \in [\alpha, \beta]$ coincide, say $z_0 = z_1$, for a family of differentiable functions f_v such that the function $v \mapsto [z_0, z_1, ..., z_t; f_v]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(t+1)$ points coincide for a family of t differentiable functions with the same property. The proofs are obtained by*

$$[\underbrace{x, ..., x}_{j\text{-times}}; f] = \frac{f^{(j-1)}(x)}{(j-1)!} \tag{4.6}$$

and suitable characterization of convexity.

Example 4.8. *Consider the family of functions*

$$\Omega = \{\varphi_u : [0,\infty) \to \mathbb{R} : u > 0\},$$

defined by

$$\varphi_u(x) = \begin{cases} \frac{x^u}{u(u-1)\cdots(u-n+1)}, & u \notin \{1,...,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & u = j \in \{1,...,n-1\}, \end{cases}$$

with $0 \log 0 :\ = 0$.
Since $\frac{d^n \varphi_u}{dx^n}(x) = x^{u-n} = e^{(u-n)\log x} > 0$, *it follows that* φ_u *is* n*-convex on* $[0,\infty)$ *for every* $u > 0$. *The function* $u \mapsto \frac{d^n \varphi_u}{dx^n}(x)$ *is exponentially convex. Therefore, using the same arguments as in proof of Theorem 4.4, we conclude that the function* $u \mapsto [z_0, z_1, ..., z_t; \varphi_u]$ *is exponentially convex (and so exponentially convex in the Jensen sense). From Corollary 4.5 it follows that* $u \mapsto A_1(\varphi_u)$ *is exponentially convex (since it is exponentially convex in the Jensen sense and continuous). If* $[\alpha,\beta] \subset [0,\infty)$ *and* $A_1(\varphi_u) > 0$, *for this family of functions, (4.4) becomes*

$$M_{r,s}(A_1,\Omega) = \begin{cases} \left(\frac{s(s-1)...(s-n+1)}{r(r-1)...(r-n+1)} \frac{\sum_{j=1}^{l} a_j x_j^r - \sum_{i=1}^{m} b_i y_i^r}{\sum_{j=1}^{l} a_j x_j^s - \sum_{i=1}^{m} b_i y_i^s} \right)^{\frac{1}{r-s}}, & r \neq s, \\ \exp\left(\frac{\sum_{j=1}^{l} a_j x_j^r \log x_j - \sum_{i=1}^{m} b_i y_i^r \log y_i}{\sum_{j=1}^{l} a_j x_j^r - \sum_{i=1}^{m} b_i y_i^r} + \sum_{k=0}^{n-1} \frac{1}{k-r} \right), \\ r = s \notin \{1,...,n-1\}, \\ \exp\left(\frac{\sum_{j=1}^{l} a_j x_j^r \log^2 x_j - \sum_{i=1}^{m} b_i y_i^r \log^2 y_i}{2\left(\sum_{j=1}^{l} a_j x_j^r \log x_j - \sum_{i=1}^{m} b_i y_i^r \log y_i\right)} + \sum_{k=0, i\neq r}^{n-1} \frac{1}{k-r} \right), \\ r = s \in \{1,...,n-1\}. \end{cases} \tag{4.7}$$

For a different choice of parameters $r, s \in \mathbb{R}$, *extensions are obtained by continuity. Choosing* $f = \varphi_r$ *and* $g = \varphi_s$, *where* $r, s \in \mathbb{R}$, $r \neq s$, $r, s \neq 0, 1$, *and setting* $\alpha = \min_{i,j}\{x_j, y_i\}$, $\beta = \max_{i,j}\{x_j, y_i\}$, *by Theorem 4.2 there exists* $\xi \in [\alpha,\beta]$ *such that*

$$\alpha \leq \xi = \left(\frac{A_1(\varphi_r)}{A_1(\varphi_s)} \right)^{\frac{1}{r-s}} \leq \beta.$$

This shows that we obtain the new class of two-parameter Cauchy-type means which are obviously symmetric and monotonicity over both parameters.

Remark 4.9. *Using the linear functional* A_2, *defined by (4.1), we can on analog way derive another class of two-parameter Cauchy-type means with the same properties.*

References

[1] M. Adil Khan, S. Ivelić Bradanović and J. Pečarić, *Generalizations of Sherman's inequality by Hermite's interpolating polynomial,* Math. Inequal. Appl. **19** (4) (2016), 1181-1192.

[2] M. Adil Khan, S. Ivelić Bradanović and J. Pečarić, *Generalizations of Sherman's inequality by Hermite's interpolating polynomial and Green function,* Konuralp J. Math. **4**(2) (2016), 255-270.

[3] M. Adil Khan, J. Khan and J. Pečarić, *Generalizations of Sherman's inequality by Montgromey Idenity and Green function,* Mongolian Mathematical Journal **19** (2017), 46-63.

[4] R. P. Agarwal, S. Ivelić Bradanović and J. Pečarić, *Generalizations of Sherman's inequality by Lidstone's interpolating polynomial,* J. Inequal. Appl. 6, 2016 (2016)

[5] Albert W. Marshall, Ingram Olkin and Barry C. Arnold, *Inequalities: Theory of Majorization and Its Applications,* (Second Edition), Springer Series in Statistics, New York 2011.

[6] J. Borcea, *Equilibrium points of logarithmic potentials,* Trans. Amer. Math. Soc. **359** (2007) 3209-3237.
[7] A.-M. Burtea, *Two examples of weighted majorization*, Annals of the University of Craiova, Math. Comp. Sci. Ser. **37** (2) (2010) 92–99.
[8] P. Cerone, S. S. Dragomir, *Some new Ostrowski-type bounds for the Čebyšev functional and applications*, J. Math. Inequal. **8** (1) (2014), 159-170.
[9] J. Jakšetić, J. Pečarić, *Exponential Convexity Method*, J. Convex Anal. **20** (2013), No. 1, 181-197.
[10] M. Niezgoda, *Remarks on Sherman like inequalities for* (α, β)*-convex functions,* Math. Ineqal. Appl. **17** (4) (2014) 1579–1590.
[11] M. Niezgoda, *Remarks on convex functions and separable sequences*, Discrete Math. **308** (2008), 1765-1773.
[12] M. Niezgoda, *Vector majorization and Schur-concavity of some sums generated by the Jensen and Jensen-Mercer functionals*, Math. Inequal. Appl., **18** (2) (2015), 769-786.
[13] M. Niezgoda, *Jessen's functional and majorization*, Math. Inequal. Appl. **18** (3) (2015), 1003-1011.
[14] M. Niezgoda, *Vector joint majorization and generalization of Csiszar-Korner's inequality for f-divergence*, Discrete Appl. Math. **198** (2016), 195-205.
[15] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.
[16] S. Sherman, *On a theorem of Hardy, Littlewood, Pólya and Blackwell,* Proc. Nat. Acad. Sci. USA **37** (1) (1957), 826-831.

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